On the singularities of foliations and of vector bundle maps

Bruce F. Golbus *

Abstract. Let F be a (smooth) Γ_q^∞-structure (often called a codimension-q Haefliger structure) on a compact manifold X^n. Cohomological invariants associeted to the singularities of F are defined whose vanishing is shown to be a necessary condition for deforming F to a codimension-q foliation on X^n. An analagous approach to vector bundle maps is then utilized to prove a general theorem concerning the possibility of embedding a vector bundle in the tangent bundle of X^n, and applications to the planefield problem are given. In the final section geometric realizations of the singularity classes associated to F are constructed.

0. Introduction.

In 1947, G. Reeb posed the following fundamental question in the theory of (smooth) foliations [cf. 19, p. 95]. If the manifold X^n admits a field of $(n-q)$-planes ξ does it also admit a completely integrable field of $(n-q)$-planes ξ_0, i.e., a field of planes tangent to the leaves of a codimension q foliation E? In [5, p. 10], A. Haefliger refined the problem somewhat by asking if ξ_0 did exist, could it be chosen to lie in the homotopy class of ξ?

In the last several years, a great deal of attention has been paid to these two questions. (For a survey of known results, see [12]) On *open manifolds*, a general construction due to Haefliger [cf. 6] has reduced the problem to a question in homotopy theory. Basically, his approach is to consider the so-caled "Γ_q^∞-structures" F (often called codimension q Haefliger structures in the literature) and to ask when such "singular foliations" are homotopic to actual codimension-q foliations. (Note that Γ_q^∞-structures exist in abundance on any manifold; in particular any smooth function $f\colon X^n \to R^q$ will induce such a structure.) Using the Gromov-Phillips Transversality Theorem [cf. 6, p. 188] as his main tool, he was able to demonstrate: (See Section 1 for definitions)

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Theorem (H). [6, *p.* 190] *Let X^n be an open manifold (every compact component has a boundary) and F a Γ_q^∞-structure on X^n with normal bundle $\nu(F)$. Suppose that there exists a maximal rank bundle map ϕ: $TX \longrightarrow \nu F \longrightarrow 0$. Then F is homotopic to a foliation E.*

As an immediate application, there is the following important corollary:

Corollary. *Suppose ξ is an $(n-q)$-subbundle of TX^n (X^n open). A neccessary and sufficient condition for ξ to be homotopic to a foliation is that TX/ξ be vector bundle isomorphic to $\nu(F)$, F a Γ_q^∞-structure on X^n.*

In this work our primary purpose is to provide further insight into the existence problem for foliations (and plane fields) on compact manifolds X^n through a study of the singularities of Γ_q^∞-structures F and of bundle maps ϕ: $TX^n \longrightarrow \nu^q$, ν^q a vector bundle over X^n. That is, we consider the problem of finding neccessary and sufficient conditions for the elimination of these singularities within the homotopy class (resp. isomorphism class) of F (resp. ν^q).

In Section 1, we give the basic definitions and constructions for the convenience of readers.

In Section 2, the graph construction of [6] is recalled, and some further properties of this foliated bundle are established. This in turn makes it possible to show, in Sections 3.2, 3.3, that a (smooth) Γ_q^∞-structure F on a paracompact manifold does indeed have a well-defined global singular set in the sense of Thom's theory of singularities of differentiable mappings [cf. 14]. In 3.3 the existence of this global singular set is exploited to produce our first new result (Thm. 3.3.2). That is, we define for each such F three sets of "singularity classes" with varying assumptions of orientability on X^n and F) $b_i(F) \varepsilon H^{i(n-q+i)}(X^n; Z_2)$, $b_i^C(F) \varepsilon H^{2i(n-q+i)}(X^n; Z)$ and $r_j(F) \varepsilon H^{j(n-q+j)}(X^n; Z)$, for $0 \le i, j \le q, j = 2k$, such that each is an invariant of the homotopy class of F and if F is homotopic to a foliation E, all of the classes vanish $(i, j > 0)$.

