Galerkin Methods Applied to the Benjamin-Bona-Mahony Equation

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1. Introduction

Consider the equation for $u = u(x, t)$,

$$u_t + \beta u_x + \gamma u u_x - \delta u_{xxt} = 0, \quad -\infty < x < +\infty, \quad 0 < t \leq T, \tag{1.1}$$

subject to the initial condition

$$u(x, 0) = u_0(x), \quad -\infty < x < +\infty, \tag{1.2}$$

and the periodicity condition

$$u(x+1, t) = u(x, t), \quad -\infty < x < +\infty, \; 0 \leq t \leq T, \tag{1.3}$$

with β, γ and δ as positive constants.

This equation was proposed by T. B. Benjamin, J. L. Bona and J. J. Mahony as a model for the propagation of long waves in non-linear dispersive systems. In [1] they solved the initial value problem in a class of real non-periodic functions defined for $-\infty < x < +\infty$, $t \geq 0$. Existence of a unique periodic solution of the initial value problem was proved with different techniques by L. A. Medeiros and G. Perla Menzala in [3] and M. Milla Miranda in [4]. The result in [3] is: if u_0 is three times differentiable and u_0''' is square integrable on [0,1], then there exists only one u, having all derivatives in t and twice continuously differentiable in x, satisfying (1.1)-(1.3). From their proof we can extract the following lemma which will be useful in our analysis later.

Lemma 1.1. *Let u be the solution of* (1.1)-(1.3). *Then, for* $t \geq 0$,

$$\max_{x \in [0,1]} |u(x,t)| + \max_{x \in [0,1]} |u_x(x,t)| \leq K, \tag{1.4}$$

K depending only on the data.

In this paper we shall be concerned with the numerical solution of (1.1)-(1.3) by methods of Galerkin type in which the approximate solutions are periodic cubic splines in the space variable x at each time level.

Recebido em 23 de Janeiro de 1976.

More precisely, given positive integers N and M, we define $h = 1/N + 1$, $x_j = jh$, $\Delta t = T/M + 1$, $t_n = n\Delta t$, for $j = 0, 1, \ldots, N + 1$, $n = 0, 1, \ldots, M + 1$, and $S_h = \{\phi \varepsilon C^2([0, 1]) \mid \phi$ is a cubic polynomial on each $[x_{j-1}, x_j]$ and $d^k\phi dx^k(0) = d^k\phi/dx^k(1)$, for $k = 0, 1, 2\}$. Then we shall seek sequences $\{U_n(x) \mid n = 0, 1, \ldots, M + 1\}$ in S_h such that $U_n(x)$ is a good approximation of $u(x, t_n)$.

We recall that a basis for S_h can be constructed starting from the characteristic function $\chi(x)$ of the interval $[0, 1]$, in the following way. We take

$$\psi(x) = \chi * \chi * \chi * \chi(x)$$

where $*$ is the convolution product

$$f * g(x) = \int_{-\infty}^{\infty} f(x - y)\, g(y)\, dy,$$

and then define

$$\psi_0(x) = \psi\left(\frac{x}{h} + 3\right)\chi_{[0, x_1]}(x) + \psi\left(\frac{x}{h} - N + 2\right)\chi_{[x_{N-2}, 1]}(x),$$

$$\psi_1(x) = \psi\left(\frac{x}{h} + 2\right)\chi_{[0, x_2]}(x) + \psi\left(\frac{x}{h} - N + 1\right)\chi_{[x_{N-1}, 1]}(x),$$

$$\psi_2(x) = \psi\left(\frac{x}{h} + 1\right)\chi_{[0, x_3]}(x) + \psi\left(\frac{x}{h} - N\right)\chi_{[x_N, 1]}(x),$$

$$\psi_j(x) = \psi\left(\frac{x}{h} - j + 3\right),\ j = 3, \ldots, N,$$

where $\chi_{[x_j, x_k]}$ is the characteristic function of the interval $[x_j, x_k]$. We have

$$S_h = \text{span}\,\{\psi_0, \ldots, \psi_N\}.$$

The first scheme we consider is a predictor-corrector. At each time level we initially compute the predicted approximation $\hat{U}_n \in S_h$ by the Galerkin condition

