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Hyperbolic sets for semilinear parabolic equations

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Abstract

In this note we consider C^r semiflows on Banach spaces, roughly speaking C^r flows defined only for positive values of time. Such semiflows arise as the "general solution" of a large class of partial differential equations that includes the Navier-Stokes equation. Our main result (Proposition B) is that under certain assumptions on the P.D.E. (satisfied by the Navier--Stokes equation) a hyperbolic set for the corresponding semiflow (hyperbolicity is defined following closely the finite dimensional case) is always ε-equivalent to a hyperbolic set for an ordinary differential equation that can be easily deduced from the P.D.E. As an example we consider the P.D.E.

(0) $$\frac{\partial u}{\partial t} = -\Delta u + \varepsilon F(x, u, u')$$

where $u: M \to \mathbb{R}^k$ and M is a closed smooth Riemannian manifold. Applying normal hyperbolicity techniques the phase portrait of (0) can be analyzed proving that every example of hyperbolic set for O.D.E. can appear as a hyperbolic set for the semiflow generated by (0).

1. Semiflows

Let E be a Banach space and $U \subset E$ an open subset. A C^r semiflows on U is a map $\varphi: [0, T] \times U \to E$ satisfying:

a) $\varphi(0, x) = x \ \forall x \in U$

b) $\varphi(t_1 + t_2, x) = \varphi(t_1, \varphi(t_2, x)) \ \forall 0 \le t_1 \le T,\ 0 \le t_2 \le T$ and $x \in U$ such that $\varphi(t_2, x) \in U$

c) $\varphi: (0, T) \times U \to E$ is a C^r map

d) $\lim_{t \to 0+} \varphi(t, x) = x \ \forall x \in U$

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Let $\mathscr{F}_r^T(U)$ be the set of C^r semiflows defined on $[0, T] \times U$ with the topology induced by the metrics:

$$d_{\tau_1\tau_2}(\varphi, \psi)\ U \sup\{\|(j^r\varphi)(t,x) - (j^r\psi)(t,x)\| \mid \varsigma_1 \leq t \leq \varsigma_2,\ x \in U\}$$

where $0 < \varsigma_1 < \varsigma_2 < T$ and j^r denotes the r-jet. The motivation for these definition comes from the following class of partial differential equations called semilinear parabolic equations. Consider a Banach space X. We say that a linear operator A densely defined on X is sectorial ([3], [7]) if there exist $0 < \alpha < \pi/2$ and $M > 0$ such that the spectre of A is contained in the sector $S_\alpha = \{z \in C \mid |\arg(z)| \leq \alpha\}$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{\lambda}$$

for all λ in the resolvent of A.

Strongly elliptic operators with regular boundary value conditions define sectorial operators [7]. Let X^a $0 \leq a$ be the domain of A^a [3] endowed with the graph norm $\|x\|_a = \|x\| + \|A^a x\|$. Let V be an open set in X^a for some $0 \leq a < 1$ and $F: V \to X$ a C^r map such that F and its derivative F' are bounded on V. Then if $U \subset V$ is open and $\inf\{\|x - y\|_a \mid x \in U, y \notin V\} > 0$ there exists $T > 0$ and a unique semiflow $\varphi \in \mathscr{F}_r^T(U)$ satisfying

(1) $$\varphi((0, T) \times U) \subset V \cap X^1$$

(2) $$\frac{\partial \varphi}{\partial t}(t,x) = -A\varphi(t,x) + F(\varphi(t,x))$$

for all $x \in U$, $0 < t < T$. Moreover if $C^r(V, X)$ is the space of C^r maps $F: V \to X$ such that $\sup\{\|j^r F(x)\| \mid x \in V\} < +\infty$ with the topology defined by the norm $\|F\| = \{\sup \|j^r F(x)\| \mid x \in V\}$ then given $F_0 \in C^r(V, X)$ there exists $T > 0$ and a neighborhood $\mathscr{U}$ of F_0 such that the map $\mathscr{U} \ni F \to \varphi \in \mathscr{F}_r^T(U)$ where φ satisfies (1) and (2) is well defined and continuous. When A has compact inverse φ is also compact i.e. $\varphi(t, S)$ is compact for all bounded $S \subset U$ and $0 < t < T$.