In Section 4, using a theorem due to R. Thom, I. Porteous and F. Ronga, analogous "singularity classes" are defined for mappings of vector bundles ν^q over X^n. A general theorem relating the vanishing of these classes to the existence of a bundle monomorphism ϕ: $\nu^q \longrightarrow TX^n$ is then proved, which is our main result (Theorem (4.2)). Applying this theorem to the situation where ν^q is isomorphic to the trivial q-plane bundle over X^n, we are able to derive the following corollary:

Corollary 4.3. *For each $q \ge 9$, there is a closed, orientable manifold X^n with zero Euler characteristic and signature such that X^n has a q-frame field over its 3-skeleton but no global q-frame field.*

In the final section, we consider $T-\Gamma_q^\infty$-structures, i.e., ones satisfying a certain set of first order transversality conditions [cf. Def. 3.2.1]. The main resul here, accomplished via a partial extension of the Thom transversality theorem, is that the classes $B(F)$ and $R(F)$ can always be "realized geometrically" in X^n. [cf. Def. 5.2.1].

As this work was being completed, W. Thurston succeeded in extending Theorem (H) to the case of compact manifolds. [See 25].

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1. General Definitions

Reference for this section are [3, p. 39-42] and [6, p. 184-187].

1.1. Let X^n be an n-dimensional, paracompact, connected C^∞ manifold. A (smooth) Γ_q^∞-cocycle c on X^n $(q \le n)$ is given by:

1) An open cover $\bar{U} = \{U_i\}_{i\varepsilon J}$ of X^n;
2) smooth maps $f_i : U_i \longrightarrow R^q$ called the *local projections*; and
3) for each $i, j \varepsilon J$ and $x \varepsilon U_i \cap U_j$ a diffeomorphism γ_{ij}^x from a neighborhood of $f_j(x) \subset R^q$ onto a neighborhood of $f_i(x)$ satisfying:

a) $f_i = \gamma_{ij}^x \circ f_j$ in a neighborhood of x, and
b) for each $x \varepsilon U_i \cap U_j \cap U_k$, one has $\gamma_{ki}^x = \gamma_{kj}^x \circ \gamma_{ji}^x$ in a neighborhood of $f_i(x)$. b) is called the *cocycle condition and the* $\{\gamma_{ij}^x\}_{i,j\varepsilon J}$ are called the *transition functions.*

Two such cocycles $c = \{U_i, f_i, \gamma_{ij}^x\}_{i,j\varepsilon J}$, $c' = \{U'_\alpha, f'_\alpha, \gamma_{\alpha\beta}^{x'}\}_{\alpha,\beta\varepsilon J'}$ are said to be *equivalent* if there is a cocycle corresponding to the covering $\{U_\gamma\}_{\gamma\varepsilon J \cup J'}$ which restricts to c on $\{U_i\}_{i\varepsilon J}$ and c' on $\{U'_\alpha\}_{\alpha\varepsilon J'}$. Phrased in terms of the transition functions, this says that there exists a collection of local diffeomorphisms of R^q, $\bar{\gamma}_{\alpha i}$, $\alpha \varepsilon J'$, $i \varepsilon J$ that together with the $\{\gamma_{ij}\}$ and the $\{\gamma_{\alpha\beta'}\}$ satisfy condition b on the union of the covers $\{U_i\}$ and $\{U'_\alpha\}$. Then a (smooth) Γ_q^∞*-structure* F on X^n is an equivalence class of Γ_q^∞-cocycles.

1.2. Let $F = \{U_i, f_i, \gamma_{ij}^x\}_{i,j\varepsilon J}$ be as above. If all the local projections f_i: $U_i \longrightarrow R^q$ are of maximal rank, i.e. submersions then F is a codimension-q foliation in the standard sense, the γ_{ij}^x are completely determined by the f_i's and b) follows from a). Recall that if F is a foliation, there is defined the *normal bundle* of F, $Q(F)$, which can be described (via a Riemannian metric on X^n) as the set of vectors $v \varepsilon TX^n$ orthogonal to the leaves of F.

