Stratification of GL$_+$(2, $\mathbb{R}$) by Topological Classes*

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Introduction

Given a class of elements G and an equivalence relation $\mathscr{R}$ in G one has the general problem of classifying the elements in G up to $\mathscr{R}$-equivalence. If G has some manifold structure and the $\mathscr{R}$-equivalence classes are submanifolds, it is natural to look at the stratification of G by these classes in order to study the bifurcation of families of elements of G.

In this paper we study this problem for G equals $GL_+(2, \mathbb{R})$ and $\mathscr{R}$ equals to topological conjugacy through an orientation preserving homeomorphism. We refer to [1], [2] and [4] as studying a similar, but not identical problem.

Our main results are contained in the two theorems in the last section of this paper.

We hope that the detailed analysis, given here for the case $n = 2$, will contribute to the understanding of the general problem of topological classification of elements in $GL_+(n, \mathbb{R})$, for $n \geq 3$, and of the geometry of the classes and also to the stratification of germs of diffeomorphisms with a fixed point in $0 \in \mathbb{R}^2$.

1. The classes are smooth manifolds

Definition: $G = GL_+(2, \mathbb{R})$ = the set of 2×2 real matrices with positive determinant.

Definition: $M \overset{t}{\sim} M'$ if there exists a homeomorphism $h: R^2 \longrightarrow R^2$ such that $h \circ M' = M \circ h$. If h is orientation preserving, we say $M \overset{t}{\sim} M'$ through an orientation preserving homeomorphism. Here $M, M' \in G$.

The topological classification of linear isomorphisms in $\mathbb{R}^n$ is still an open problem. It has been solved for $n = 2$ (classifical) and for $n = 3$ (classification of Lens spaces) but is unknown for $n \geq 4$. See Kuiper and Robbin [2] for further details.

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For the case $n = 2$, it follows from [2] that the classes (through an orientation preserving homeomorphism) in G are:

expansions:	$\lvert\lambda_1\rvert,\ \lvert\lambda_2\rvert > 1$
contractions:	$0 < \lvert\lambda_1\rvert < 1;\ \ 0 < \lvert\lambda_2\rvert < 1$
saddles:	$\lambda_1 > 1;\ \ 0 < \lambda_2 < 1$
twisted saddles:	$\lambda_1 < -1;\ \ -1 < \lambda_2 < 0$
$1 \times$ expansions:	$\lambda_1 = 1;\ \ \lambda_2 > 1$
$1 \times$ contractions:	$\lambda_1 = 1;\ \ 0 < \lambda_2 < 1$
$-1 \times -$ expansions:	$\lambda_1 = -1;\ \ \lambda_2 < -1$
$-1 \times -$ contractions:	$\lambda_1 = -1;\ \ -1 < \lambda_2 < 0$
sheers:	$[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}];\ \ [\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}];\ \ [\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}];\ \ [\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}]$
rotations:	$\left[\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right]$ for $0 < \theta < \pi,\ \ \pi < \theta < 2\pi$
I and $-I$.	

Here, λ_1, λ_2 denote the eigenvalues of the matrices and $[M]$ denotes the topological class of M. Here, also, I denotes the identity matrix. Hence, we have 14 classes and two 1-parameter families of classes.

Consider the smooth map $f: G \longrightarrow \mathbb{R}^2$ given by $f(M) = (\det M, \operatorname{tr} M)$.

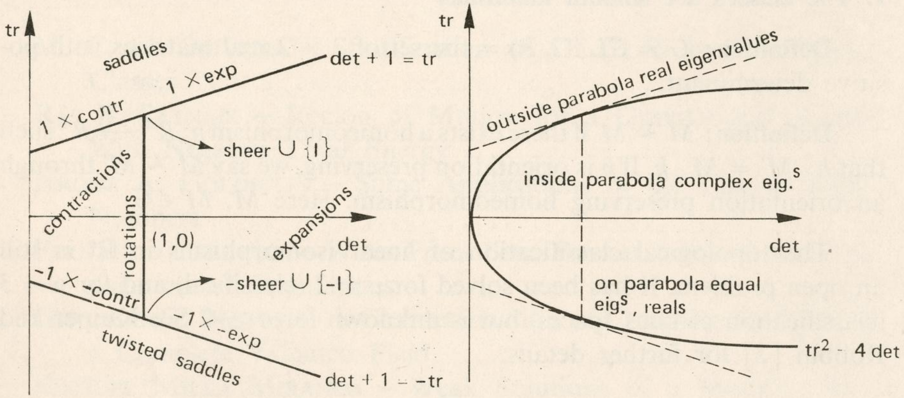


Fig. 1

A straightforward computation leads to:

(1.1) **Lemma:** *If* M *has different eigenvalues, that is,* $M \notin f^{-1}(P)$, *where* P *is the parabola* $y^2 = 4x$, *or* $M \in f^{-1}(P)$ *and is not a diagonal matrix then* M *is a regular point of* f.

