The geometry of the structural stability proof using unstable disks

*by Clark Robinson**

§ 1. Introduction.

In this paper we give an exposition of the geometric ideas of the proof of structural stability using compatible families of unstable disks. Hopefully, this introduction will help the reader understand our earlier papers which carried out the analysis carefully. Also in this paper, we treat some simpler cases of the proof first where the proof is less complicated. In some of these cases we can prove the conjugacy is one to one directly without using the d_f metric of Robbin.

Throughout the paper we have ignored taking local coordinates on the manifold and identified the hyperbolicity of the derivative with the behavior of the diffeomorphism in a neighborhood. For more careful definitions the reader should consult [10] or [12]. We note here that for a diffeomorphism the stable manifold is defined by $W^s(x) = \{y: d(f^n(x), f^n(y)) \to 0$ as $n \to \infty\}$. When the nonwandering set of f is hyperbolic, this set is locally equal to $\{y: d(f^n(x), f^n(y)) \leq \varepsilon$ for $n \geq 0\}$. Similarly for $W^u(x)$ and $\{y: d(f^n(x), f^n(y)) \leq \varepsilon$ for $n \leq 0\}$.

Because our paper [10] includes a discussion of the historical roots of the proof of structural stability, we omit this discussion here. We only mention that it is definitely in the spirit of Anosov, [1], using the compatibility of Palis on basic sets, [7] and [8].

The exposition here is influenced by conversations with Conley and especially his talk at Brown University, [2]. It is based on lectures given at Northwestern University and Instituto de Matematica Pura e Aplicada in Rio de Janeiro. We would like to take this occasion to thank the people at IMPA for their hospitality.

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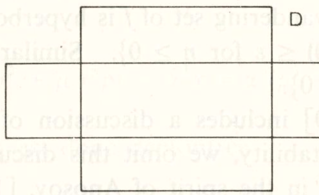
The next section first discusses the stability of a hyperbolic fixed point and then the structural stability of a diffeomorphism that has a hyperbolic structure on all of M (an Anosov diffeomorphism.) The third section proves the stability in a neighborhood of a contracting fixed point and then in a neighborhood of a general hyperbolic attractor. Section four proves structural stability on a manifold with only two invariant nonwandering sets when there is strong transversality (northpole southpole diffeomorphism.) All these proofs are done without using the d_f metric, $d_f(x, y) = \sup\{d(f^n(x), f^n(y)) : n \in Z\}$. The fifth section proves the stability in a neighborhood of a hyperbolic set and sketches how the conjugacy might be proven to be one to one without using the d_f metric. Section six considers the general Axiom A and strong transversality diffeomorphism. Finally, section seven discusses some of the changes necessary to prove stability of flows.

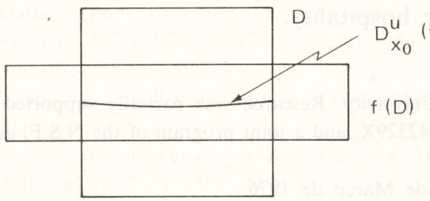
2. Stability of Anosov diffeomorphisms

2A. Before treating the case where there is hyperbolic behavior everywhere, we look at the permanence of a hyperbolic fixed point. Let $D = D^u \times D^s$ be a neighborhood of x_0 where D^u is a disk of dimension u and D^s a disk of dimension s.

Let f be a local diffeomorphism that expands in the D^u direction and contracts in the D^s direction.



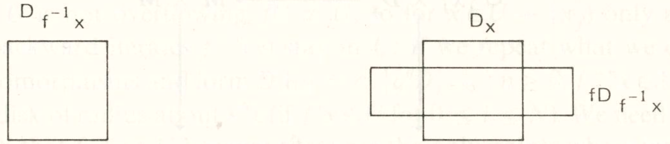
Then under iteration $f^n(D)$ becomes thinner in the s direction and longer in the u-direction. The intersection $\cap\{f^n(D) : n \geq 0\}$ is a u dimensional disk, $D^u_{x_0}(f)$. Also f expands $D^u_{x_0}$, so f^{-1}: $D^u_{x_0}(f) \to D^u_{x_0}(f)$ is a contraction.



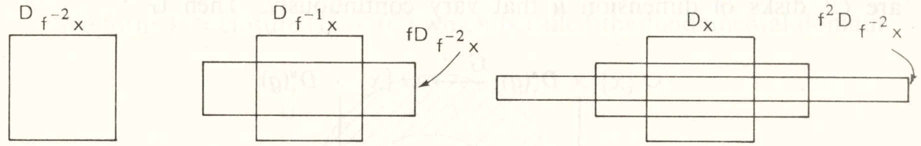
Therefore $\cap\{f^n(D^u_{x_0}(f)) : n \leq 0\}$ is the unique fixed point x_f. Now, if g is a local diffeomorphism that is C^1 near f, then we have the same picture. Again $\cap\{g^n(D) : n \geq 0\} = D^u_{x_0}(g)$ is a u-disk and $\cap\{g^n(D^u_{x_0}(g)) : n \leq 0\}$ is the unique fixed point of g, x_g.

2B. We turn to the case when $f : M \to M$ is a diffeomorphism on a compact manifold M that has a hyperbolic structure on all of M. (These are called Anosov diffeomorphisms.)