1.3. More generally, a normal bundle $\nu(F)$ can be associated to any Γ_q^∞-structure F on X. This is a q-dimensional vector bundle over X^n, defined as follows: Over U_i, $\nu(F)$ is $f_i^* TR^q$, the pull-back by f_i of the tangent bundle of R^q. Over $U_i \cap U_j$, identify $(x, v) \sim (x', v') \Leftrightarrow x = x'$ and $v' = T\gamma_{ij}^x(f_j(x)) \bullet v$ where $T\gamma_{ij}^x(f_j(x)) \bullet v$ is the tangent map of γ_{ij}^x at the point $f_j(x)$ acting on the vector v. Then $\nu(F)$ satisfies the following:

(1) if F is a foliation, $Q(F) \cong \nu(F)$ (that is, they are isomorphic as vector bundles over X^n)

(2) this construction is functorial: in particular if $f: X^n \longrightarrow Y$ is a smooth map between manifolds, F a Γ_q^∞-structure on Y, then $\nu(f^{-1}(F)) = f^*(\nu(F))$ where $f^{-1}(F)$, *the* Γ_q^∞*-structure induced by f from* F is given by $f^{-1}(F) = \{f^{-1}U_i, f_i \circ f,\ \gamma_{ij}^{x'} = \gamma_{ij}^{f(x)}\}$. Note that if F is a foliation, this will not be true in general for $f^{-1}F$. It will be the case if f is transverse to F, i.e., transverse to all the leaves of F.

Finally, F is said to be *transversally oriented* if $\nu(F)$ is an oriented vector bundle over X^n. This is equivalent to demanding that all the transition functions be orientation-preserving.

1.4. Let F_0, F_1 be two Γ_q^∞-structures on X^n. Then F_0 and F_1 are *homotopic*, written $F_0 \simeq F_1$, if there is a Γ_q^∞-structure $\bar{F}$ on $X^n \times I$ such that $i_0^{-1}(\bar{F}) = F_0$ and $i_1^{-1}(\bar{F}) = F_1$ where $i_t(x) = (x, t)$, $t = 0, 1$. Then the following are readily verified:

1) Homotopy is an equivalence relation of the set of smooth Γ_q^∞-structures on X^n.

2) If $f: X^n \longrightarrow Y^m$ is C^∞, F_0, F_1 homotopic Γ_q^∞-structures on Y^m, them $f^{-1}(F_0) \simeq f^{-1}(F_1)$.

3) If f, g: $X^n \longrightarrow Y^m$ are differentiably homotopic, F a Γ_q^∞-structure on Y^m, then $f^{-1}(F) \simeq g^{-1}(F)$.

2. Foliated Microbundles.

Given a Γ_q^∞-structure F on a paracompact manifold X^n, it is an observation due to A. Haefliger and J. Milnor [cf. 6, p. 188] that F can be obtained as the pull-back of a smooth codimension-q foliation $\bar{E}$ on a certain open manifold M^{n+q}.

This section will be devoted to establishing some further properties of the foliated manifold M^{n+q}. In section 3, these results will be applied to define a section $\sigma_F \varepsilon \Gamma^\infty(\mathrm{Hom}(TX^n, \nu F))$ (the C^∞ sections of the vector bundle $\mathrm{Hom}(TX^n, \nu F) \longrightarrow X^n$), which is a key step in defining cohomological invariants for the pair (X^n, F).

Let X^n be a C^∞ paracompact manifold, and let m be a C^∞ microbundle of fibre dimension q over $X^n \cdot m$ is given by a smooth manifold M^{n+q} together

with smooth maps i: $X^n \longrightarrow M$, p: $M^{n+q} \longrightarrow X^n$ satisfying certain local triviality conditions [cf. 16, p. 54]. If M^{n+q} possesses a C^∞ codimension-q foliation $\bar{E}$ whose leaves are everywhere transverse to the submanifolds $p^{-1}(x)$, $x \varepsilon X^n$, the collection $(M, \bar{E}, i, p)$ is called a *codimension-q foliated microbundle* (over X^n).