Example I. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ be a C^r function. Consider the P.D.E.:

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} + f(x, u, \operatorname{grad} u) \text{ in } \Omega \times \mathbb{R}^+$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+$$

Let A be the closure in $X = L^p(\Omega)$ of $-\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$ on $C_0^\infty(\Omega)$ with $p \geq 2$ and $n < p < \infty$. Then A is sectorial with compact inverse [7] and X^a is contained in $C^{1,\delta}(\bar{\Omega})$, $1 < 1 + \delta < 2a - n/p$ [1] and the inclusion $X^a \subset C^{1,\delta}(\bar{\Omega})$ is continuous. Hence taking $1 > a > 1/2(1 + n/p)$ the map $F: X^a \to X$ defined as $F(u)(x) = f(x, u(x), \operatorname{grad} u(x))$ is C^r. Let $U \subset X^a$ be a bounded subset. Taking $V = X^a$ the aformentioned result proves that there exists $\varphi \in \mathscr{F}_r^T(U)$ satisfying (1) and (2). Hence the function $u(t,x) = \varphi(t, u_0)(x)$ where $u_0 \in U$, $x \in \Omega$ and $0 \leq t < T$ can be considered as a weak solution of the partial differential equation with initial condition u_0.

A similar analysis can be applied to the Navier-Stokes equation (see [1], [9], [10]) even when in this case the nonlinear part of the equation (the operator F) is not a composition operator.

2. Hyperbolic sets

In this note we shall study the qualitative perturbation theory of compact invariant sets of a semiflow $\varphi \in \mathscr{F}_r^T$(U) i.e. compact subsets $\Lambda \subset U$ such that $\varphi(t, \Lambda) = \Lambda$ for all $0 \leq t < T$. Examples of such sets can be obtained, when φ is compact, if there exists a closed bounded subset $\sum$ such that $\sum^+ = \bigcap_{n>0} \varphi_\tau^{-n}(\Sigma)$ is non empty for some $0 < \tau < T$ where $\varphi_\tau(\cdot) = \varphi(\tau, \cdot)$. If $x \in \Sigma^+$ define $\varphi(t,x)$ for $t > 0$ as $\varphi(t,x) = \varphi(s, \varphi_\tau^n(x))$ where $t = n\tau + s$, $n \in \Gamma^+$, $0 \leq s < \tau$. Then we can define the ω-limit set of x, $\omega(x)$, as usual as for flows and follows easily from the compacity of φ that $\omega(x)$ is compact and $\varphi(t, \omega(x)) = \omega(x)$ for all $t \geq 0$.

When Λ is a compact invariant set for φ we can define $\varphi(t,x)$ for $x \in \Lambda$ and $t > 0$ as above. If φ_t/Λ is one to one for all $t > 0$ (it is sufficient to require φ_t/Λ to be one to one for some positivo value of t) we say that Λ is reversible. In this case we define φ_t/Λ for $t < 0$ as $\varphi(-t, x) = (\varphi_t)^{-1}(x)$. When F is analytic the semiflow satisfying (1) (2) is an analytic map. Hence φ_t is one to one to one for all $0 < t < T$ and all invariant set is reversible.

The main idea in qualitative perturbation theory of flows is that of stability, that can be applied without changes to reversible compact sets. When the hypothesis of the reversibility fails, even for discrete semiflows on finite reversibility fails, even for discrete semiflows on finite dimensional manifolds, the usual definition of stability doesn't work [5].

Definition. If Λ is compact reversible invariant set for $\varphi \in \mathscr{F}_r^T(U)$, we say that Λ is stable if for all $\varepsilon > 0$ there exist neighborhoods $\mathscr{U}$ of φ and V of Λ such that if $\psi \in \mathscr{U}$ and $\bigcap_{-\infty}^{+\infty} \psi_t(V)$ is reversible then there exist a continuous injective map $h: \Lambda \to U$ such that if γ is a φ-orbit in Λ i.e. $\gamma = \{\varphi(t, x) \mid t \in R\}$ for some $x \in \Lambda$, $h(\gamma)$ is a ψ-orbit and $\|h(x) - x\| < \varepsilon$ for all $x \in \Lambda$. We say that Λ and $h(\Lambda)$ are ε-equivalent.