(1.2) **Corollary:** *The classes* $1 \times$ exp, $1 \times$ contr, $-1 \times (-$ exp), $-1 \times (-$ contr) *and the set of all rotations in* $\mathbb{R}^2$ *are smooth submanifolds of codimencion* 1 *of* G. *The class of each rotation is a smooth submanifold of codimension* 2. *The sheers are smooth submanifolds of codimension* 2.

Expansions, contractions, saddles and twisted saddles are open subsets of G and therefore of codimension zero.

Diagram

Class	*Cod.*	*Top.^{al} type*	*Boundary*
expansions	0	$S^1 \times \mathbb{R}^3$	$(\pm 1 \times \pm \text{exp}) \cup (\pm \text{sheer}) \cup$ $\cup$ rotations $\cup \pm I$
contractions	0	$S^1 \times \mathbb{R}^3$	$(\pm 1 \times \pm \text{contr}) \cup (\pm \text{sheer}) \cup$ $\cup$ rotations $\cup \pm I$
saddles	0	$S^1 \times \mathbb{R}^3$	$(1 \times \text{exp}) \cup (1 \times \text{contr}) \cup$ $\cup (+ \text{sheer}) \cup I$
twisted saddles	0	$S^1 \times \mathbb{R}^3$	$(-1 \times -\text{exp}) \cup (-1 \times -\text{contr})$ $\cup (- \text{sheer}) \cup -$ I
$1 \times$ expansions	1	$S^1 \times \mathbb{R}^2$	$+$ sheer $\cup I$
$1 \times$ contractions	1	$S^1 \times \mathbb{R}^2$	$+$ sheer $\cup I$
$-1 \times -$ expansions	1	$S^1 \times \mathbb{R}^2$	$-$ sheer $\cup -I$
$-1 \times -$ contractions	1	$S^1 \times \mathbb{R}^2$	$-$ sheer $\cup -I$
rotations $0 < \theta < \pi$ 1-parameter family of classes	1	$\mathbb{R}^3$	$\pm I \cup [\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}] \cup [\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}]$
rotations $\pi < \theta < 2\pi$ 1-parameter family of classes	1	$\mathbb{R}^3$	$\pm I \cup [\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}] \cup [\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}]$
each rotation	2	$\mathbb{R}^2$	$\varnothing$
$[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}]$, $[\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}]$	2	$S^1 \times \mathbb{R}$	I
$[\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}]$, $[\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}]$	2	$S^1 \times \mathbb{R}$	$-I$
I and $-I$	4	point	$\varnothing$

2. Pictures

Let $$M = \begin{pmatrix} p+q & r+s \\ r-s & p-q \end{pmatrix};$$

then $$M \in G \Leftrightarrow \det M > 0$$
$$\Leftrightarrow q^2 + r^2 < p^2 + s^2.$$

We may embed G in $\mathbb{R}^4$ by sending $M \in G$ to (p, q, r, s). Let $\pi : G \longrightarrow \mathbb{R}^2$ denote the restriction to G of the projection onto (p, s)-plane. Then for each point $(p, s) \in \mathbb{R}^2$, $\pi^{-1}(p, s)$ is a set as follows:

$\varnothing$, when $(p, s) = (0, 0)$
open disc, when $(p, s) \neq (0, 0)$

$\therefore\ G \cong (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \cong S^1 \times \mathbb{R}^3.$

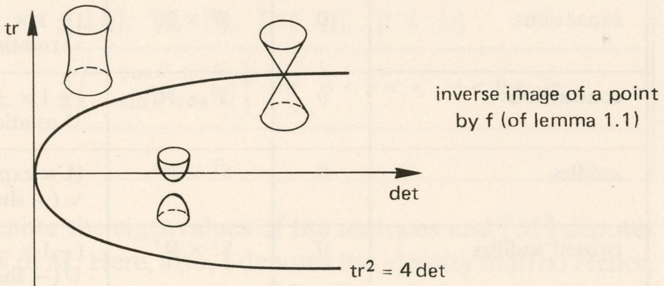
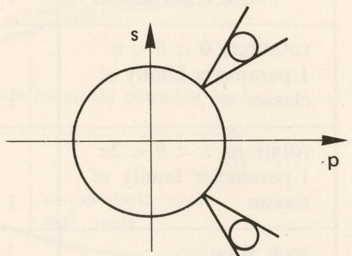