One can now state the

Theorem 2.1. [*cf.* 6, *p.* 188]: *Let* X^n *be a paracompact* C^∞ *manifold. Then given any smooth* Γ_q^∞*-structure* F *on* X^n*, there exists a foliated microbundle* (M, E, i_F, p_F) *over* X^n *such that* $i_F^{-1}(E) = F$. *M is called the graph of F. Further, this foliated microbundle satisfies:*

1) *The microbundle* $m = (M, i_F, p_F)$ *is unique up to isomorphism.* [*cf.* 16, *p.* 56]

2) *There is a vector bundle isomorphism* A: $TM \,|\, i_F(X^n) \longrightarrow T(i_F(X)) \oplus \nu(\bar{E}) \,|\, i_F(X)$, *where* $T(i_F(X^n))$ *is the tangent bundle of the sub-manifold* $i_F(X^n)$, *and* $TM \,|\, i_F(X)$, $\nu(\bar{E}) \,|\, i_F(X^n)$ *are the tangent bundle of M and the normal bundle of* $\bar{E}$, *respectively, restricted to this sub-manifold; and*

3) *If* $F_1 \simeq F_2$ *are homotopic* Γ_q^∞*-structures on* X^n, *then* $m(F_1) \sim m(F_2)$, *i.e., they are isomorphic.*

Proof. The construction of the graph of F is given in the above reference together with the assertion that property (1) is valid. In order to prove (1) through (3), and for use in Section 5, we introduce the following theorem, due to J. M. Kister [9, Theorem 2, p. 96]:

Theorem 2.2. *Let* m: $B \xrightarrow{i} E \xrightarrow{j} B$ *be a (topological microbundle of fibre dimension q over a locally-finite, n-dimensional complex B. Then there exists an open set* $E_1 \subset E$ *with* $i(B) \subset E_1$ *such that* $j \,|\, E_1 : E_1 \to B$ *is a fibre bundle with fibre* R^q *and group* $H_0(q) = \{$*homeomorphisms* h: $R^q \to R^q \,|\, h(0) = 0\}$. *Further, if* E_2 *is any other such open set of E, there exists a homeomorphism* g: $E_1 \to E_2$ *such that g is the identity on the zero-section* $i(B)$.

Let $M_1 \subset M$ be the open set given by this theorem, and set $m_1(F) =$ the corresponding fibre bundle with distinguished zero-section, $m_1(F) = (M_1, i_F, \bar{}_F \,|\, M_1)$. It is clear that $m_1(F)$ is in fact a $C^\infty R^q$-bundle over X^n, with group the set of origin-preserving diffeomorphism of R^q. To establish property (1), suppose $(M', i', p', \bar{E}')$ is another codimension-q foliated microbundle over X^n satisfying $i'^{-1}(\bar{E}') = F$. Let $M_1' \subset M'$ be the total space of the R^q-bundle contained in M' and let $\bar{E}_1'$ be the foliation of M_1' given by the restriction of $\bar{E}'$. Since $i'^{-1}(\bar{E}') = F$, the transition function $\bar{\gamma}_{ij}^y$ for $\bar{E}'$ at the point y in $p'^{-1}(x)$, $x \varepsilon U_i \cap U_j$, is exactly γ_{ij}^x. Also, since $\bar{E}_1'$ has codimension $= q$ it follows that $\bar{E}_1' \,|\, p'^{-1}(U_i)$ is diffeomorphic to a family of n-planes in R^{n+q}, and as $\bar{E}'_1$ is transverse to the fibres of p', the coordinates $z^i_1, \dots, z^i_{n+q}$ in $p'^{-1}(U_i)$ in which $\bar{E}'_1$ takes on the above form may be chosen to be trivializing coordinates for bundle M'_1 over U_i. Hence, over $x \varepsilon U_i \cap U_j$, with respect to the two sets of coordinates $(z^i_1, \dots, z^i_{n+q})$, $(z^j_1, \dots, z^j_{n+q})$ chosen in the above manner, the transition function $\bar{\gamma}^y_{ij} = \gamma^x_{ij}$ for the foliation $\bar{E}'_1$ will coincide with the transformation g_{ij} for the bundle M'_1, i.e., the group of this bundle can be reduced to the "group" $\{\gamma^x_{ij}\}_{i,\, j \varepsilon J}$. Since the same reduction is possible for $m_1(F)$, it follows that there is a fibre bundle equivalence h: $M'_1 \longrightarrow M_1$ which must necessarily preserve zero-sections of each bundle, and so (M', i', p') is isomorphic to (M, i_F, p_F).