$$(1.5)\qquad < \frac{\hat{U}_n - U_{n-1}}{\Delta t} + \beta\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x - \delta\frac{(\hat{U}_n - U_{n-1})_{xx}}{\Delta t} + \gamma U_{n-1}\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x,\ \phi > = 0,\ \phi \in S_h,$$

and then correct it to the final approximation $U_n \in S_h$ by

$$(1.6)\qquad < \frac{U_n - U_{n-1}}{\Delta t} + \beta\left(\frac{U_n + U_{n-1}}{2}\right)_x - \delta\frac{(U_n - U_{n-1})_{xx}}{\Delta t} + \gamma\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)\left(\frac{U_n + U_{n-1}}{2}\right)_x\quad \phi > = 0,\ \phi \in S_h.$$

Here $\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$.

If integration by parts of the third term in the sum is performed, the Galerkin conditions (1.5) and (1.6) are written in the form

$$(1.5)'\qquad < \frac{\hat{U}_n - U_{n-1}}{\Delta t}, \phi > + \delta < \frac{(\hat{U}_n - U_{n-1})_x}{\Delta t}, \phi_x > + \beta < \left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x, \phi > + \gamma < U_{n-1}\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x, \phi > = 0,\quad \phi \in S_h,$$

$$(1.6)'\qquad < \frac{U_n - U_{n-1}}{\Delta t}, \phi > + \delta < \frac{(U_n - U_{n-1})_x}{\Delta t}, \phi_x > + \beta < \left(\frac{U_n + U_{n-1}}{2}\right)_x, \phi > + \gamma < \left(\frac{\hat{U}_n + U_{n-1}}{2}\right)\left(\frac{U_n + U_{n-1}}{2}\right)_x, \phi > = 0,\ \phi \in S_h.$$

Those equations, for $n = 1, 2, \ldots, M + 1$, together with an initial condition U_0 given in S_h, for example, one characterized by

$$(1.7)\qquad \langle U_0, \phi\rangle = \langle u_0, \phi\rangle, \phi \in S_h,$$

define the numerical process. We start from the data at $t = 0$ and then compute the approximations step by step. If we write

$$U_n(x) = \sum_{j=0}^{N} C_j^n \psi_j(x) \quad , \quad n = 0, 1, \ldots, M + 1,$$

$$\hat{U}_n(x) = \sum_{j=0}^{N} \hat{C}_j^n \psi_j(x) \quad , \quad n = 1, \ldots, M + 1,$$

then we can see that (1.7), (1.5)′ and (1.6)′ are linear equations for the vectors $C^0 = (C_j^0)$, $\hat{C}^n = (\hat{C}_j^n)$, $C^n = (C_j^n)$, $n = 1, 2, \ldots, M + 1$, with associated matrices A, $A + \delta B + \beta\Delta t/2\, D + \gamma\Delta t/2\, E(C^{n-1})$ and $A + \delta B + \beta\Delta t/2\, D + \gamma\Delta t/2\, E(\hat{C}^n + C^{n-1}/2)$, respectively, which are invertible, at least for Δt sufficiently small.

Here $[A]_{ij} = \langle\psi_i, \psi_j\rangle$, $\quad [B]_{ij} = \langle\psi_{ix}, \psi_{jx}\rangle$, $\quad [D]_{ij} = \langle\psi_{ix}, \psi_j\rangle$,

$$[E(a)]_{ij} = \sum_{k=0}^{N} a_k \langle\psi_k \psi_{ix}, \psi_j\rangle, \quad \text{where } a = (a_k).$$

The second scheme we propose to approximate $u(x, t_n)$ involves only one linear system per time level. The procedure is defined by the following conditions

(1.8) $V_n \in S_h$,

(1.9) $$< \frac{V_n - V_{n-1}}{\Delta t}, \phi > + \delta < \frac{(V_n - V_{n-1})_x}{\Delta t}, \phi_x > + \beta < \left(\frac{V_n + V_{n-1}}{2}\right)_x, \phi > + \gamma < \left(\frac{3}{2} V_{n-1} - \frac{1}{2} V_{n-2}\right) \cdot \left(\frac{V_n + V_{n-1}}{2}\right)_x, \phi > = 0, \ \phi \in S_h,$$

(1.10) V_0 and V_1 given in S_h,

where $n = 2, 3, \ldots, M + 1$.