Technically this means the tangent bundle splits into two continuous subbundles E^u and E^s such that the sum at each point equals all of T_xM, $E^u_x \oplus E^s = T_xM$, and the derivative of f preserves these bundles and expands vectors in E^u_x and contracts vectors in E^s_x. The geometric idea is that if $D_x = D^u_x \times D^s_x$ is a neighborhood of x of size ε, then f takes the dis $D_{f^{-1}x}$



at $f^{-1}(x)$ across in the u direction disk at x. Similarly f^2 takes $D_{f^{-2}x}$ across D_x in even a thinner strip. Continuing we get that, $D^u_x(f) = \cap\{f^n(D_{f^{-n}x}) : n \geq 0\}$



is a u disk. If g is C^1 near f we get that g takes $D_{f^{-1}x}$ across D_x. Since $g(D_{f^{-2}x}) \cap D_{f^{-1}x}$ lies in $D_{f^{-1}x}$, $g^2(D_{f^{-2}x}) \cap D_x$ is even a thinner strip. Continuing, we get that $\cap\{g^n(D_{f^{-n}x}) : n \geq 0\} = D^u_x(g)$ is a u disk. Notice that the point x is not necessarily a point in $D^u_x(g)$. These disks do turn out to be unstable manifolds of g for some point but at this step of the proof we do not know which point. Also $y \in D^u_x(g)$ if and only if $y \in g^n D_{f^{-n}x}$ for all $n \geq 0$ if and only if $g^{-n}(y) \in D_{f^{-n}x}$ for all $n \geq 0$. This means that the backward g orbit of y stays near the backward f orbit of x.

Since $D^u_x(g)$ is in the unstable direction, g^{-1}: $D^u_{fx}(y) \to D^u_x(g)$ is a contraction. Therefore the intersection $\cap\{g^n D^u_{f^{-n}x}(g) : n \leq 0\}$ is a unique point which we call $h(x)$. Notice that $h(x)$ is the unique point in $D^u_x(g)$ such that $g^n h(x) \in D^u_{f^{-n}x}(g)$ for all $n \leq 0$. From this and above, it follows that $h(x)$ is the unique point y such that $g^n(y) \in D_{f^n x}$ for all $-\infty < n < \infty$. Since $g^n h(x) \in D_{f^n x}$ and $g^{n-1} hf(x) \in D_{f^{n-1}f(x)} = D_{f^n x}$, by uniqueness $gh(x) = hf(x)$.

Why is h one to one? If $h(x) = h(y)$ then $hf^n(x) = g^n h(x) = g^n h(y) = hf^n(y)$. Therefore $f^n(y)$ is near $f^n(x)$ for all n. If g is close enough to f, then $h(x)$ is within $\varepsilon/2$ of x so we can get that $f^n(y) \in D_{f^n x}$ for all n. By the uniqueness argument above when $g = f$, we have that x is the unique point such that $f^n(x) \in D_{f^n x}$ for all n. Therefore $x = y$. This property of f is called expansiveness.

The only question that remains is, why is h continuous. We have considered the neighborhoods D_x of x in M. The union $\cup \{x\} \times D_x$ is a neighborhood of the diagonal Δ in $M \times M$. By projecting on the first factor this gives a disk bundle over M. The map $F = f \times f: M \times M \to M \times M$ leaves the diagonal invariant and is hyperbolic on fibers $\{x\} \times D_x$. For g C^1 near f f, $G = f \times g: M \times M \to M \times M$ still covers $f: M \to M$ and is hyperbolic on fibers.

$$\begin{array}{ccc} \cup \{x\} \times D_x & \xrightarrow{G = f \times g} & M \times M \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M \end{array}$$

The stable manifold theory proves that $D_x^u(g) = \cap \{g^n D_{f^{-n}x}: n \geq 0\}$ are C^1 disks of dimension u that vary continuously. Then G^{-1}

$$\begin{array}{ccc} \cup \{x\} \times D_x^u(g) & \xrightarrow{G^{-1}} & \cup \{x\} \times D_x^u(g) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f^{-1}} & M \end{array}$$

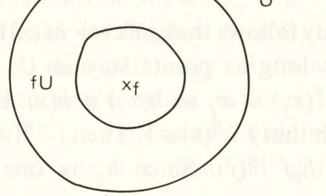
contractors the fibers $\{x\} \times D_x^u(g)$ and so has a unique continuous invariant cross section $x \to (x, h(x))$. The invariance of the section means that $G^{-1}(x, h(x)) = (f^{-1}(x), hf^{-1}(x))$, or $(f^{-1}(x), g^{-1}h(x)) = (f^{-1}(x), hf^{-1}(x))$, or $g^{-1}h(x) = hf^{-1}(x)$.

This completes the proof of the stability of Anosov diffeomorphisms, c.f. [2].

3. Conjugacy on a neighborhood of attractors

3A. Contracting fixed point. Again to facilitate understanding, we begin with a very simple case of a contraction near a fixed point. Let $U \subset R$

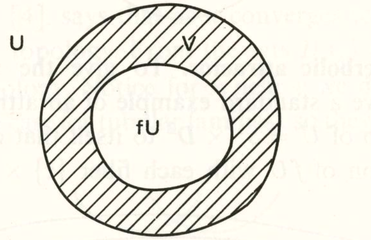
be a open set of R and $f: U \to U$ a local diffeomorphism that is a contraction, i.e. $|f(y) - f(z)| \leq \lambda |y - z|$ where $0 < \lambda < 1$. Then $\cap \{f^n U: n \geq 0\} = \{x_f\}$ is fixed point.



