BOL. SOC. BRAS. MAT. 6(1975), 53–56

A Characterization of Complete Subfields of a Complete Valuated Field

Gervasio G. Bastos(*)

Let (K, φ) be a non-trivial valuated field, where φ is a real valuation (archimedean or non-archimedean). The question of a subfield F of a henselian valuated field (K, φ) (i.e., φ is henselian) being also a henselian valuated field with respect to the restriction $\varphi|_F$ was studied by Endler in [3] for non-archimedean valuations. Besides the trivial case when K/F is purely inseparable he gave other sufficient conditions under which the property of (K, φ) being henselian is inherited by $(F, \varphi|_F)$, where $\varphi|_F$ stands for the restriction of φ to F. Such conditions were:

1) L/K is normal and $L_s \neq L$
2) $[L : K]_s < \infty$ and $L_s \neq L$

where L_s is a separable closure of L and $[:]_s$ the separability degree.

In this paper we intend to study the analogous problem for an important class of henselian valuated fields, namely for complete valuated fields. In other words we shall try to give an answer to the question: when is a subfield F of a complete valuated field (K, φ) itself complete?

Initially, it will be shown that neither condition 1) nor condition 2) stated above by Endler are sufficient in our present problem. In fact, we shall give an example of a complete valuated field (K, φ) which has a subfield F which is not complete with respect to $\varphi|_F$ and such that K/F is purely inseparable. Futhermore K will be not separably closed.

For fields of characterestic zero we shall characterize the subfields F of a complete valuated field (K, φ) which are complete with respect to $\varphi|_F$, within the class of subfields F such that K/F is algebraic. It will be proven in this case: $(F, \varphi|_F)$ is complete if and only if K/F is finite.

In order to give the announced example we start with a complete valuated field (K, φ), where φ is discrete, K has characteristic $p \neq 0$ and infinite imperfection degree (i.e., K contains an infinite set T such that $x_m \notin K^p(x_1, \ldots, x_{m-1})$, for every finite subset $\{x_1, \ldots, x_{m-1}, x_m\}$ of T). Such

---

Recebido em 24 de Setembro de 1975.
(*) The autor was partially supported by project 171/CT UFC/FINEP.

a valuated field can be worked out as following. Let F_p be the Galois field with p elements, $\{X_\alpha\}_{\alpha\in I}$ an infinite set of independent variables over F_p and $K_1 = F_p(\{X_\alpha\})$ the field of the rational functions in the variables X_α having coefficients in F_p. It is obvious that K_1 has infinite imperfection degree. Now we define $K_2 = K_1(Y)$ where Y is a variable over K_1 and consider the Y-adic valuation φ_y of K_2. Let (K, φ) be a completion of (K_2, φ_y). So, (K, φ) is complete and φ is discrete. We claim that K has also infinite imperfection degree.

Now, let's recall that elements $w \neq 0$ of K can be written uniquely in the form $w = \sum_{n=r}^{\infty} a_n Y^n$, where r is an integer, $a_r \neq 0$ and $a_n \in F_p(\{X_\alpha\})$ (cf. [6], 1-9-1). One can easily show that $K^p = \left\{\sum_{n=r}^{\infty} c_n Y^{pn};\ c_n \in F_p(\{X_\alpha^p\})\right\}$. Now let's take from $\{X_\alpha\}_{\alpha\in I}$ a finite number of variables $X_1, \dots, X_m$ and suppose we have $X_m \in K^p(X_1, \dots, X_{m-1})$ or equivalently that $X_m = f(X_1, \dots, X_{m-1})/g(X_1, \dots, X_{m-1})$, where $f, g \in K^p[X_1, \dots, X_{m-1}]$. We have $f(X_1, \dots, X_{m-1}) = \sum_{n=r}^{\infty} c_n Y^{pn}, g(X_1, \dots, X_{m-1}) = \sum_{n=r}^{\infty} d_n Y^{pn}$ with $c_n = f_n/g_n$, $d_n = h_n/k_n$, where $f_n, g_n, h_n, k_n \in F_p[\{X_\beta\}_{\beta\neq m}][X_m^p]$. Then $\sum_{n=s}^{\infty} X_m \cdot d_n Y^{pn} = \sum_{n=r}^{\infty} c_n Y^{pn}$, and so $x_m \cdot f_s/g_s = h_s/k_s$ that implies $X_m \cdot k_s = g_s h_s$. Now, by looking at the degrees of both sides of the last equality we get in $F_p[\{X_\beta\}_{\beta\neq m}][X_m] : 1 \equiv deg\ (X_m \cdot f_s \cdot k_s) = deg(g_s \cdot h_s) \equiv 0$ (mod p), which is an absurdity. So K has an infinite imperfection degree.

Since φ is discrete, K cannot be separably closed (cf. [4], (27. 10) to (27. 12)).