Although there are alternative possibilities, the initial conditions V_0 and V_1 can be taken from equations (1.7) and (1.5)′-(1.6)′ with $n = 1$, for example, and that is the choice we shall proceed with, for the sake of definiteness.

Observe that if we expand the new approximation as

$$V_n(x) = \sum_{j=0}^{N} C_j^n \psi_j(x), \ n = 0, 1, 2, \ldots, M + 1,$$

we see that the matrices involved in this scheme are the same as before, with $E(a)$ evaluated at $a = 3/2\, C^{n-1} - 1/2\, C^{n-2}$. Hence the approximations are well defined if we take Δt small enough.

The object of this paper is to analyse the convergence of both $U_n(x)$ and $V_n(x)$ to $u(x, t_n)$ when h and Δt go to zero. We shall demonstrate that constants C_1 and C_2 can be found, depending on the data of the problem, on u and certain of its derivatives, such that

$$||u - U_n||_1 \le C_1 [h^3 + (\Delta t)^2],$$
$$||u - V_n||_1 \le C_2 [h^3 + (\Delta t)^2],$$

if U_0, V_0 and V_1 are chosen as indicated above or in an equivalent way. Those estimates are true for $n = 1, 2, \ldots, M + 1$, and h and Δt sufficiently small. They give optimal order of convergence with respect to the norm $||\cdot||_1$, the usual norm of the classical Sobolev space $H^1(0, 1)$, since we are using cubic splines approximations.

The proof of this main result will be presented in section 3. In section 2 we discuss a basic stability lemma and in section 4 present numerical results.

2. A Preliminary Lemma

The functions considered here are real valued and C will denote a generic constant. We adopt the usual notation

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx,$$
$$\langle f, g \rangle_1 = \int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx,$$
$$||f|| = \sqrt{\langle f, f \rangle},$$
$$||f||_1 = \sqrt{\langle f, f \rangle_1},$$

and

$$|f|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

A priori estimates for equations (1.5)′, (1.6)′ and (1.9) will be derived for use in the convergence analysis. We summarize them in the following.

Lemma 2.1. *Any possible solutions of* (1.5)′, (1.6)′ *and* (1.9) *satisfy, respectively, for* Δt *sufficiently small,*

(2.1) $$|\hat{U}_n|_\infty \le C,$$

(2.2) $$|U_n|_\infty \le C,$$

(2.3) $$|V_n|_\infty \le C,$$

for $n = 1, 2, \ldots, M + 1$, *where the C's depend only on the data.*

In proving the lemma we shall need some well known results from the theory of Sobolev spaces which we state now. Proofs are given in [6]. Let

$$H^0(0,1) = \{u : [0,1] \longrightarrow R \,|\, ||u|| < \infty\},$$
$$H^1(0,1) = \{u \in H^0(0,1) \,|\, u' \in H^0(0,1)\}$$

and

$$L^\infty(0, T; H^1(0,1)) = \{u : [0, T] \longrightarrow H^1(0,1) \,|\, \sup_{t \in [0,T]} ||u(., t)||_1 = ||u||_{L^\infty(0,T;H^1(0,1))} < \infty\}.$$

Then the following is true:

(i) If $u \in H^1(0,1)$, there exists a constant C, independent of u, such that

(2.4) $$|u|_\infty \leq C\,||u||^{1/2}\,||u||_1^{1/2};$$

(ii) Let $f : R \longrightarrow R$ be a continuously differentiable function with $f(0) = 0$. If $u \in L^\infty(0, T; H^1(0,1))$ then $f(u) \in L^\infty(0, T; H^1(0,1)$, and

(2.5) $$||f[u(.,t)]||_1 \leq C\,||u(.,t)||_1,\ t \in [0, T],$$

where C is a constant independent of u.