Fig. 2

$$M \in SL(2, \mathbb{R}) \Leftrightarrow \det M = 1$$
$$\Leftrightarrow q^2 + r^2 = p^2 + s^2 - 1.$$

Then, for each point $(p, s) \in \mathbb{R}^2$, $\pi^{-1}(p, s)$ is a set as follows:

$\varnothing$ when $p^2 + s^2 < 1$
point when $p^2 + s^2 = 1$
circle when $p^2 + s^2 > 1$



$\therefore\ SL(2, \mathbb{R}) \cong S^1 \times \mathbb{R}^2$

M is a rotation $\Leftrightarrow \begin{cases} \det M = 1 \\ |p| < 1 \end{cases}$ $\therefore$

$\therefore$ rotations $\cong 2 \times \mathbb{R}^3$ (foliated by copies of $\mathbb{R}^2$).
$SL(2, \mathbb{R}) \cap$ (saddles $\cup$ twisted saddles) $\cong 2 \times \mathbb{R}^2 \times S^1$.
Sheers $\cup \{I\} \cup \{-I\} \cong 2 \times$ cone.

Picture of $SL(2, \mathbb{R}) \cong$ open solid torus

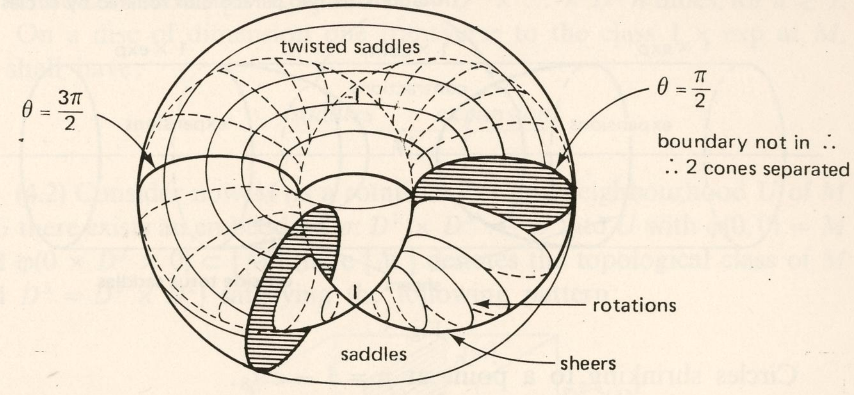


Fig. 3

Consider $S = \{M \in G \mid \det M = c > 1,$ fixed $c\}$.

$S \cong S^1 \times \mathbb{R}^2$

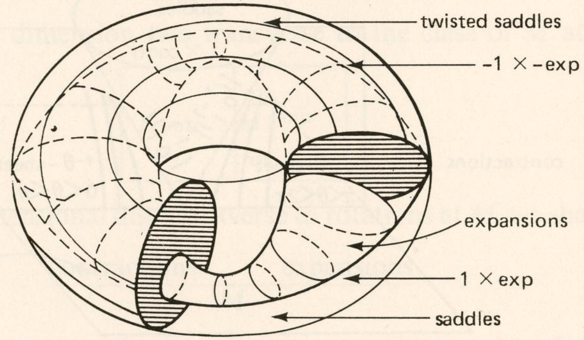


Fig. 4

When $c \longrightarrow 1$, boundary of expansions, $2 \times S^1 \times \mathbb{R}$, deepens to cones at $c = 1$, but saddles and twisted saddles remain $S^1 \times \mathbb{R}^2$. For $c \in (0,1)$, we get the same picture, just change the word expansion to contraction.