If g is C^1 near f, the $\cap \{g^n U: n \geq 0\} = \{x_g\}$ is the fixed point for g. Next we let D_x be small ε disks near each point $x \in U$.

The set U is not overflowing, $fU \not\supset U$, so for $x \in U - \{x_f\}$ only a finite number of backward iterates $f^{-n}(x)$ stay in U. If we repeat what we did for Anosov diffeomorphisms and form $D'_x(g) = \cap \{g^n D_{f^{-n}x}: n \geq 0, f^{-n}x \in U\}$, this only gives a disk of radius about $\lambda^N \varepsilon$ (if $f^i x \in U$ for $0 \leq i \leq N$). We need to restrict to n such that $f^{-n}x \in U$ because these are the only points where we know f is a contraction. For x_f $D'_{x_f}(g)$ is a point because $f^{-n}(x_f) \in U$ for all $n \geq 0$, and $D'_{x_f}(g) = \{x_g\}$. For other points we need to make a choice of a point in $D'_x(g)$. (Notice backward iterates are expansions so they don't help.)

We form $V =$ closure $(U - fU)$ which is called the fundamental domain.



We make choices of $h(x)$ on V. Let $h_{ex}(x) = x$ and $h_{in}(x) = gf^{-1}(x)$. We use h_{ex} near the exterior boundary of V, or ∂U, and h_{in} near the inner boundary of V, or $\partial f U$. Let $\beta(x)$ be a bump function such that $\beta(x) = 1$ for x near ∂U and $\beta(x) = 0$ for x near $\partial f U$. Define $h_0(x) = \beta(x) h_{ex}(x) + (1 - \beta(x) h_{in}(x))$. For g C^1 near f, h_0 is C^1 near identity on a neighborhood of V, and so if $x \in V$, y near V, and $h_0(x) = h_0(y)$, then $x = y$.

Also if x and $f(x)$ are both in a neighborhood of V then $\beta(x) = 1$ and $\beta f(x) = 0$ so $h_0(x) = x$, $h_0 f(x) = h_{in} f(x) = gf^{-1}f(x) = gh_0(x)$, and so h_0 is a conjugacy on a neighborhood of V.

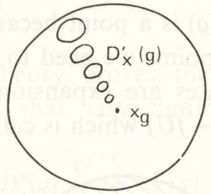
Extend h_0 to all of U by

$$h(x_f) = x_g.$$

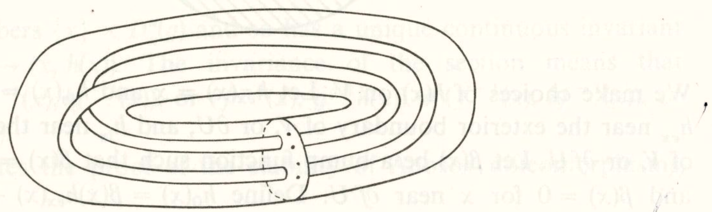
$$h(x) = g^N h_0 f^{-N}(x) \text{ if } x \neq x_f$$

where $f^{-N}(x) \in V$. It easily follows that $gh(x) = hf(x)$ for $x \in U$. Also if $h(x) = h(y)$ then $hf^n(x) = hf^n(y)$ as long as points stay in U. If $x \neq x_f$, then $f^{-N}(x) \in V$ is a long way from $f^{-N}(x_f) = x_f$ so $h(x_f) \neq h(x)$. If $x \in U - \{x_f\}$ and $h(x) = h(y)$ then take N such that $f^{-N}(x) \in V$. Then $f^{-N}(y)$ is near V, and $h_0 f^{-N}(x) = hf^{-N}(x) = hf^{-N}(y) = h_0 f^{-N}(y)$. Since h_0 is one to one, $f^{-N}(x) = f^{-N}(y)$ and $x = y$.

To show h is continuous, notice this is easily true at points $x \in U - \{x_f\}$ since $h(y) = g^N h_0 f^{-N}(y)$ for y near x. To get continuity at x_f, we have that $h_0(x) \in D'_x(g)$ for $x \in V$, so $h(x) \in D'_x(g)$ for all $x \in U$. As x converges to x_f, the N such that $f^{-N}(x) \in V$ goes to infinity, so the radius of $D'_x(g)$ goes to zero. Also $h(x) \in D_x(g)$ and $D'_x(g)$ converges to $D'_{x_f}(g) = \{x_g\}$, so $h(x)$ converges to $x_g = h(x_f)$. This gives continuity at x_f.

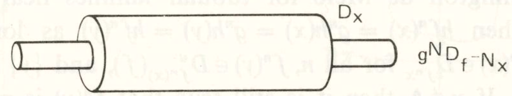


3B. **General hyperbolic attractor.** To give the reader some examples to think about, we give a standard example of an attractor, a solenoid, [12]. It is formed by a map of $U = S^1 \times D^2$ to itself that covers that map $z \to z^2$ on S^1. The intersection of fU with each fiber $\{z\} \times D^2$ is two small disks.

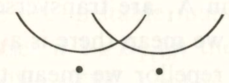


The diffeomorphism has a one dimensional expanding direction along S^1 and a two dimensional contracting direction along D^2. The invariant set $\Lambda = \cap \{f^n U: n \geq 0\}$ is locally a Cantor set cross an interval. In fact $\{z\} \times D \cap \Lambda$ is a Cantor set for each fiber over z.