Following the finite dimensional case we define hyperbolic sets.

Definition. A compact reversible invariant set Λ for a semiflow $\varphi \in \mathscr{F}^T_r(U)$ is hyperbolic if there exists a continuous splitting $E = E^s_x \oplus E^\varphi_x \oplus \oplus E^\varphi_x$, $x \in \Lambda$, where E^φ_x is the one dimensional subspace spanned by $\partial\varphi/\partial t$ $(0, x)$ and E^s_x, E^u_x satisfy:

a) $\varphi'_t(x)\, E^s_x \subset E^s_x$
$\varphi'_t(x)\, E^u_x \subset E^u_x$

b) There exist $K > 0$, $\sigma > 0$ such that:
$\| \varphi'_t(x)/E^s_x \| \le K\, e^{-\sigma t}$
$\| \varphi'_t(x) v \| \ge K\, e^{\sigma t} \| v \|$

for all $x \in \Lambda$, $t > 0$, $v \in E^u_x$.

The subbundle defined by the subspaces E^s_x (E^u_x) is called the stable (unstable) subbundle of Λ. The derivative $\dfrac{\partial\varphi}{\partial t}(0, x)$ exists and satisfies

$$\frac{\partial\varphi}{\partial t}(0, x) = \frac{\partial\varphi}{\partial t}(1, \varphi(-1, x)).$$

Example II. Let M be a closed smooth Riemannian manifold and $X = L^p(M, \mathbb{R}^k)$ $p > \dim M$. Let Δ denote the Laplacian operator densely defined on X and $F: J^1(M, \mathbb{R}^k) \to \mathbb{R}$ a C^r map, $r \ge 2$, where $J^1(M, \mathbb{R}^k)$ denotes the space of 1-jets such that $\{\| (j^r F)(x) \| \mid x \in M\}$ is bounded. Take $0 \le a < 1$, $0 < v < 1$ satisfying $1 + v < 2a - \dim M/p$. Then it is known that X^a is continuously embedded in $C^{1,v}(M, \mathbb{R}^k)$. Hence the map $X^a \notin u \to F \circ j^1 u \in X$ is C^r. Since Δ is sectorial [7] there exists $T > 0$ such that for all ε small enough there exists $\varphi^\varepsilon \in \mathscr{F}^T_r(X^a)$ satisfying:

$$\frac{\partial\varphi^\varepsilon}{\partial t}(t, x) = -\Delta\varphi^\varepsilon(t, x) + \varepsilon F \circ j^1 \varphi^\varepsilon(t, x)$$

The kernel of Δ is the space of constants maps, so it can be identified R^k, and it is a normally hyperbolic [2] invariant manifold for φ^0. Applying the methods of [2] and [4] there exists for all ε near to 0 a map $h_\varepsilon: \mathbb{R}^k \to X^\alpha_0$, where X^α_0 is the space of maps in X^α with mean value 0 (observe that $X^\alpha = \mathbb{R}^k \oplus X^\alpha_0$), such that $V_\varepsilon = \text{graph}\,(h_\varepsilon)$ is a normally hyperbolic invariant manifold for φ^ε. Let X_ε be the C^{r-1} vectorfield on V_ε induced by φ^ε and let Y_ε be the vectorfield obtained projecting $1/\varepsilon\, X_\varepsilon$ on $\mathbb{R}^k$ by the projection $\pi: X^\alpha \to \mathbb{R}^k$ associated to the splitting $X^\alpha = X^\alpha_0 \oplus \mathbb{R}^k$. Let Y_0 be the vectorfield on $\mathbb{R}^k$ defined as:

$$Y_0(x) = \pi(F \circ j^1 \tilde{x})$$

where $\tilde{x}: M \to \mathbb{R}^k$ is the constant map $\tilde{x}(p) = x\ \forall\, p \in M$. Then $Y_\varepsilon \to Y_0$ in $C^{r-1}(\mathbb{R}^k, \mathbb{R}^k)$ when $\varepsilon \to 0$. If the flow generated by the differential equation $x = Y_0(x)$ has a hyperbolic set Λ, by the local stability of hyperbolic sets, it follows that Y_ε has an equivalent hyperbolic set Λ_ε. Therefore the lifting of Λ_ε to V_ε by the projection π is hyperbolic for $\varphi^\varepsilon/V_\varepsilon$ and since V_ε is normally hyperbolic is hyperbolic for φ^ε.