Proof of Lemma 2.1. In equation (1.5)′ we take $\phi = 1/2(\hat{U}_n + U_{n-1})$, so that

(2.6) $$\frac{1}{2\Delta t}\{[||\hat{U}_n||^2 + \delta||(\hat{U}_n)_x||^2] - [||U_{n-1}||^2 + \delta\,||(U_{n-1})_x||^2]\}$$
$$= -\beta < \left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x, \left(\frac{\hat{U}_n + U_{n-1}}{2}\right) >$$
$$- \gamma < U_{n-1}\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x, \frac{\hat{U}_n + U_{n-1}}{2} >.$$

The last term in the second member of this equation can be bounded as

$$\left|\gamma < U_{n-1}\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x, \frac{\hat{U}_n + U_{n-1}}{2} > \right| =$$
$$= \gamma\left|\int_0^1 U_{n-1}\left[1/2\left(\frac{\hat{U}_n + U_{n-1}}{2}\right)^2\right]_x dx\right| \leq C\,\{||\hat{U}_n||_1^2 + ||U_{n-1}||_1^2\},$$

if we use Cauchy-Schwartz inequality and formula (2.5) with $f(s) = s^2/2$.

Hence from (2.6) we get

(2.7) $$\{[||\hat{U}_n||^2 + \delta||(\hat{U}_n)_x||^2] - [||U_{n-1}||^2 + \delta||(U_{n-1})_x||^2]\}$$
$$\leq C\,\Delta t\,\{||\hat{U}_n||_1^2 + ||U_{n-1}||_1^2\}.$$

Now in equation (1.6)′ we choose $\phi = 1/2(U_n + U_{n-1})$ and by the same reasoning conclude that

(2.8) $$\{[||U_n||^2 + \delta\,||(U_n)_x||^2] - [||U_{n-1}||^2 + \delta\,||(U_{n-1})_x||^2]\} \leq$$
$$\leq C\,\Delta t\,\{||U_n||_1^2 + ||\hat{U}_n||_1^2 + ||U_{n-1}||_1^2\}.$$

If we take Δt small enough in (2.7), it implies

(2.9) $$||\hat{U}_n||_1 \leq C\,||U_{n-1}||_1.$$

Equations (2.8) and (2.9) combined give

(2.10) $$\{[||U_n||^2 + \delta||(U_n)_x||^2] - [||U_{n-1}||^2 + \delta||(U_{n-1})_x||^2]\}$$
$$\leq C\,\Delta t\,\{||U_n||_1^2 + ||U_{n-1}||_1^2\}.$$

Adding from 1 to m, we get

(2.11) $$||U_m||_1^2 \leq \frac{\max(1,\delta)}{\min(1,\delta)}\,||U_0||_1^2 + C\sum_{j=1}^{m}\Delta t\,||U_j||_1^2,$$

for $m = 1, 2, \ldots, M+1$. Applying Gronwall's lemma to (2.11) we finally obtain

(2.12) $$||U_m||_1 \leq \text{const.}\,||U_0||_1, \qquad m = 1, 2, \ldots, M+1,$$

which concludes the proof of formula (2.2) if we combine it with (2.4). Estimate (2.1) results from (2.9), (2.12) and (2.4).

After taking $\phi = 1/2(U_n + U_{n-1})$ in (1.9), the argument for the justification of (2.3) is completely analogous to the one presented above.

In effect, this lemma implies that the two schemes we are proposing are unconditionally stables, in the sense defined in Lions [2].

3. Convergence Analysis

Having the necessary tools to prove it we can now state formally the main result of this paper. The approximations will be seen to satisfy the following error bound.

Theorem 3.1. *Suppose the exact solution u is four times continuously differentiable in x and three times in t over $(0, 1) \times (0, T)$. Then there exist constants τ_0 and C, depending on the data and $\partial^{p+q}/\partial x^p \partial t^q\, u(x, t)$, $p = 0, 1, \ldots, 4, q = 0, \ldots, 3$, such that, for $\Delta t \leq \tau_0$,*

(3.1) $$\sup_{0 \leq n \leq M+1} ||U_n - u(.,t_n)||_1 \leq C\,[h^3 + (\Delta t)^2],$$

(3.2) $$\sup_{0 \leq n \leq M+1} ||V_n - u(.,t_n)||_1 \leq C\,[h^3 + (\Delta t)^2].$$

Proof. We initially write the equation for the exact solution at $t = t_{n-1/2} = (n - 1/2)\,\Delta t$ in the weak finite difference form

(3.3) $$< \frac{u(.,t_n) - u(.,t_{n-1})}{\Delta t}, \phi > + \delta < \frac{u_x(.,t_n) - u_x(.,t_{n-1})}{\Delta t}, \phi_x >$$
$$+ \beta < \frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}, \phi > +$$
$$+ \gamma < u(.,t_{n-1})\,\frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}, \phi >$$
$$= < A_n(.), \phi >, \ \phi \in H_p^1(0,1),$$

with the initial condition

(3.4) $$\langle u(.,0), \phi\rangle = \langle u_0, \phi\rangle, \phi \in H_p^1(0,1)$$

where

$$H_p^1(0,1) = \{u \in H^1(0,1) \mid u(0) = u(1)\},$$

and $A_n(x) = 0(\Delta t)$ pointwise.