For the proof in the neighborhood of this attractor, we need to find unstable disks for g near f. As in the case of a point attractor, U is not overflowing. We need to make choices on the fundamental domain V = closure $(U - fU)$. Notices that V is a set whose intersection with each fiber $\{z\} \times D^2$ is a disk with two open disks removed. We let $D_x = D^u_x \times D^s_x$ be an ε neighborhood of each point $x \in U$. Then let $D'_x(g) = \cap \{g^n D_{f^{-n}x}: n \geq 0, f^{-n}(x) \in U\}$. If $f^{-n}(x) \in U$ for $0 \leq n \leq N$, then $g^n D_{f^{-N}x} \cap D_x$ is a cylinder of radius of about $\varepsilon\lambda^N$ where λ is the contraction constant on fibers. If $x \in \Lambda$, then $f^{-n}(x) \in U$ for all $n \geq 0$ so $D'_x(g) = D^u_x(g)$ is a u dimensional disk in the unstable direction. For $f = g$, $D^u_x(f)$ is the local unstable manifold of f at x.



Following the method near a fixed point, we construct unstable disks $D^u_x(g)$ for $x \in V$. We won't do this in general, but for the example above. If $x = (z, y)$ we could let $D^u_x(g) = [z - \varepsilon, z + \varepsilon] \times \{y\}$ be the disk for x near ∂U and $D^u_{f(x)}(g) = gD^u_x(g)$ for $f(x)$ near $\partial f U$. Using a C^1 bump function there is a way averaging between these two choices. Then for $x \in U - \Lambda$, take N such that $f^{-N}(x) \in V$ and let $D^u_x(g) = g^N D^u_{f^{-N(x)}}(g) \cap D_x$ (Actually a component of this intersection.) For $x \in \Lambda$ we let $D^u_x(g) = D'_x(g)$ as above. The stable manifold theory, [4], says that as x converges to $x_0 \in \Lambda$, $D^u_x(g)$ converges to $D^u_{x_0}(g)$ in the C^1 topology. From the sets $D'_x(g)$ above, it is clear they converge in the C^0 topology. Notice for x near y we don't assume the disks are compatible (as is the case for tubular families), so they don't form a foliation.



To get the conjugacy we use g^{-1} on these disks. The map $G^{-1} = f^{-1} \times g^{-1}: \cup \{x\} \times D^u_x(g) \to \cup \{x\} \times D^u_x(g)$ contracts fibers $\{x\} \times D^u_x(g)$ and is overflowing on the base space (first factor), $f^{-1}(U) \supset U$. Therefore $\{h(x)\} = \cap \{g^n D^u_{f^{-n}x}(g): n \leq 0\}$ is well defined and a point. It is unique once the choices of $D^u_x(g)$ on V are made. So, $\{h(x)\} \subset \cap \{g^n D_{f^{-n}x}: n \in Z, f^n(x) \in U\}$ but not equal unless $x \in \Lambda$.

To see that h is continuous, we can use the bundle map $G^{-1} = (f^{-1}, g^{-1})$: $M \times M \to M \times M$ restricted to unstable disks:

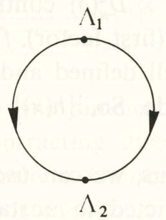
$$\begin{array}{ccc} \cup\{x\}\times D^u_x(g) & \xrightarrow{G^{-1}} & \cup\{x\}\times D^u_x(g) \\ \downarrow & & \downarrow \\ U & \xrightarrow{f^{-1}} & f^{-1}U \end{array}$$

Since f^{-1} is overflowing ($f^{-1}U \supset U$) we get there is a unique continuous invariant section $\{(x, h(x))\}$. The invariance of the section means $g^{-1}h(x) = hf^{-1}(x)$ so h is a semiconjugacy.

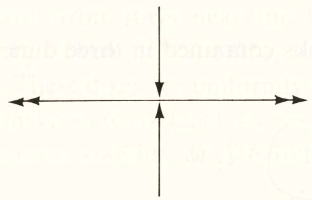
To prove that h is one to one, we make a trivial adaptation of the one given by Welington de Melo for tubular families near attractors, [3]. If $h(x) = h(y)$, then $hf^n(x) = g^nh(x) = g^nh(y) = hf^n(y)$ as long as $f^n(x) \in U$. If $x \in \Lambda$, then $f^n(x) \in D_{f^nx}$ for all n, $f^n(y) \in D^u_{f^n(x)}(f)$, and $\{y\} = \cap\{f^{-n}D^u_{f^n(x)}(f)\colon n \geq 0\} = \{x\}$. If $x \notin \Lambda$ then it is still true that $f^n(y)$ is near $f^n(x)$ for $n \geq 0$ so $y \in W^s_{loc}(x,f) = \cap\{f^nD_{f^{-n}x}\colon n \leq 0\}$. This says that the stable manifolds of x are the points whose forward f orbit stays near the forward f orbit of x. On V, the unstable disks $D^u_x(g)$ are C^1 and transverse to $W^s(z,f)$. Therefore the unstable disks form a tubular neighborhood of $W^s(z,f)$. If $z_2 \in W^s(z_1,f)$ and $D^u_{z_1}(g) \cap D^u_{z_2}(g) \neq \emptyset$, then $z_1 = z_2$. If $h(x) = h(y)$ take N such that $f^N(x) \in V$. Then $f^N(y) \in W^s(f^N(x), f)$, $hf^N(x) \in D^u_{f^Nx}(g)$, and $hf^N(y) \in D^u_{f^Ny}(g)$ so $f^N(x) = f^N(y)$ and $x = y$. This completes the proof for attractors.