Proposition A. Hyperbolic sets are stable.

Proof: Let Λ be a hyperbolic set for $\varphi \in \mathscr{F}^T_r(U)$ and let B be the vector bundle on Λ defined as $B = \{(p, v) \mid p \in \Lambda, v \in E^s_p \oplus E^u_p\}$. Let $\pi: B \to \Lambda$ be the canonical projection. Define $B_k = \{(p, v) \in B \mid \|v\| \le k\}$ and Γ_k as the space of continuous bundle maps $f: B_k \to B$ vertically C^r. Endow Γ_k with the C^r topology. In $0 < \tau_0 < T$ there exist $k > 0$, a neighborhood $\mathscr{U}$ of φ in $\mathscr{F}^T_r(U)$, $\delta > 0$ and a map:

$$\mathscr{U} \times (0, \tau_0) \notin (\psi, t) \to \hat{\psi}_t \in \Gamma_k \tag{3}$$

defined by the properties:

a) $\hat{\psi}_t(p, v) = (\varphi(t, p), \psi(\bar{t}, p + v) - \varphi(t, p))$

where

b) $\psi(\bar{t}, p + v) - \varphi(t, p) \in E^s{}_{(t,p)} \oplus E^u{}_{(t,p)}$ (4)

c) $|\bar{t} - t| < \delta$ (5)

With a suitable choice of k, δ and $\mathscr{U}$ (4), and (5) define a unique $\bar{t}$ and the map (3) is well defined and continuous. The geometrical idea behind this definition is that of taking Poincaré maps induced by ψ between the local cross sections $E^s_p \oplus E^u_p$ and $E^s_{\varphi(t,p)} \oplus E^u_{\varphi(t,p)}$. Consider the space G^b of bounded sections of the bundle map $\pi: B \to \Lambda$ with the norm $\|\eta\| = \sup\{\|\eta(p)\| \mid p \in \Lambda\}$. If $f \in \Gamma_k$ and covers a homeomorphism $\alpha: \Lambda \supset$ define $f_*: G^b_k = \{\eta \in G^b \mid \|\eta\| \le k\} \to G^b$ defined by $f_*(\eta) = f \circ \eta \circ \alpha^{-1}$. Let $G^0 \subset G^b$ be the closed subspace of continuous sections. Take $0 < \tau < \tau_0$. Then the zero section of B is a hyperbolic fixed point for $\hat{\varphi}_{\tau *}: G^b_k \to G^b$ and $\hat{\varphi}_{\tau *}: G^b_k \cap G^0 \to G^0$. By elementary properties of hyperbolic fixed points, for all $\psi \in \mathscr{U}$ the map

$\hat{\psi}_{\tau*}: G^b_k \to G^b$ has a unique fixed point ζ_ψ near to the zero section and $\zeta_\psi \in G^0$. Define $h: \Lambda \to U$ by $h(p) = p + \zeta_\psi(p)$. It is continuous because ζ_ψ is continuous. It takes φ-orbits onto ψ-orbits because $\hat{\psi}_{\tau*}(\zeta_\psi)\ U(\hat{\psi}_{t*} \circ \hat{\psi}_{\tau*})(\zeta_\psi) = U\,\hat{\psi}_{\tau*}(\hat{\psi}_\psi))$, hence, by the unicity of the fixed point ζ_ψ it follows that that $\zeta_\psi = \hat{\psi}_{t*}(\zeta_\psi)$ for all $0 < t < \varsigma$. Finally h is one to one because if $p_1 \neq p_2$ and $h(p_1) = h(p_2)$ the intersection of the φ-orbit of p_2 with the hyperplanes $E^s_{\varphi(t,p_1)} \oplus E^u_{\varphi(t,p_2)}$ define a section $\eta \in G^b_k$ with support in the orbit of p_1 and satisfying $\varphi_\tau(\eta) = \eta$ for all $0 \le t \le \tau$ thus contradicting the unicity of the fixed point of φ_t in G^b_k.

