Weak Solutions of a Modified kdV Equation

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Introduction

In this paper we shall prove the existence and uniqueness of periodic weak solutions for the equation

$$(*) \quad u_t + uu_x - u_{xxt} = 0$$

proposed by Benjamin-Bona-Mahony [1], as modification for the well known Korteweg-de Vries equation $u_t + uu_x + u_{xxx} = 0$. The existence and uniqueness of periodic infinitely differentiable solutions for (*) has been proved by Medeiros and Perla in [4] and for very general nonlinear term by Neves in [5]. We use in the proof the result of [4] and the compactness method as in Lions [3]. Even though it is somewhat unusual we would like to show in this case that by setting up the modified *KdV* equation in more general spaces, we can still apply the classical limiting process to prove existence and uniqueness.

In section 1, we introduce the notation and prove the existence of a weak for the equation (*). In section 2, we study the regularity of these solutions and the problem of uniqueness.

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1. Existence of Weak Solutions.

If m is a non-negative integer, $\Omega = (0, 1)$ we represent by $H^m(\Omega)$ the usual Sobolev space of order m. Represent $\partial^k/\partial x^k$ by D^k and by V^m the subspace of $H^m(\Omega)$ of all functions v such that $D^k v(0) = D^k v(1)$, for $k = 0, 1, \ldots, m-1$. Note that $H^0(\Omega) = L^2(\Omega)$. The inner product and the norm of this space will be denoted by $(.|.)$ and $||.||$, respectively. By V^∞ we represent the space of all functions $v \in C^\infty(\bar{\Omega})$ such that $D^k v(0) = D^k v(1)$

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for all k. If E is a Banach space, T a positive real number and $1 \le p \le +\infty$. We represent by $L^p(0,T;E)$ the Banach space of all measurable vector functions $u:(0,T) \longrightarrow E$, such that $\|u(t)\|_E \in L^p(0,T)$, with the norm:

$$\|u\|^p_{L^p(0,T;E)} = \int_0^T \|u(t)\|^p_E\,dt, \qquad 1 \le p < \infty$$

$$\|u\|_{L^\infty(0,T;E)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_E$$

By $C^m(0,T;E)$ we represent the space of all vector functions $u:(0,T) \longrightarrow E$ m times continuously differentiable.

The following lemmas are known from the study of Sobolev spaces [7].

Lemma 1.1. *If $u \in H^1(0,1)$, there exists a constant $k > 0$, independent of u, such that*

$$\sup_{0\le x\le 1} |u(x)| \le k\|u\|^{1/2}(\|u\| + \|u_x\|)^{1/2}$$

Lemma 1.2. *Suppose $f \in C^m(\mathbb{R})$ and $f(0)=0$. If $u(x,t) \in L^\infty(0,T;H^m(\Omega))$, then $f(u(x,t)) \in L^\infty(0,T;H^m(\Omega))$ and the following inequalities are true:*

(1.1) $$\|f(u(t))\|_{H^1(\Omega)} \le M_1\|u(t)\|_{H^1(\Omega)}$$

and

(1.2) $$\|f(u(t))\|_{H^m(\Omega)} \le c_m M_m(1 + \|u(t)\|^{m-1}_{H^{m-1}(\Omega)})\|u(t)\|_{H^m(\Omega)}$$

where $M_m = \max_{j=1,\dots,m} \sup_U \left|\frac{d^j f(s)}{ds^j}\right| \qquad U = \{s; |s| \le \sup_{\substack{0\le x\le 1\\ 0\le t\le T}} |u(x,t)|\}$

and $c_m > 0$ is a constant independent of u.

In the next theorem we prove the existence of weak solutions for (*). The idea of the proof can be summarized as follows. We take the initial data u_0 in V^1 and we approximate u_0 in $H^1(\Omega)$ norm by elements $u_{0\varepsilon}$ of V^∞. It is known that V^∞ is dense in V^1, (see Teman [8]). For each initial data $u_{0\varepsilon} \in V^\infty$ there exists a unique solution u^ε of (*) with this initial data, as in Medeiros and Perla [4]. By a priori estimates independent of ε on the solutions u^ε, it is possible to pass the limit when ε converges to zero.