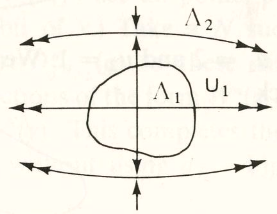
4. Northpole southpole diffeomorphism

A northpole southpole diffeomorphism (NS diffeomorphism) is a diffeomorphism f such that the nonwandering set $\Omega(f) = \Lambda_1 \cup \Lambda_2$ where Λ_1 is an repellor and Λ_2 is an attractor, each has a dense periodic point, and the unstable manifolds of points in Λ_1 are transverse to the stable manifolds of points in Λ_2. By an attractor, we mean there is a neighborhood U_2 such that $\cap\{f^nU_2\colon n \geq 0\} = \Lambda_2$. By a repellor we mean there is a neighborhood U_1 such that $\cap\{f^nU_1\colon n \leq 0\} = \Lambda_1$. One example is the time one map of a gradient flow on a sphere of any dimension.



For a second example, let $f_1\colon S^1 \to S^1$ be a NS diffeomorphism and $f_2\colon T^2 \to T^2$ be an Anosov diffeomorphism. Then $f = (f_1, f_2)\colon S^1 \times T^2 \to S^1 \times T^2$ is a NS diffeomorphism. A third example is a DA on T^2. To construct it, take an Anosov diffeomorphism and push out at the fixed point along the stable foliation to change the hyperbolic fixed point into a source Λ_1. If U_1 is a neighborhood of Λ_1 then $\cap\{f^n(T^2 - U_1)\colon n \geq 0\}$ is a hyperbolic attractor with one dimensional stable manifold and one dimensional





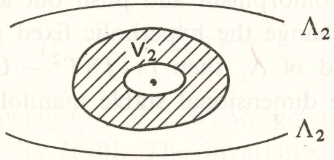
unstable manifold. See [12], [13] for a more complete description and discussion.

For a general NS diffeomorphism, $W^u(x_1,f) \cap W^s(x_2,f) \neq \emptyset$ for some $x_1 \in \Lambda_1$ and $x_2 \in \Lambda_2$. Therefore $u_1 + s_2 \geq \dim M$ where u_i is the unstable dimension on Λ_i and similarly s_i. Also $s_2 + u_2 = \dim M$. Therefore $u_1 \geq u_2$.

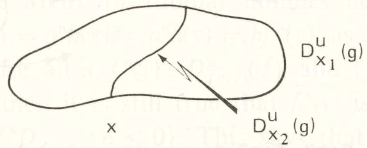
The stability proof goes much as for attractors but introduces the concept of compatibility of unstable disks for Λ_2 and those for Λ_2. The general case of Axiom A and strong transversality has this compatibility but is more complicated.

We start by constructing unstable disks for Λ_1. Let U_1 be a neighborhood of Λ_1 where there is hyperbolic behavior. The neighborhood is overflowing, $fU_1 \supset U_1$. Therefore for $x \in U_1$, $f^{-n}(x) \in U_1$ for $n \geq 0$, and $D^u_{x1}(g) = \cap\{g^nD_{f^{-n}x}\colon n \geq 0\}$ is a u_1 disk for all $x \in U_1$. (Here $D_x = D^{u_1}_x \times D^{s_1}_x$ is an ε-neighborhood at x.) These are unstable manifolds of g and are uniquely determined. As in earlier cases if $y \in D^u_{x1}(g)$ then $g^{-n}(y) \in D_{f^{-n}(x)}$ for $n \geq 0$. We can define $D^u_{x1}(g)$ for all $x \in \mathcal{O}(U_1) = \cup\{f^nU_1\colon n \geq 0\} = M - \Lambda_2$ by $D^u_{x1}(g) = g^ND^u_{f^{-N}x_1}(g)$ where $f^{-N}(x) \in U_1$.

We want to get unstable disks $D^u_{x_2}(g)$ of dimension u_2 in a neighborhood of Λ_2. We first define these on a fundamental domain, $V_2 = \text{closure}\,(U_2 - fU_2)$ where U_2 is a neighborhood of Λ_2. If $y \in D^u_{x1}(g)$, we want $g^{-n}(y)$ to be near $f^{-n}(x)$ for all $n \geq 0$ and so we need $D^u_{x_2}(g) \subset D^u_{x_1}(g)$. If $f\colon S^k \to S^k$ has two fixed points then the $D^u_{x_1}(g)$ are merely neighborhoods of x. On V_2 we want to pick out points that lie in these neighborhoods. This is exactly like the case of stability in a neighborhood of a contracting fixed point. If $f\colon T^2 \to T^2$ is the DA diffeomorphism, then Λ_1 is a point source again so $D^u_{x_1}(g)$ are neighborhoords. Now, $u_2 = 1$ so we want to pick out intervals that lie in this neighborhoord for point of V_2. This is similar to the general attractor consider above. Lastly, for the example of $f = (f_1, f_2)$: $S^1 \times T^2 \to S^1 \times T^2$ considered



above, $u_1 = 2$ and $u_2 = 1$. We have two disks contained in three dimensions. We pick



out a one dimensional disk that is approximately the unstable direction of Λ_2. We can do this in a manner so they vary uniformly C^1 on $W^u(x,f) \cap V_2$ because the disks $D^u_{x_2}(g)$ are uniformly C^1 on these sets.