Let X, A, X^α, V, F and U as in section 1 and let $\varphi \in \mathscr{F}^T_r(U)$ be the semiflow satisfying (1) and (2). Assume that X has a Schauder basis.

Proposition B. *Assume that A has compact inverse and let E^λ, $\lambda > 0$ be the subspace of X spanned by the generalized eigenvectors associated to eigenvalues of modulus $\ge \lambda$. Then if $\cup\{E^\lambda \mid \lambda > 0\}$ is dense in X, for all hyperbolic set for φ such that every point of Λ is non wandering for φ/Λ and $\varepsilon > 0$, there exists $\lambda > 0$ and a continuous projection $\hat{\pi}^\lambda: X \to E^\lambda$ such that the ordinary differential equation on $E^\lambda \cap V$:*

$$\dot{x} = -Ax + \hat{\pi}^\lambda F(x)$$

has a hyperbolic set ε-equivalent to Λ.

Proof: For all $\gamma > 0$ the space $\cup\{E^\lambda \mid \lambda > 0\}$ is dense in X^γ because $\cup\{E^\lambda \mid \lambda > 0\}$ is dense in X, invariant under $A^{-\gamma}$ and $A^{-\gamma}$ is an isomorphism between X and X^γ. It is easy to see that there exists a family of continuous projectios $\pi^\lambda: X^\alpha \to E^\lambda$, $\hat{\pi}^\lambda: X \to E^\lambda$ such that $\|\hat{\pi}^\lambda v - v\| \to 0$ and $\|\hat{\pi}^\lambda v - v\|^\alpha \to 0$ when $\lambda \to +\infty$. We shall need the two following elementary properties of the semiflow φ:

(I) – For all $\alpha < \beta < 1$ and $0 < t < T$ $\varphi_t(U)$ is contained in X^β and $\varphi_t: U \to X^\beta$ is continuous.

(II) – For all $\alpha < \beta < 1$, $0 < t < T$ and $x \in U$ $\varphi_t'(x)X^\alpha \subset X^\beta$ and there exists $K_\beta > 0$ such that

$$\|\varphi_t'(x)v\|_\beta \le K_\beta t^{\alpha-\beta} \|v\|_\alpha$$

for all $v \in X^\alpha$.

Take $\alpha < \beta < 1$. By (I) it follows that $\Lambda = \varphi_t(\Lambda)$ is compact in X^β, hence bounded. Take a bounded neighborhood U_1 of Λ in X^β. The existence of a compact inverse of A implies that $A^{-\gamma}$ is compact $\forall\, \gamma > 0$. Since $X^\beta = A^{-(\beta-\alpha)} X^\alpha$ it follows that bounded sets of X^β have compact closure in X^α. From this it is easy to prove that $\cup\{\pi^\lambda U_1 \mid \lambda > 0\}$ has compact closure in X^α. Hence its closed convex hull W in X^α is compact. A similar argument proves that the closed convex hull B of $\cup\{\pi^\lambda B^\beta \mid \lambda > 0\}$ in X^α, where $B^\beta = \{v \in X^\beta \mid \|v\|_\beta\ U\ 1\}$, is compact. Then for all $\delta_1 > 0$ there exists $\delta_2 > 0$ such that:

$$\|F(x_1) - F(x_2)\| \le \delta_1$$

$$\|F'(x_1) - F'(x_2)\| \le \delta_1$$

for all $x_i \in W \cap V$ $i = 1, 2$, where $F'(x): X^\alpha \to X$ denotes the derivative of $F: X^\alpha \to X$ at x. From Banach-Steinhaus theorem follows that there exists $\lambda > 0$ satisfying:

$$\pi^\lambda(U_1) \subset V$$

$$\|\pi^\lambda x - x\|^\alpha \le \delta_2 \quad \forall\ x \in W \cup B$$

$$\|(\hat{\pi}^\lambda - I)F(x)\| \le \delta_2 \quad \forall\ x \in W \cap V$$

$$\|(\hat{\pi}^\lambda - I)F'(x)v\| \le \delta_2 \quad \forall\ x \in W \cap V,\ v \in B$$