Theorem 1.1. *If $u_0 \in V^1$, there exists $u(x,t)$ satisfying the following conditions:*

(1.3) $$u \in L^\infty(0,T;V^1)$$

(1.4) $$u_t + uu_x - u_{xxt} = 0 \quad \textit{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \textit{weakly star}$$

(1.5) $$u(x,0) = u_0(x) \quad \textit{on } \Omega$$

The equality in (1.4) is intended in the sense that for all $v \in L^1(0,T;L^2(\Omega))$ we have:

$$\int_0^T (u_t\,|\,v)\,dt + \int_0^T (uu_x\,|\,v) - \int_0^T (u_{xxt}\,|\,v)\,dt = 0$$

Proof: If $u_0 \in V^\infty$, there exists an unique $u:[0,1]\times[0,T] \longrightarrow LR$, $T > 0$, such that:

(1.6) $u \in C^\infty([0,1]\times[0,T])$ and $D^k u(0,t) = D^k u(1,t),\ \forall\ 0 \le t \le T,$ and $k \ge 0$

(1.7) $u_t + uu_x - u_{xxt} = 0$ pointwise in $[0,1]\times[0,T]$.

(1.8) $u(x,0) = u_0(x)$ on $[0,1]$.

Since $u_0 \in V^1$, there exists a family $\{u_{0\varepsilon}\}$, $u_{0\varepsilon} \in V^\infty$ such that $u_{0\varepsilon}$ converges to u_0 in $H^1(\Omega)$ as ε converges to zero. Therefore, for each $\varepsilon > 0$ we obtain a function u^ε satisfying the conditions (1.6), (.7) and (1.8). If we write (1.7) for u^ε and take the inner product in $L^2(\Omega)$ with u^ε, we obtain:

$$\frac{d}{dt}(\|u^\varepsilon\|^2 + \|u^\varepsilon_x\|^2) = 0,$$

from which we obtain:

(1.9) $$\|u^\varepsilon(t)\| < c_1,\ \|u^\varepsilon_x(t)\| < c_2$$

independent of ε, for all $0 \le t \le T$.

It follows from (1.9):

(1.10) u^ε is bounded in $L^\infty(0,T;L^2(\Omega))$

for any $\varepsilon > 0$ such that $u_{0\varepsilon} \longrightarrow u_0$

(1.11) u^ε_x is bounded in $L^\infty(0,T;L^2(\Omega))$

for any $\varepsilon > 0$ such that $u_{0\varepsilon} \longrightarrow u_0$

Now, if we take the inner product of both sides of (1.7) with u^ε_t, we obtain:

(1.12) $$\|u^\varepsilon_t\|^2 + (u^\varepsilon u^\varepsilon_x\,|\,u^\varepsilon_t) + \|u^\varepsilon_{xt}\|^2 = 0$$

Remark 1: We have $((u^\varepsilon)^2 u^\varepsilon_t)_x = 2u^\varepsilon u^\varepsilon_x u^\varepsilon_t + (u^\varepsilon)^2 u^\varepsilon_{xt}$ and if we integrate on $[0,1]$, we obtain $2(u^\varepsilon u^\varepsilon_x\,|\,u^\varepsilon_t) = -((u^\varepsilon)^2\,|\,u^\varepsilon_{xt})$.

Therefore, (1.12) can be written in the following form:

$$\|u^\varepsilon_t\|^2 - \tfrac{1}{2}((u^\varepsilon)^2\,|\,u^\varepsilon_{xt}) + \|u^\varepsilon_{xt}\|^2 = 0,$$

that is,

(1.13) $$4\|u^\varepsilon_t\|^2 + 3\|u^\varepsilon_{xt}\|^2 \le \int_0^1 (u^\varepsilon)^4\,dx$$

By Lemma 1.1 and estimates (1.9), we obtain

$$\sup_{0\le x\le 1} |u^\varepsilon(x,t)| < \text{constant}$$

independent of ε, for all t. It follows from (1.13), that

(1.14) $$\|u_t^\varepsilon\|^2 + \|u_{xt}^\varepsilon\|^2 < \text{constant}$$

independent of ε, for all t. This implies:

(1.15) $$u_t^\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\Omega))$$

for any $\varepsilon > 0$ such that $u_{0\varepsilon} \longrightarrow u_0$

Because the dual of $L^1(0,T;L^2(\Omega))$ is $L^\infty(0,T;L^2(\Omega))$ (see Bochner-Taylor [2]) it follows from (1.10), (1.11) and (1.15) that we can obtain a subsequence of u^ε, which we will still represent by u^ε, satisfying the following conditions:

(1.16) $$u^\varepsilon \text{ converges weak star to } u \text{ in } L^\infty(0,T;L^2(\Omega))$$

(1.17) $$u_x^\varepsilon \text{ converges weak star to } u_x \text{ in } L^\infty(0,T;L^2(\Omega))$$

(1.18) $$u_t^\varepsilon \text{ converges weak star to } u_t \text{ in } L^\infty(0,T;L^2(\Omega))$$

If $f(s) = s^2/2$, since $u^\varepsilon \in L^\infty(0,\ T;\ H^1(\Omega))$, it follows from Lemma 1.2 that $f(u^\varepsilon) \in L^\infty(0,T;H^1(\Omega))$, and:

$$\|f(u^\varepsilon)\|_{H_1(\Omega)} \le M\,\|u^\varepsilon(t)\|_{H_1(\Omega)} \quad \text{or} \quad \|f(u^\varepsilon)\|_{H_1(\Omega)} < \text{constant}$$

independent of ε for all t. It follows that

(1.19) $$f(u^\varepsilon)_x \text{ converges weak star to some } X \text{ in } L^\infty(0,T;L^2(\Omega))$$

By (1.9), (1.14) and the compactness theorem of Rellich, it follows that u^ε converges strongly to u in $L^2(Q)$, $Q = (0,1) \times (0,T)$. Therefore, pointwise a.e in Q and it follows by the continuity of $f(s)$ that $f(u^\varepsilon)$ converges to $f(u)$ in the same sense.

We also have

$$\|f(u^\varepsilon)\|_{L^2(Q)} = \int_0^T\int_0^1 |f(u^\varepsilon)|^2\,dx\,dt = \int_0^T\int_0^1 ((u^\varepsilon)^4/4)\,dx\,dt < c$$

It follows ([3], p. 12, lemma 1.3) that $f(u^\varepsilon)$ converges fo $f(u)$ weakly in $L^2(Q)$ and, therefore, $(f(u^\varepsilon))_x$ converges to $(f(u))_x$ in $\mathscr{D}'(Q)$, where $\mathscr{D}'(Q) = (C_0^\infty(Q))'$. We can say that

(1.20) $$X = (f(u))_x.$$

By the equation (1.7) and the estimates obtained above, we have the following:

(1.21) $$u_{xxt}^\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\Omega))$$

when ε converges to zero.

It follows from (1.21) that:

(1.22) $$u_{xxt}^\varepsilon \text{ converges weak star to } u_{xxt} \text{ in } L^\infty(0,T;L^2(\Omega))$$

From (1.18), (1.20), (1.22) we can say that u satisfies the conditions (1.4) of Theorem 1.1.

Let us show that u satisfies the initial condition (1.5). In fact, from (1.16) and (1.18), we have:

(1.23) $$\lim_{\varepsilon\to 0} \int_0^T (u^\varepsilon, v)\,dt = \int_0^T (u, v)\,dt$$

(1.24) $$\lim_{\varepsilon\to 0} \int_0^T (u_t^\varepsilon, v)\,dt = \int_0^T (u_t, v)\,dt$$

for all v in $L^1(0,T;L^2(\Omega))$, where each integral in (1.23) and (1.24) is the duality between $L^\infty(0,T;L^2(\Omega))$ and $L^1(0,T;L^2(\Omega))$. Take $v(x,t) = \theta(t)\,w(x)$, with $w \in L^2(\Omega)$ and $\theta \in C^1([0,T])$ such that $\theta(0) = 1$, $\theta(T) = 0$. If we consider $v = \theta' w$ in (1.23), $v = \theta w$ in (1.24), and add both equalities, we obtain $\lim_{\varepsilon\to 0}(u^\varepsilon(0), w) = (u(0), w)$ for all $w \in L^2(\Omega)$, that is, $u^\varepsilon(0)$ converges to $u(0)$ weakly in $L^2(\Omega)$. But $u^\varepsilon(0) = u_{0\varepsilon}$ converges to u_0 strongly in $L^2(\Omega)$, therefore, $u(0) = u_0$.