So assume we can choose disks $D^u_{x_2}(g)$ for $x \in V_2$ such that they are in approximately the unstable direction. These disks extend to a family $D^u_{x_2}(g)$ for $x \in U_2$ that are unique once the choices on V_2 are made. This follows from the ideas in the attractor case. To get the conjugacy h we use G^{-1} on the set of unstable disks:

$$\begin{array}{ccc} \cup\{x\} \times D^u_{x_2}(g) & \xrightarrow{G^{-1}} & \cup\{x\} \times D^u_{x_2}(g) \\ \downarrow & & \downarrow \\ fU_2 & \xrightarrow{f^{-1}} & U_2 \end{array}$$

This bundle map contracts fibers so has a unique invariant continuous section $h(x)$. Notice f^{-1} is overflowing, $U_2 \supset fU_2$. We have $gh(x) = hf(x)$ by invariance. We extend h to $M - \Lambda_1$ by $h(x) = g^{-n}hf^n(x)$ where $f^n(x) \in U_2$. This defines h on $V_1^u = \text{closure } (U_1 - f^{-1}U_1)$. Then

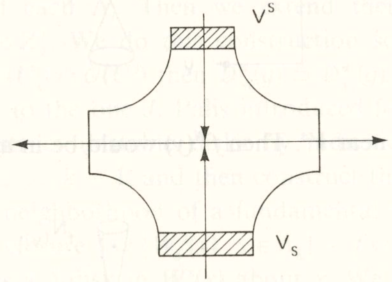
$$\begin{array}{ccc} \cup\{x\} \times D^u_{x_1}(g) & \xrightarrow{G^{-1}} & \cup\{x\} \times D^u_{x_1}(g) \\ \downarrow & & \downarrow \\ U_1 & \xrightarrow{f^{-1}} & f^{-1}U_1 \end{array}$$

is not overflowing but we already have h defined on the difference, V_1^u. Also G^{-1} contracts fibers. Therefore there is a unique extension to an continuous invariant section $h(x)$ over U_1. This defines h on all of M and shows it is continuous.

To show h is one to one if $x \in \Lambda_1 \cup \Lambda_2$ and $h(x) = h(y)$ then $x = y$ as before. Otherwise, $hf^n(x) = hf^n(y)$ for all n so $f^n(y)$ is near $f^n(x)$. Therefore $y \in W^s(x,f) \cap W^u(x,f)$. (It is true that $W^s(x,f)$ are all points whose forward orbit stays near the forward orbit of x and $W^u(x,f)$ are all points p whose backward orbit stays near the backward orbit of x.) Take a N such that $f^N(x) \in V_2$. Then $hf^N(x) \in D^u_{f^N x_2}(g)$ and $hf^N(y) \in D^u_{f^N x_2}(g)$ so these disks intersect. These disks are uniformly C^1 on intersections of the form $W^u(z,f) \cap V_2$ and transverse to $W^s(z,f)$, so we get $f^N(x) = f^N(y)$. This completes the proof of structural stability of $N\,S$ diffeomorphism without using d_f – Lipschitz.

§ 5. Neighborhood of a hyperbolic set

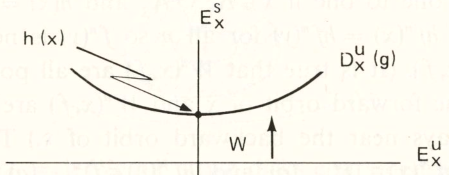
Let Λ be an invariant set for f that has a hyperbolic structure (and local product structure). Let U be a neighborhood where h still has hyperbolic behavior. By choosing U carefully we can make $V^s = \text{closure } (U - fU)$ a set such that if $x \in V$ then $f^2(x) \notin V^s$. To prove this exists we use the ideas of [5], see [10, Lemma 4.4].



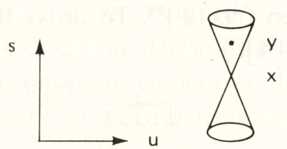
To construct unstable disks over V^s we first do it in a neighborhood of $\{x \in V : f(x) \in V\}$ and then extend to $\{x \in V^s : f^{-1}(x) \in V\}$ by invariance by g. We need these sets disjoint. These disks extend to all of U as before. Then

$$\begin{array}{ccc} \cup\{x\} \times D^u_x(g) & \xrightarrow{G^{-1}} & \cup\{x\} \times D^u_x(g) \\ \downarrow & & \downarrow \\ U \cap f(U) & \xrightarrow{f^{-1}} & f^{-1}(U) \cap U \end{array}$$

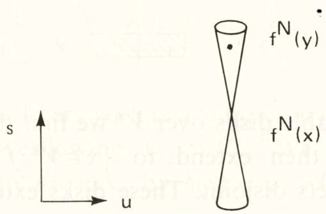
is contracting on fibers but not overflowing on the base space. We need to choose h on $V^u =$ closure $(U - f^{-1}U)$. Each disk $D_x^u(g)$ can be represented by a graph in terms of a splitting at x (extended from Λ): $T_xM = E_x^u \oplus E_x^s$, $w_x: E_x^u \to E_x^s$.