Now we choose ϕ in $S_h \subset H_p^1(0,1)$ and take the difference between (3.3) and (1.5)'. to obtain

$$< \frac{\hat{e}_n - e_{n-1}}{\Delta t}, \phi > + \delta < \frac{(\hat{e}_n - e_{n-1})_x}{\Delta t}, \phi_x > + \beta < \frac{(\hat{e}_n + e_{n-1})_x}{2}, \phi >$$

$$+ \gamma < u(., t_{n-1}) \frac{u_x(., t_n) + u_x(., t_{n-1})}{2}, \phi > - \gamma < U_{n-1} \left(\frac{\hat{U}_n + U_{n-1}}{2}\right)_x,$$

$$\phi > = < A_n, \phi > \quad , \quad \phi \in S_h,$$

where $e_n = u(., t_n) - U_n$ and $\hat{e}_n = u(., t_n) - \hat{U}_n$.

If we add and subtract $< U_{n-1} \cdot u_x(., t_n) + u_x(., t_{n-1})/2, \phi >$ to the above equation we get the following relation for the errors $\hat{e}_n$ and e_n :

(3.5) $$< \frac{\hat{e}_n - e_{n-1}}{\Delta t}, \phi > + \delta < \frac{(\hat{e}_n - e_{n-1})_x}{\Delta t}, \phi_x >$$

$$+ \beta < \frac{(\hat{e}_n + e_{n-1})_x}{2}, \phi > + \gamma < e_{n-1} \left(\frac{u_x(., t_n) + u_x(., t_{n-1})}{2}\right), \phi >$$

$$+ \gamma < U_{n-1} \left(\frac{\hat{e}_n + e_{n-1}}{2}\right)_x, \phi > = < A_n, \phi > \ , \ \phi \in S_h .$$

We claim then that the estimate

(3.6) $$\|\hat{e}_n\|_1 \le C \{(\Delta t)^2 + \|e_{n-1}\|_1 + \\ + \|\tfrac{1}{2} [(u(., t_n) - \theta_h u(., t_n)) + (u(., t_{n-1}) - \\ - \theta_h u(., t_{n-1}))] \|_1\}$$

is implied by (3.5) for Δt sufficiently small. Here θ_h is the interpolation map

$$\theta_h : H^1(0,1) \longrightarrow S_h$$
$$u \longrightarrow \theta_h u$$

with $\theta_h u$ uniquely defined by the conditions $u(jh) = \theta_h u(jh)$, $j = 0, 1, \ldots, N$.

To see this let us take $\phi = (\hat{e}_n + e_{n-1})/2 + \frac{1}{2}[\theta_h u(., t_n) + \theta_h u(., t_{n-1})] - \frac{1}{2}[u(., t_n) + u(., t_{n-1})] = \hat{e}_n + e_{n-1}/2 + \eta_{n-1/2}$ in (3.5). It follows

(3.7) $$\frac{1}{2\Delta t} \{[\|\hat{e}_n\|^2 + \delta \|(\hat{e}_n)_x\|^2] - [\|e_{n-1}\|^2 + \delta \|(e_{n-1})_x\|^2]\} =$$

$$= - < \frac{\hat{e}_n - e_{n-1}}{\Delta t}, \eta_{n-1/2} > - \delta < \frac{(\hat{e}_n - e_{n-1})_x}{\Delta t}, (\eta_{n-1/2})_x >$$

$$- \beta < \left(\frac{\hat{e}_n + e_{n-1}}{2}\right)_x, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} >$$

$$- \gamma < e_{n-1} \left(\frac{u_x(., t_n) + u_x(., t_{n-1})}{2}\right), \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} >$$

$$- \gamma < U_{n-1} \left(\frac{\hat{e}_n + e_{n-1}}{2}\right)_x, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} >$$

$$+ < A_n, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} > .$$

The terms in the right hand side of (3.7) can be bounded in the following way:

$$| < A_n, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} > | \le \|A_n\| (\|\hat{e}_n\| + \|e_{n-1}\|$$

$$+ \|\eta_{n-1/2}\|) \le C \, \Delta t \, (\|\hat{e}_n\| + \|e_{n-1}\| + \|\eta_{n-1/2}\|)$$

$$\le \Delta t \, [\varepsilon(\|\hat{e}_n\| + \|e_{n-1}\| + \|\eta_{n-1/2}\|)^2 + \frac{C^2}{4\varepsilon}]$$

$$\le \frac{\alpha}{\Delta t} [\|\hat{e}_n\|^2 + \|e_{n-1}\|^2 + \|\eta_{n-1/2}\|^2] + \frac{C^2}{4\alpha} \Delta t^3$$

where we had chosen $\varepsilon = \alpha \, \Delta t^{-2}$, and α is a positive number to be chosen conveniently later;

$$|\gamma < U_{n-1} \left(\frac{\hat{e}_n + e_{n-1}}{2}\right)_x, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} > | \le C \, |U_{n-1}|_\infty$$

$$\{\|\hat{e}_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|^2\} \le C\{\|\hat{e}_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|^2\},$$

where we used lemma 2.1 to bound $|U_{n-1}|_\infty$;

$$|\gamma < e_{n-1} \left(\frac{u_x(., t_n) + u_x(., t_{n-1})}{2}\right), \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} > | \le C\{\|\hat{e}_n\|^2$$

$$+ \|e_{n-1}\|^2 + \|\eta_{n-1/2}\|^2\},$$

where we used lemma 1.1 to bound $|u_x|_\infty$;

$$|\beta < \left(\frac{\hat{e}_n + e_{n-1}}{2}\right)_x, \frac{\hat{e}_n + e_{n-1}}{2} + \eta_{n-1/2} > | \le C\{\|\hat{e}_n\|_1^2 +$$

$$+ \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|^2\};$$

$$|<\frac{\hat{e}_n - e_{n-1}}{\Delta t}, \eta_{n-1/2}> + \delta < \frac{(\hat{e}_n - e_{n-1})_x}{\Delta t}, (\eta_{n-1/2})_x>|$$

$$\leq \frac{C}{\Delta t}\|\hat{e}_n - e_{n-1}\|_1 \, \|\eta_{n-1/2}\|_1$$

$$\leq \frac{C}{\Delta t}\{\varepsilon\|\hat{e}_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|_1^2\},$$

for any $\varepsilon > 0$.

Combining the estimates above into (3.7) we get

$$(1 - \alpha - C\varepsilon - C\Delta t)\|\hat{e}_n\|_1^2 \leq C\{\|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|_1^2 + (\Delta t)^4\},$$

which imply (3.6) by an appropriate choice of α, ε and Δt.

We can now focus attention on the corrector scheme. We write the equation for u at $t_{n-1/2} = (n - 1/2)\Delta t$ now in a different form:

$$(3.8)\quad <\frac{u(.,t_n) - u(.,t_{n-1})}{\Delta t}, \phi> + \delta < \frac{u_x(.,t_n) - u_x(.,t_{n-1})}{\Delta t}, \phi_x>$$
$$+ \beta < \frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}, \phi> + \gamma < \left(\frac{u(.,t_n) + u(.,t_{n-1})}{2}\right) \cdot \left(\frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}\right), \phi> = <B_n(.), \phi>, \ \phi \in H_p^1(0,1),$$

where $B_n(x)$ is $0((\Delta t)^2)$ both pointwise and in $L^2(0,1)$. As initial condition we remain with (3.4).

Choosing ϕ in S_h, taking the difference between (3.8), (3.4) and (1.6)′, (1.7), respectively, and adding and subtracting $<(\hat{U}_n + U_{n-1})/2 \cdot (u_x(.,t_n) + u_x(.,t_{n-1})/2), \phi>$ to the first result, we obtain the evolutionary difference equation for e_n:

$$(3.9)\quad <\frac{e_n - e_{n-1}}{\Delta t}, \phi> + \delta < \frac{(e_n - e_{n-1})_x}{\Delta t}, \phi_x>$$
$$+ \beta < \frac{(e_n + e_{n-1})_x}{2}, \phi> + \gamma < \frac{\hat{e}_n + e_{n-1}}{2}\left(\frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}\right), \phi>$$
$$+ \gamma < \frac{\hat{U}_n + U_{n-1}}{2}\left(\frac{e_n + e_{n-1}}{2}\right)_x, \phi> = <B_n, \phi>, \ \phi \in S_h,$$

$$(3.10)\quad \langle e_0, \phi\rangle = 0\ , \ \phi \in S_h.$$

Now we take the test function $\phi = 1/2\,(e_n + e_{n-1}) + \eta_{n-1/2}$, where $\eta_{n-1/2}$ is defined as before. In view of the stability property of the scheme and lemma 1.1 we can arrive at the following estimates:

$$(3.11)\quad |<B_n, \phi>| \leq C\{\|e_n\|^2 + \|e_{n-1}\|^2 + \|\eta_{n-1/2}\|^2 + (\Delta t)^4\},$$

$$(3.12)\quad |\gamma < \frac{\hat{U}_n + U_{n-1}}{2}\left(\frac{e_n + e_{n-1}}{2}\right)_x, \phi>| \leq C\{\|e_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|^2\},$$

$$(3.13)\quad |\beta < \left(\frac{e_n + e_{n-1}}{2}\right)_x, \phi>| \leq C\{\|e_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|^2\},$$

$$(3.14)\quad |\gamma < \frac{\hat{e}_n + e_{n-1}}{2}\left(\frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}\right), \phi>| \leq$$
$$\leq C\|\hat{e}_n\|^2 + \|e_n\|^2 + \|e_{n-1}\|^2 + \|\eta_{n-1/2}\|^2\}.$$

Using (3.6) to bound $\hat{e}_n$ in (3.14),

$$(3.15)\quad |<\frac{\hat{e}_n + e_{n-1}}{2}\left(\frac{u_x(.,t_n) + u_x(.,t_{n-1})}{2}\right), \phi>| \leq$$
$$\leq C\{\|e_n\|^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|_1^2 + (\Delta t)^4\},$$

for Δt small enough.

Carrying estimates (3.11), (3.12), (3.13) and (3.15) back to equation (3.9) we have

$$\frac{1}{2\Delta t}\{[\|e_n\|^2 + \delta\|(e_n)_x\|^2] - [\|e_{n-1}\|^2 + \delta\|(e_{n-1})_x\|^2]\}$$
$$\leq - < \frac{e_n - e_{n-1}}{\Delta t}, \eta_{n-1/2}> - \delta < \frac{(e_n - e_{n-1})_x}{\Delta t}, (\eta_{n-1/2})_x>$$
$$+ C\{\|e_n\|_1^2 + \|e_{n-1}\|_1^2 + \|\eta_{n-1/2}\|_1^2 + (\Delta t)^4\}.$$

Hence, if we multiply this relation by Δt and sum from 1 to $m \in \{1, 2, \ldots, M+1\}$, doing summation by parts in the first and second term of the right hand side, we obtain

$$\inf(1,\delta)\|e_m\|_1^2 \leq \|e_m\|^2 + \delta\|(e_m)_x\|^2 \leq C\{\|e_0\|_1^2 + \|\eta_{1/2}\|_1^2 + \|\eta_{m-1/2}\|_1^2 + \sum_{j=1}^{m}\Delta t\|\eta_{j-1/2}\|_1^2 + \sum_{j=1}^{m-1}\Delta t\left\|\frac{\eta_{j+1/2} - \eta_{j-1/2}}{\Delta t}\right\|_1^2 + (\Delta t)^4\} + C'\sum_{j=1}^{m}\Delta t\|e_j\|_1^2,$$

so that, by Gronwall's lemma,

$$(3.16)\quad \|e_m\|_1^2 \leq C\{\|e_0\|_1^2 + \|\eta_{1/2}\|_1^2 + \|\eta_{m-1/2}\|_1^2 + \sum_{j=1}^{m}\Delta t\|\eta_{j-1/2}\|_1^2 + \sum_{j=1}^{m-1}\Delta t\left\|\frac{\eta_{j+1/2} - \eta_{j-1/2}}{\Delta t}\right\|_1^2 + (\Delta t)^4\}.$$

Now e_0, by (3.10), is the error committed in the approximation of u_0 by its L^2-projection into S_h. From Schultz [5] we know that $\|e_0\|_1 = 0(h^3)$.