3. Neighbourhood of I

Let $M = \left(\begin{smallmatrix} p+q & r+s \\ r-s & p-q \end{smallmatrix}\right)$. The sphere S_ε of centre I and radius ε ($\varepsilon > 0$, small) is given by:

$$(p-1)^2 + s^2 + q^2 + r^2 = \varepsilon^2/_2.$$

Consider now the torus T_ε contained in S_ε given by: $(p-1)^2 + s^2 = q^2 + r^2 = \varepsilon^2/_4$. On T_ε we have:

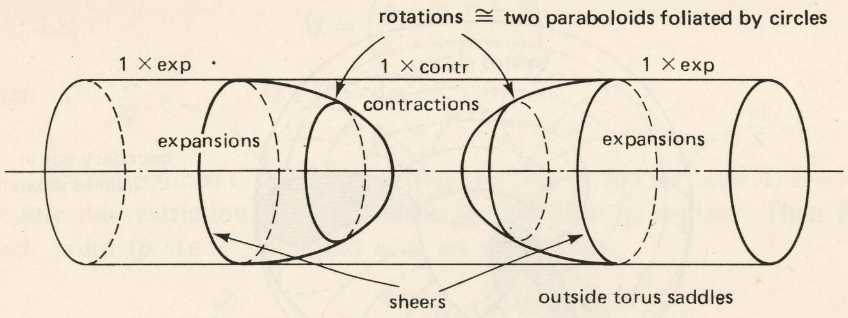


Fig. 5

Circles shrinking to a point at $p = 1 - \varepsilon^2/_8$.
Picture for a neighbourhood of I in G.

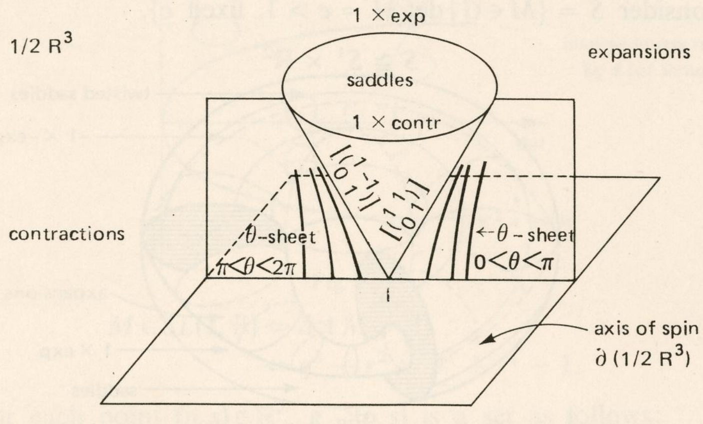
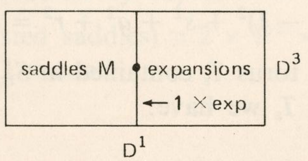


Fig. 6. Spin 1/2 $\mathbb{R}^3$ about its boundary plane

4. How the classes meet together locally

(4.1) Let us consider first the classes of codimension one. To fix our ideas let M be a matrix in $1 \times \exp$, the other cases are treated similarly.

For any neighbourhood U of M in G there exists an embedding $\varphi: D^1 \times D^3 \longrightarrow U$ into U with $\varphi(0,0) = M$ and $\varphi(0 \times D^3) \subset 1 \times \exp$, satisfying the following pattern:

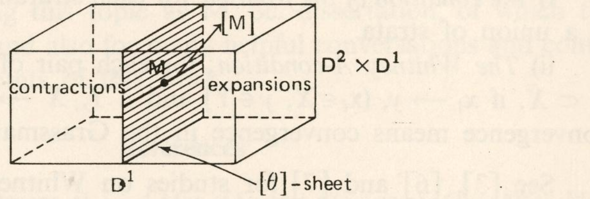


Here D^1 denotes $[-1, 1]$ and $D^n \doteq D^1 \times \ldots \times D^1$ n times, for $n \geq 1$.

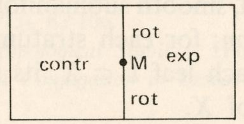
On a disc of dimension one transverse to the class $1 \times \exp$ at M, we shall have:

$$\underset{M}{\overline{\textit{saddles} \quad \cdot \quad \textit{expansions}}}$$

(4.2) Consider now M as a rotation. For any neighbourhood U of M in G there exists an embedding $\varphi: D^1 \times D^3 \longrightarrow U$ into U with $\varphi(0,0) = M$ and $\varphi(0 \times D^2 \times 0) \subset [M]$ (here $[M]$ denotes the topological class of M and $D^3 = D^2 \times D^1$) satisfying the following pattern:



On a disc of dimension two transverse to the class of M at M, we shall have:



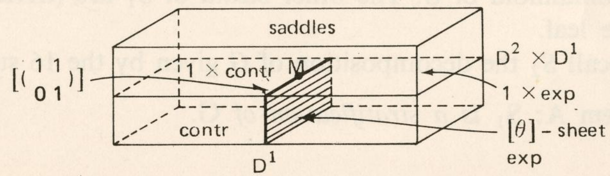
On a one-dimensional disc transverse to rotations at M, we shall have:

$$\underset{M}{\text{contractions} \quad \cdot \quad \text{expansions}}$$

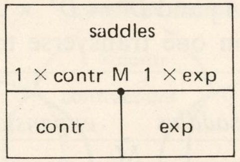
(4.3) For the classes of codimension two, consider M in $[(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix})]$. As before, the other classes are treated similarly.