Near $\{x \in V^u : f^{-1}(x) \in V^u\}$ let $h_{ex}(x)$ U $(0_x, w(0_x))$ where $0_x \in E_x^u$ is the zero vector. Here we identify T_xM with a neighborhood of x. Near $\{x \in V^u : f(x) \in V^u\}$ let $h_{in}(x) = g^{-1}h_{ex}f(x)$. Construct h_0 on V^u using a bump function combining h_{ex} and h_{in}. This h_0 extends to h on U which conjugates f and g. With this construction it should be true that if $h_0(x) = h_0(y)$ then y is in a cone centered at x about the stable direction. Then take N such that



$f^N(x) \in V^s$ and $f^N(x)$ is near V^s. Then $f^N(y)$ would be in a very thin cone about the stable direction.



The unstable disks are C^1 in a neighborhood of V^s and transverse to the stable direction, E^s. Therefore for $f^N(y)$ in this thin cone $D^u_{f^N(y)}(g)$ would be disjoint from $D^u_{f^N(x)}(g)$ unless $f^N(y) = f^N(x)$. These disks are not disjoint because $hf^N(x) \in D^u_{f^N(x)}(g)$ and $hf^N(y) \in D^u_{f^N(y)}(g)$. Therefore $f^N(x) = f^N(y)$ and $x = y$.

We believe that it should be possible to write down the details of the above sketch of a proof that h is one to one. Also the same proof should work when there are one fixed point for a source, one fixed point for a sink, and one other hyperbolic basic set. In other words, it should work for the horseshoe on the two sphere as described in [12].

§ 6. Axiom A and strong transversality

Let $f: M \to M$ satisfy hyperbolicity on the nonwandering set Ω, have periodic points dense in Ω, and satisfy the strong transversality condition. If f satisfies the first two conditions it is said to satisfy Axion A (of Smale). The third condition means that for any point $x \in M$ the unstable manifold of x is transverse to the stable manifold of x, $W^u(x)$ transverse at x to $W^s(x)$. Under these hypotheses, the nonwandering set breaks up into a finite union of disjoint compact sets each of which has a dense orbit in it. We can index these sets, $\Omega = \Lambda_1 \cup \ldots \cup \Lambda_K$, so that if $W^u(\Lambda_i) \cap W^s(\Lambda^j) \neq \phi$ then $i \leq j$. Here $W^u(\Lambda_i) = \cup\{W^u(x): x \in \Lambda_i\}$ etc. By strong transversality, if $x_i \in \Lambda_i$ and $x_j \in \Lambda_j$ then $W^u(x_i)$ is transverse to $W^s(x_j)$, so $\dim W^u(x_i) + \dim W^s(x_j) \geq$ $\geq \dim M$. Since $\dim W^u(x_j) + \dim W^s(x_j) = \dim M$, we have $u_i = \dim W^u(x_i) \geq \dim W^u(x_j) = u_j$.

We want to construct unstable disks, for g C^1 near f, $D^u_{xi}(g)$, for x in a neighborhood U_i of each Λ_i. Then we extend then to the orbit of U_i, $\mathcal{O}(U_i) = \cup\{f^nU_i : n \in Z\}$. We do this construction so they are compatible, i.e. if $i < j$ and $x \in \mathcal{O}(U_i) \cap \mathcal{O}(U^j)$ then $D^u_{xi}(g) \supset D^u_{xj}(g)$. The method is to use an induction similar to the one J. Palis introduced for Morse Smale diffeomorphism. We construct unstable disks by induction assuming they are constructed for $i = 1, \ldots, k-1$, and then construct then for a neighborhood of Λ_k. Let V_k^s be a neighborhood of a fundamental domain of $W^s(\Lambda_k)$, i.e. a neighborhood of closure $[\cup\{W_\delta^s(x): x \in \Lambda_k\} - f\cup\{W^\delta(x): x \in \Lambda_k\}]$ for δ small where $W_\delta^s(x)$ is a δ-disk in $W^s(x)$ about x. We construct the disks on V_k^s by induction on $j = k-1, \ldots, 1$. First we construct disks in a neighborhood of $W^u(\Lambda_{k-1}) \cap V_k^s$ contained in $\mathcal{O}(\Lambda_{k-1})$ that are compatible with the $D^u_{xk-1}(g)$. Then we extend these to a neighborhood of $W^u(\Lambda_{k-2}) \cap V_k^s$ contained in $\mathcal{O}(\Lambda_{k-1})$ compatible with the $D^u_{x_{k-2}}(g)$. Continuing we get then on all of V_k^s. This induction is somewhat complicated. The reader should probably read the proof in [8], before reading the corresponding construction in [10]. Once we have the disks on V_k^s then extend to a neighborhood U_k of Λ_k as before.

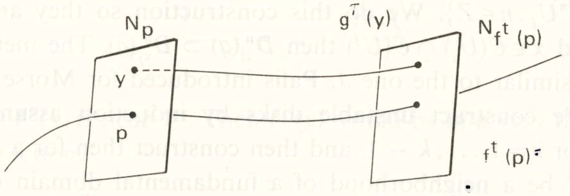
Once the unstable disks are constructed, we get the conjugacy h starting in U_K and working backward. The neighborhoods of sinks are overflowing for f^{-1}, so there are no choices there. When we get to U_j we already have h defined on $U_j - f^{-1}(U_j) \subset \{\mathcal{O}(U_i) : j < i \leq K\}$. Therefore there are no more choices. In other words once the choices of the unstable disks are made, the conjugacy is determined.