Then if $x \in U_1$, $v \in B^\beta$:

$$\|(\hat{\pi}^\lambda F \pi^\lambda)(x) - F(x)\| \le \|(\hat{\pi}^\lambda - I)F(\pi^\lambda x)\| + \|F(\hat{\pi}^\lambda x) - F(x)\| \le \delta_2 + \delta_1$$

$$\|((\hat{\pi}^\lambda F \pi^\lambda)'(x) - F'(x))v\| \le \|(\hat{\pi}^\lambda - I)F'(\hat{\pi}^\lambda x)\pi^\lambda v\| + \|(F'(\hat{\pi}^\lambda x) - F'(x))\hat{\pi}^\lambda v\| + \|F'(x)(\hat{\pi}^\lambda - I)v\| \le \delta_2 + \delta_1 K_1 + \delta_2 K_2$$

where $K_1 = \sup\{\|\pi^\lambda v\|_\alpha \mid \|v\|_\alpha \le 1,\ \lambda > 0\}$ $K_2\ U \sup\{\|F'(x)\| \mid x \in U_1\}$. Hence we can chose $\lambda > 0$ such that F and $\hat{\pi}^\lambda F \pi^\lambda$ are near in $C^1(U_1, X)$. Take a neighborhood U_2 of Λ such that $U_2 \subset U_1$ and $\inf\{\|x - y\|_\beta \mid x \in U_2, y \notin U_1\} > 0$ and a unique semiflow $\tilde{\varphi} \in \mathscr{F}^\tau_r(U_2)$ satisfying:

$$\varphi((0, \tau) \times U_2) \subset U_1 \cap X^\beta$$

$$\frac{\partial \tilde{\varphi}}{\partial t}(t, x) = -A\tilde{\varphi}(t, x) + F(\tilde{\varphi}(t, x))$$

for all $x \in U_2$, $0 < t < \tau$. Observe that $\tilde{\varphi}(t, x) = \varphi(t, x)\ \forall\ x \in U_2$. We claim that Λ is a hyperbolic set for $\tilde{\varphi}$. We have already proved that Λ is compact in X^β. By the definition of hyperbolicity:

(1) $$\varphi_t'(\varphi(-t,x))\, E^u_{\varphi(-t,x)} = E^u_x$$

hence by (I) $E^u_x \subset X^\beta$. Therefore $X^\beta = E^u_x \oplus (X^\beta \cap E^s_x)$ $\forall\, x \in \Lambda$. Moreover since $X^\beta \subset X^\alpha$ is compact property (II) implies that $\varphi_t'(x) \colon X^\alpha \to X^\alpha$ is compact. Then, by (1), E^u_x is finite dimensional. By the compacity of Λ in X^β there exist constants $C_i > 0$ $i = 1,2$ satisfying $C_1 \| v \|_\beta \leq \| v \|_\alpha \leq C_2 \| v \|_\beta$ for all $v \in U\{E^u_x \mid x \in \Lambda\}$. Then if $x \in \Lambda$, $v \in E^u_x$ we have

$$\| \tilde{\varphi}_t'(x) v \|_\beta = \| \varphi_t'(x) v \|_\beta \geq \frac{1}{C_2} \| \varphi_t'(x) v \|_\alpha \geq$$

$$\geq \frac{1}{C_2} K e^{\sigma t} \| v \|_\alpha \geq \frac{C_1}{C_2} K e^{\sigma t} \| v \|_\beta$$