To prove the theorem, we need to show that $u \in L^\infty(0,T;V^1)$. In fact, from (1.16) and (1.17), it follows that u^ε converges to u and u_x^ε converges to u_x weakly in $L^2(0,1;L^2(0,T))$. Therefore, u and u^ε belong to $C^0([0,1],\ L^2(0,T))$ ([3], pag. 7, lemma 1.2) and

$$\lim_{\varepsilon\to 0} \int_0^1 (u^\varepsilon, v)\,dx = \int_0^1 (u, v)\,dx$$

$$\lim_{\varepsilon\to 0} \int_0^1 (u_x^\varepsilon, v)\,dx = \int_0^1 (u_x, v)\,dx$$

for all v in $L^2(0,1;L^2(0,T))$. Take $v(x,t) = \theta(t)\,w(x)$ with $\theta \in L^2(0,T)$ and and $w \in C^1([0,1]$, $w(0) = 1$, $w(1) = 0$, to obtain, by the same argument used above, $u^\varepsilon(0,t)$ converges to $u(0,t)$ weakly in $L^2(0,T)$. Also, $u^\varepsilon(1,t)$ converges to $u(1,t)$ weakly in $L^2(0,T)$. Since $u^\varepsilon(0,t) = u^\varepsilon(1,t)$ for all $0 \le t \le T$, it follows that $u(0,t) = u(1,t)$ a.e. in $0 \le t \le T$ and (1.16), (1.17) implies that u belong to $L^\infty(0,T;V^1)$.

2. Regularity and Uniqueness.

In this section we study the solutions of equation (*) when we take the initial data in a Sobolev space of order $k \ge 1$. If we increase the order k where we choose the initial data, we obtain solutions of (*), each time more regular. To complete the study of this type of question for the equation (*), we prove the uniqueness of such solutions for $k > 1$.

Theorem 2.1. *If $u_0 \in V^k$, $k \geq 1$, then there exists a function u in $L^\infty(0, T; V^k)$, satisfying equation (1.4) and the initial data (1.5). Furthermore, if $k > 1$, this u is unique.*

Proof of Existence. Since $u_0 \in V^k$, there exists a family $u_{0\varepsilon} \in V^\infty$ such that $u_{0\varepsilon}$ converges to u_0 in $H^k(\Omega)$. Let us prove the theorem by induction. If $k = 1$ we have the Theorem 1.1; suppose the Theorem 2.1 is true for $k > 1$ and let us prove it is true for $k + 1$. If $f(s) = s^2/2$, from (1.4) we have:

$$\frac{d}{dt}(\|D^k u^\varepsilon\|^2 + \|D^{k+1} u^\varepsilon\|^2) \leq (\|D^k f(u^\varepsilon)\|^2 + \|D^{k+1} u^\varepsilon\|^2 \tag{2.1}$$

By the hypotheses of induction, since $D^k u^\varepsilon$ is bounded in $L^\infty(0, T; L^2(\Omega))$, we have by the Lemma 1.2.

$$\|D^k f(u^\varepsilon)\|^2 \leq \text{constant}$$

independent of ε for all t.

Therefore, by (2.1) we see that $D^{k+1} u^\varepsilon$ is bounded in $L^\infty(0, T; L^2(\Omega))$, that is, taking subsequence $D^{k+1} u^\varepsilon$ converges to $D^{k+1} u$ in $L^\infty(0, T; L^2(\Omega))$ in the weak star convergence. By the same argument used in Theorem 1.1 we obtain $D^k u(0, t) = D^k u(1, t)$ a.e. in $[0, T]$, that is, $u \in L^\infty(0, T; V^{k+1})$.

Proof of Uniqueness. Let u, v be two solutions of (1.4) corresponding to the same initial data u_0 and $w = u - v$. Therefore, $w_t - w_{xxt} + uu_x - vv_x = 0$, $w(0) = w_x(0) = 0$. It follows that

$$\frac{d}{dt}(\|w\|^2 + \|w_x\|^2) = -2(uu_x - vv_x \mid w) \tag{2.2}$$

A short calculation shows that $(uu_x - vv_x \mid w) = (uu_x - vu_x \mid w) + (vu_x - vv_x \mid w) = \int_0^1 u_x w^2\, dx + \frac{1}{2}\int_0^1 v w_x^2\, dx = \int_0^1 u_x w^2\, dx - \frac{1}{2}\int_0^1 v_x w^2\, dx$, and therefore by Lemma 1.1 and (2.2) we get

$$\frac{d}{dt}(\|w\|^2 + \|w_x|^2) \leq c\,\|w\|^2$$

which implies $w \equiv 0$.

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