To prove h is one to one, we need to introduce the metric $d_f(x, y) = \sup\{d(f^n(x), f^n(y)) : n \in Z\}$, see [9]. Locally on components of $W^s(x) \cap W^u(x)$ d_f is equivalent to the usual metric. Distinct such components are at a discrete distance apart. We then prove h is d_f Lipschitz in the following sense. If $v(x) = h(x) - x$ (in local coordinates), then $|v(x) - v(y)| \leq \eta d_f(x, y)$ for η small. This is like saying h is Lipschitz close to the identity. Using this it can be shown that h is one to one.

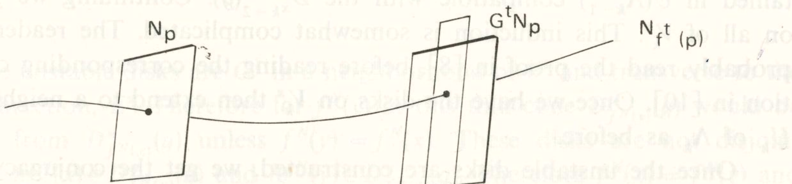
§ 7. Flows

Let $f : R \times M \to M$ be a flow on M and $X(p) = d/dt\, f^t(p)|_{t=0}$ its vector field. We assume f is hyperbolic on its nonwandering set. This means f contracts some directions E^s_x, expands others E^u_x, and preserves $X(x)$ if $X(x) \neq 0$.

If $X(p) \neq 0$ for all p on M, we can introduce transversal disks to X at each point $p \in M$, N_p, $\dim N_p = \dim M - 1$. We let $X^\perp = \cup\{N_p : p \in M\}$ be the normal bundle to X on M. For g^t a flow near f^t, we introduce a flow on $X^\perp$, $G^t : X^\perp \to X^\perp$, by letting $G^t(p, y) = (f^t(p), g^\tau(y))$ where $\tau = \tau(t, p, y)$ is the

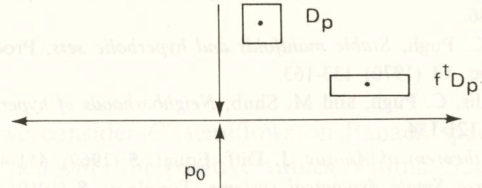


time such that $g^\tau(y) \in N_{f^t(p)}$. (G^t does not preserve the transversal disks of a given radius but ignore this problems here.) If $g = f$ we denote the flow on $X^\perp$ by F^t. Both F^t and G^t are hyperbolic on fibers of $X^\perp$ so we can proceed with a proof much as in the diffeomorphism case.

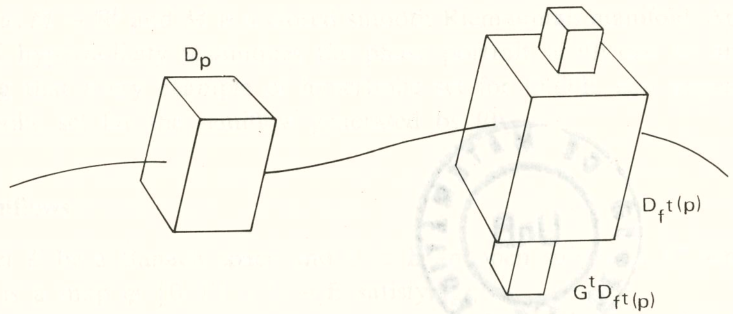


In this manner we solve the equation $hf^t(p) = g^\tau h(p)$ where $\tau = \tau(t, p, h(p))$. Letting $\alpha(t, p)$ be the inverse of τ we get $hf^{\alpha(t,p)}(p) = g^t h(p)$. The proof that h is one to one is harder than the one for diffeomorphisms, see [11].

Near a fixed point of f^t, $X(p_0) = 0$, the flow f^t is hyperbolic on all of M and not just in the normal directions to the flow.



Therefore, we need to use the flow $G^t(p, y) = (f^t(p), g^t(y))$ on all of $M \times M$ near p_0 and not just on $X^\perp$. (Note that transverse disks can not be defined in a continuous manner in a neighborhood of p_0.) There is no reparameterization here. To make a smooth transition between the two methods, we need to extend the flow away from $X(p) = 0$ to a neighborhood of the diagonal in $M \times M$. Geometrically the idea is that if D_x is a neighborhood of x, we add a artifical contraction in the X direction by means of a reparameterization to define G^t.



Using this flow G^t we construct unstable disks. On a basic set $\Lambda_i \subset \Omega$ that is not a fixed point N_p is preserved, so $D^u_{pi}(g) \subset N_p$ for $p \in \Lambda_i$. For p near Λ_i, $D^u_{pi}(g)$ is near N_p.

By this means we solve $hf^t(p) = g^\tau h(p)$ where $\tau = \tau(t, p, h(p))$. Letting $\alpha(t, p)$ be the inverse of τ we get $hf^{\alpha(t,p)}(p) = g^t h(p)$ where $\alpha'(t, p) = 1$ for p in a neighborhood of $\{y : X(y) = 0\}$. We prove h is one to one using d_f Lipschitz. See [11].

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