On the other hand, η_j is the error in approximating $u(x, t_j)$ by $\theta_h u(x, t_j) \in S_h$, which is $0(h^3)$ if we measure it with the H^1-norm (see again [5]).

Hence (3.16) implies (3.1).

The proof of (3.2) follows in exactly the same line of reasoning, starting with the equation for the exact wave form at instant $t = (n - 1/2)\Delta t$ written in the following way:

$$< \frac{u(., t_n) - u(., t_{n-1})}{\Delta t}, \phi > +$$

$$+ \delta < \frac{u_x(., t_n) - u_x(., t_{n-1})}{\Delta t}, \phi_x > +$$

$$+ \beta < \frac{u_x(., t_n) + u_x(., t_{n-1})}{2}, \phi > +$$

$$+ < [\tfrac{3}{2} u(., t_{n-1}) - \tfrac{1}{2} u(., t_{n-2})] \frac{u_x(., t_n) + u_x(., t_{n-1})}{2}, \phi >$$

$$= < C_n, \phi > \quad , \quad \phi \in H_p^1(0, 1),$$

where $\|C_n\| = 0[(\Delta t)^2]$.

The a priori bound for the error brought about in this scheme is the same (3.16) with one more term in the right hand side: $C\|\eta_{3/2}\|_1^2$. Lemmas 1.1 and 2.1 are used in the argument in the same way as in the predictor-corrector analysis.

4. Numerical Results

Since equation (1.1), as a model, is only correct for small amplitudes, in our calculations the initial velocity field was taken to be $u_0(x) = \frac{1}{20} \sin 2\pi x$. The physical parameters were $\beta = 1$, $\gamma = 3/2$ and $\delta = 1/6$. The time interval was 30 seconds long, $M = 149$ and $N = 19$.

The result of a representative calculation for V_n is shown in Figure 1. It exhibits the velocity field profiles at various time levels. For each n, the difference between U_n and V_n is less than 10^{-5}, that is $0(h^4)$.

We can see the propagation of the initial function from right to left, a damping effect in the first five seconds and after that a wave train being generated whose amplitudes are of the order of 1/5 of the initial one.

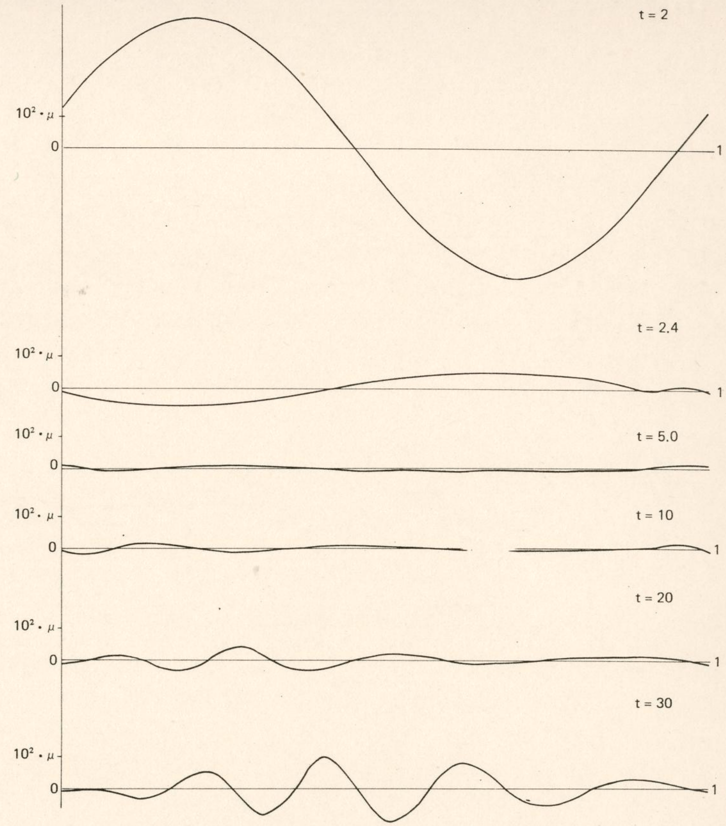

Fig. 1. $V_n(x)$ at time levels 0.2, 2.4, 5, 10, 20, and 30 Sec.

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