Therefore E^u_x defines the unstable subbundle of Λ for $\tilde{\varphi}$. Finally we prove that $X^\beta \cap E^s_x$ defines the stable subbundle. By (II), if $0 < t < t_0$, we have

$$\| \tilde{\varphi}_t'(x) v \|_\beta = \| \varphi_t'(x) v \|_\beta = \| \varphi_{t_0}'(\varphi(t - t_0, x))\, \varphi_{t-t_0}'(x) v \|_\beta \leq$$
$$\geq K_\beta t_0^{\alpha-\beta}\, C e^{-\sigma(t-t_0)} \| v \|_\beta = K_\beta C t_0^{\alpha-\beta} e^{\sigma t_0} e^{-\sigma t} \| v \|_\beta$$

for all $x \in \Lambda$, $x \in X^\beta \cap E^s_x$. Then if $C_1 = \sup\{ \| \varphi_t'(x) v \|_\beta e^{\sigma t} \mid x \in \Lambda,\ \| v \|_\beta \leq 1,\ 0 \leq t \leq t_0\}$ and $C_2 = \sup\{C_1, K_\beta C t_0^{\alpha-\beta} e^{\sigma t_0}\}$ it is easy to see that:

$$\| \tilde{\varphi}_t'(x) v \|_\beta \leq C_2\, e^{-\sigma t} \| v \|_\beta$$

for all $x \in \Lambda$, $v \in X^\beta \cap E^s_x$. Now take $\psi \in \mathscr{F}^\tau_1(U_2)$ satisfying:

$$\psi((0,\tau) \times U_2) \subset X^1 \cap U_1$$

$$\frac{\partial \psi}{\partial t}(t,x) = -A\psi(t,x) + (\hat{\pi}^\lambda F \pi^\lambda)(\psi(t,x)).$$

Since $\hat{\pi}^\lambda F \pi^\lambda$ is near to F in $C^1(U_1, X)$ ψ is near to φ in $\mathscr{F}^\tau_1(U_2)$. Hence by Proposition A ψ has a hyperbolic set Λ_ψ ε-equivalent to Λ. Let $X = E^\lambda \oplus \tilde{E}^\lambda$ where $\tilde{E}^\lambda = \ker \hat{\pi}^\lambda$ and $\tilde{\pi}^\lambda \colon X \to \tilde{E}^\lambda$ the associated projection. Defining $\psi_1 = \hat{\pi}^\lambda \psi$, $\psi_2 = \tilde{\pi}\psi$ we have:

(2) $$\frac{\partial \psi_1}{\partial t} = -A\psi_1 - \hat{\pi}^\lambda A\psi_2 + \hat{\pi}^\lambda F(\pi^\lambda \psi)$$

(3) $$\frac{\partial \psi_2}{\partial t} = -\tilde{\pi}^\lambda A$$

Let $\tilde{A} \colon \tilde{E}^\lambda \to \tilde{E}^\lambda$ defined by $\tilde{A} = \tilde{\pi}^\lambda A\, \tilde{E}^\lambda$. It is easy to verify that $\tilde{A}$ is sectorial and $sp(\tilde{A}) \subset sp(A)$. Since A is sectorial and has compact inverse by hypothesis, there exists $\mu > 0$ such that: $sp(\tilde{A}) \subset \{z \mid Re(z) > \mu\}$. Then $-\tilde{A}$ generates a linear semigroup [7] $e^{-\tilde{A}t}$ such that

$$\| e^{-\tilde{A}t} v \| \leq e^{-\mu t} \| v \|.$$

From (3) follows that:

$$\psi_2(t,x) = e^{-\tilde{A}t}\, \tilde{\pi}^\lambda x$$

hence:

$$\| \psi_2(t,x) \| \leq e^{-\mu t} \| \tilde{\pi}^\lambda x \|.$$

Therefore $\Lambda_\psi \subset E^\lambda$ and from (2) it is an invariant set for the O.D.E.:

$$\dot{x} = -Ax + (\hat{\pi}^\lambda F)(x)$$

where $x = E^\lambda \cap V$. Let ζ be the flow generated by this equation. Then $\zeta = \psi / E^\lambda$. The hyperbolicity Λ_ψ implies that $\forall\, 0 \neq v \in E^\lambda$ the set $\{\| \zeta_t'(v) \| \mid t \in \mathbb{R}\}$ is unbounded. Moreover, since Λ_ψ is ε-equivalent to Λ every point of Λ_ψ is nonwandering for $\psi/\Lambda_\psi = \zeta/\Lambda_\psi$. By [8] this implies that Λ_ψ is hyperbolic for ζ.

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