Let $f: G - \{I\} \longrightarrow \mathbb{R}^2$ be given by $f(N) = (\det N, \operatorname{tr} N)$. f is transversal to the line $l = \{\det + 1 = \operatorname{tr}\}$ (lemma 1.1). So $f^{-1}(l) = (1 \times \text{contr}) \cup \cup [(\begin{smallmatrix}1 & \pm 1\\ 0 & 1\end{smallmatrix})] \cup (1 \times \exp)$ is a submanifold of codimension 1 in $G - \{I\}$ containing the submanifold $f^{-1}(1,2) = [(\begin{smallmatrix}1 & \pm 1\\ 0 & 1\end{smallmatrix})]$ in its interior.

For any neighbourhood U of M in G one can find an embedding $\varphi: D^1 \times D^3 \longrightarrow U$ into U with $\varphi(0,0) = M$ and $\varphi(0 \times D^2 \times 0) \subset [(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix})]$ satisfying the following pattern:



On a two-dimensional disc transverse to $[(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})]$ at M, we shall have:



5. The stratification of G

(5.1) **Definition:** A *stratification* S of a smooth manifold W is a cover of W by pairwise disjoint, connected, smooth submanifolds, called strata, satisfying:

i) *The condition of the frontier*; for each stratum X, its frontier $\bar{X} - X$ is a union of strata.

ii) *The Whitney A-condition*; for each pair of strata X, Y such that $Y \subset \bar{X}$, if $x_i \longrightarrow y$, $(x_i \in X,\ y \in Y)$ and if $T_{xi} X \longrightarrow \tau$, then $T_y Y \subseteq \tau$ (here convergence means convergence in the Grassmannian bundle).

See [3], [6] and [7] for studies on Whitney stratifications.

(5.2) **Definition:** A *foliated stratification* S of W is a cover of W by disjoint, connected, foliated, smooth submanifolds, called strata, satisfying:

i) *The frontier condition*; for each stratum X, its frontier $\bar{X} - X$ is union of strata, and for each leaf $L \subset X$, its frontier $\bar{L} - L$ is a union of leaves in the frontier of X.

ii) *A Whitney A^* — condition*; for each pair of strata X, Y s.t. $Y \subset \bar{X}$, if $x_i \longrightarrow y$, $(x_i \in X,\ y \in Y)$ and if $(T_{x_i}X,\ T_{x_i}L_{x_i}) \longrightarrow (\tau, \tau')$, where L_{xi} denotes the leaf containing x_i, then $(T_y\, Y,\ T_y L_y) \subseteq (\tau, \tau')$.

The importance of the A^*-condition follows from the following lemma which is easy to prove. See Trotman [5].

(4.3) **Lemma:** *If* X, Y *are two strata s.t.* Y ⊂ X̄, *and if a manifold* V *meets* Y, L_y *transversely at* y ∈ Y, *then* V *also meets* X *and its foliation transversely in a neighbourhood of* y.

Call S_1 the decomposition of G given by the topological classes using orientation preserving homeomorphisms (14 submanifolds and two 1-parameter families of submanifolds. See diagram in section 1).

Each one of the two 1-parameter families of submanifolds forms a foliated submanifold of G. The other strata of S_1 are (trivially) foliated by a single leaf.

Now, call S_2 the decomposition of G given by the 16 submanifolds.

Theorem A: S_1 *is a stratification of* G.

Theorem B: S_2 *is a foliated stratification of* G *with finitely many strata.*

Theorem A follows from theorem B.

Proof of theorem B: The frontier condition is clearly satisfied. See diagram.

Consider (X, Y) a pair of strata in S_2. Suppose $Y \subset \bar{X} - X$, and Y is a point. Then Condition A^* is automatically satisfied at Y. Now, as condition A^* is local, theorem B follows from the local analysis in section 4.

Q.E.D.

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