To see (2), begin by noting that as M_1 is open in M, $TM \mid i_F(X^n) = TM_1 \mid i_F(X^n)$ and $\nu(\bar{E}) \mid i_F(X^n) = \nu(\bar{E}_1) \mid i_F(X^n)$. On the other hand, since M_1 is the total space of an R^q-bundle over X^n, by considering transition functions, it can be easily shown that as vector bundles $TM_1 \cong \mathrm{Ver}(TM_1) \oplus p^*_F(TX^n)$ where $\mathrm{Ver}(TM_1)$ denotes the tangent space to the fibres of $m_1(F)$. This isomorphism restricts to $\tilde{A}$: $TM_1 \mid i_F(X) \longrightarrow \mathrm{Ver}(TM_1) \mid i_F(X^n) \oplus T(i_F(X^n))$ or, by the above remark to $\tilde{A}$: $TM \mid i_F(X) \longrightarrow \mathrm{Ver}(TM_1) \mid i_F(X^n) \oplus T(i_F(X))$. Now an argument similar to the one given above will show that there is a vector bundle isomorphism B: $\mathrm{Ver}(TM_1) \mid i_F(X^n) \longrightarrow \nu(\bar{E}) \mid i_F(X^n)$. Then $A = (B \oplus id) \circ \tilde{A}$ is the desired map.

To conclude the proof of Theorem 2.1, suppose that $F_0 \simeq F_1$ are homotopic Γ^∞_q-structures on X^n, and $\bar{F}$ is the Γ^∞_q-structure on $X^n \times I$ satisfying $i_0^{-1}(\bar{F}) = F_0$ and $i_1^{-1}(\bar{F}) = F_1$. Let $m(\bar{F})$: $X^n \times I \overset{i_{\bar{F}}}{\rightarrow} \bar{M} \overset{p_{\bar{F}}}{\rightarrow} X^n \times I$ be the graph of $\bar{F}$ and let $i^*_\alpha(m(\bar{F}))$ be the induced microbundle over X^n, $\alpha = 0, 1$. Since the graph construction is unique up to microbundle isomorphism, it may be assumed that the microbundles $m(\bar{F})$, $m(i_1^{-1}(F))$ and $m(i_0^{-1}(\bar{F}))$ have been constructed in the manner specified in [6]. It is then straightforward to check that for $\alpha = 0, 1$, $i^*_\alpha(m(\bar{F})) \sim m(i_\alpha^{-1}(\bar{F}))$. However, according to [16, Thm. 3.1, p. 58], since X^n is paracompact, $i^*_0(m(\bar{F})) \simeq i^*_1(m(\bar{F}))$ and so there is a chain of isomorphisms: $m(F_1) = m(i_1^{-1}(\bar{F})) \sim i^*_1(m(\bar{F})) \sim i^*_0(m(\bar{F})) \sim m(i_0^{-1}(\bar{F})) = m(F_0)$, and so the theorem is proved.

3. Invariants for the pair (X^n, F).

In this section we assume throughout that the manifold X^n is compact (with or without boundary), connected, and that F is Γ^∞_q-structure on X^n.