The Frobenius map $x \longrightarrow x^p$ from K in itself defines an uniformly continuous map with respect to the topology defined by φ on K. Therefore $(K^p, \varphi|_{K^p})$ is also complete. It is interesting to observe that at this point we come across an example of an algebraic extension (M, χ) of (L, ψ) (i.e., M/L is algebraic and $\chi|_L = \psi$) where both (M, χ) and (L, ψ) are complete and $[M : L]$ infinite, which could not happen if M/L were separable (Cf. [5], § 7, pp. III – 26). Returning to our example, we are going to study the fields between K^p and K. It is well known that every finite extension (M, χ) of a complete valuated field (K, φ) is itself complete. Now we chose an infinite denumerable set $x_1, x_2, \dots$ of elements in K such that $x_m \notin \{x_1, \dots, x_{m-1}\}$ for every $n = 1, 2, \dots$. Then we have

$$K_0 = K^p \subsetneq K_1 = K^p(x_1) \subsetneq \dots \subsetneq K_j = K^p(x_1, \dots, x_j) \subsetneq \dots.$$

Define $K^* = \bigcup_{j=0}^{\infty} K_j$. Thus (K^*, φ_{K^*}) is a valuated field which we claim not to be complete. Now, if it were, by Baire's category theorem one of the K_j must have non-empty interior. Hence, $K^* = K_j = K_{j+1}$ (Cf. [2], (5.5)) which contradicts the choice of the set $\{x_1, x_2, \dots\}$. It turns out that (K, φ) is a complete valuated field which contains subfields F such that K/F is purely inseparable and $(F, \varphi|_F)$ is not complete (though henselian). Futhermore K is not separably closed.

From now on we assume that K is a field of characteristic zero and not algebraically closed. Let F be a subfield of K such that K/F is algebraic and $(F, \varphi|_F)$ is complete. Since (K, φ) is complete and K/F is separable it follows that $[K : F] < \infty$ (Cf. [5], § 7, pp. III-26). Conversely suppose we have $[K : F] < \infty$. We claim that $(F, \varphi|_F)$ is also complete. We divide the demonstration in two cases:

Case I: *φ is archimedean.* By Ostrowski's theorem (See [6], 1-8-3)

$$(K, \varphi) \cong \begin{cases} (\mathbb{C}, \phi_\infty^\rho) \\ \text{or}, \quad \text{for some} \quad \rho > 0, \quad \text{i.e., there is an} \\ (\mathbb{R}, \phi_\infty^\rho) \end{cases}$$

isomorphism σ from K onto $\mathbb{C}$ (or $\mathbb{R}$) such that $\varphi(x) = \phi_\infty(\sigma(x))^\rho$, for every $x \in K$, where $\mathbb{C}$ (resp. $\mathbb{R}$) denotes the field of the complex (resp. real) numbers and ϕ_∞ stands for the usual absolute value on $\mathbb{C}$ (or $\mathbb{R}$).

Thus, by using Artin-Schreier theorem (cf. [1] theorem 4) we conclude that F is equal to $\mathbb{R}$ or $\mathbb{C}$. Therefore $(F, \varphi|_F)$ is also complete. Observe that we did not make use of a not algebraically closed K.

Case II: *φ is non-archimedean.* By Endler's results in [3] we learned that $(F, \varphi|_F)$ is also henselian. We contend that F is closed in K with respect to the topology determined by φ. Now, let (b_n) be a sequence of elements of F which φ-converges to an element $b \in K$, that is $\varphi(b_n - b) \longrightarrow 0$ as $n \longrightarrow \infty$. By taking b_n close enough to b we have $F(b) \subseteq F(b_n) = F$, by Krasner's lemma (Cf. [6], 3-2-5), and so $(K, \varphi|_F)$ is complete. Hence the following theorem is proved.

Theorem: *Let (K, φ) be a complete valuated field with K of characteristic zero and not algebraically closed. Let F be a subfield of K such that $K \mid L$ is algebraic. Then (F, φ_F) is complete if and only if $[K : F] < \infty$.*

It is worthile to remark that for archimedean valuations the theorem above was proved also when K is algebraically closed.

On other hand for non-archimedean valuations the theorem doesn't work as the following example sugested by 0. Endler and A. Prestel shows.

Counter-example: We start with an extension to $\mathbb{C}$ of the 3-adic valuation of $\mathbb{C}$ usually denoted by φ_3 and take a completion (K, φ) of $(\mathbb{C}, \varphi_3)$. Now, (K, φ) is complete, φ is non-archimedean and K is algebraically closed, by Kurschak's theorem (Cf. [5] § 4 pp. III-25), It follows from Krull's existence theorem (Cf. [4] § 27) that the residue class field $\bar{K}$ of (K, φ) is also algebraically closed. By Artin-Schreier's theorem there exists a proper subfield F of K such that $K = F(i)$, where $i = \sqrt{-1}$ and $\bar{F}$ cannot have subfields $\bar{F}'$ with $[\bar{F} : \bar{F}'] < \infty$, for $\bar{F}$ has characteristic three. So, $\bar{F}$ is the residue class field of $(F, \varphi|_F)$ too. Suppose $(F, \varphi|_F)$ is complete. Since $\bar{F}$ is algebraically closed $x^2 + 1$ is reducible in $\bar{F}[x]$. Then, by Hensel's lemma, $x^2 + 1$ is also reducible in $F[x]$ and therefore $i \in F$ contradicting the choice F as a proper subfield of K.

References

[1] E. Artin und O. Schreier, *Eine Kennzeichung der reell abgeschlossenen Korper*. Abh. Math. Sem. Hamburg 5 (1927), 225-231.

[2] G. G. Bastos, *Sobre um problema de existência na teoria das valorizações* (Tese de doutoramento – IMPA), Rio de Janeiro, 1972.

[3] O. Endler, *A note on henselian valuation rings*, Canad. Math. Bull. 11 (1968).

[4] O. Endler, *Valuation theory*, Springer-Verlag. Berlin/Heidelberg/New York, 1972.

[5] P. Ribenboim, *Theory of Valuations* (mimeographed notes IMPA), Rio de Janeiro, 1961.

[6] E. Weiss, *Algebraic number theory*, McGraw-Hill Book Company, New York/San Francisco/Toronto/London, 1963.

Departamento de Matemática da UFC – Campus do PICI
60.000 – Fortaleza – Ceara – Brasil