3.1. Let $\xi^m \longrightarrow X^n$, $\tau^q \longrightarrow X^n$ be real vector bundles over X^n of (fibre) dimension m and q, respectively. Let $\mathrm{Hom}(\xi, \tau) \longrightarrow X^n$ denote the vector bundle over X^n of dimension $m \cdot q$ with $\mathrm{Hom}(\xi, \tau)_x$ = fibre of $\mathrm{Hom}(\xi, \tau)$ over $x \varepsilon X^n = \{R\text{-linear maps } \phi_x\colon \xi_x \longrightarrow \tau_x\}$ and let $S_i(\xi, \tau)$ denote the submanifold

of $\mathrm{Hom}(\xi, \tau)$ consisting of those elements with rank equal to $\min(m, q) - i$, for $0 \le i \le \min(m, q)$. Then the following facts about $S_i(\xi, \tau)$ are well known: [cf. 13, p. 372].

1) $S_i(\xi, \tau)$ is regularly embedded in $\mathrm{Hom}(\xi, \tau)$, that is the inclusion map is a homeomorphism into;
2) $S_i(\xi, \tau)$ is a sub-bundle of $\mathrm{Hom}(\xi, \tau)$ (although not a sub-vector bundle) with fibre equal to $S_i(m, q) = \{m \times q$ real matrices of rank $\min(m, q) - i\}$;
3) the codimension of $S_i(\xi, \tau)$ in $\mathrm{Hom}(\xi, \tau) = i(|m - q| + i)$; and
4) $\overline{S_i(\xi, \tau)} = \bigcup_{j=0}^{\min(m,q)-i} S_{i+j}(\xi, \tau)$. Thus, for example, $\overline{S_0(\xi, \tau)} = \bigcup_{j=0}^{\min(m,q)-0} S_j(\xi, \tau) = \mathrm{Hom}(\xi, \tau)$. Similarly, for $\eta^m \longrightarrow X^n$, $\omega^q \longrightarrow X^n$ complex vector bundles over X^n of complex dimension m and q, respectively, if one forms the corresponding bundle $\mathrm{Hom}_C(\eta, \omega) \longrightarrow X^n$ and then decomposes this bundle into its singularlity sub-bundles $S^C_i(\eta, \omega)$ according to complex rank, properties analogous to (1)-(4) will continue to hold.

It is a consequence of the work of Borel-Haefliger [cf. 21, p. 23-24, see also 7, p. 8-02], that for $0 \le i \le \min(m, q)$, $\overline{S_i(\xi, \tau)}$ carries a fundamental class in $H_s(\overline{Si(\xi, \tau)}; Z_2)$, singular homology with closed supports, where s denotes the dimension of $S_i(\xi, \tau)$. Also, if X^n is orientable, the complex space $\overline{S^C_i(\eta, \omega)}$ possesses such a class. (Z coefficients). Further, according to [20], if the bundle $\mathrm{Hom}(\xi, \tau) \longrightarrow X^n$ is orientable (in particular if both $\xi^m \longrightarrow X^n$, $\tau^q \longrightarrow X^n$ are orientable bundles) $m - q = 2r$, $j = 2k$, then $\overline{S_j(\xi, \tau)}$ has a fundamental class over Z.

Let $[\overline{S_i(\xi, \tau)}]$ denote the image of the fundamental class of $\overline{S_i(\xi, \tau)}$ in $H_*(\mathrm{Hom}(\xi, \tau); Z_2)$ under the homology homomorphism induced by the inclusion $\overline{S_i(\xi, \tau)} \rightarrow \mathrm{Hom}(\xi, \tau)$. (Since $\mathrm{Hom}(\xi, \tau)$ is a paracompact manifold, by [2, p. 20], singular homology with closed supports gives the ordinary singular homology, so $[\overline{S_i(\xi, \tau)}]$ is a singular homology class.) Define $P.D.[\overline{S_i(\xi, \tau)}] \,\varepsilon\, H^*_C(\mathrm{Hom}(\xi, \tau); Z_2)$ (the Z_2-cohomology with compact supports) to be the image of $[\overline{S_i(\xi, \tau)}]$ under the Poincare Duality isomorphism $P.D.$: $H_s(\mathrm{Hom}(\xi, \tau); Z_2) \rightarrow H^{m+m \cdot q-s}_C(\mathrm{Hom}(\xi, \tau); Z_2)$. [cf. 22, p. 341]. Similarly define $P.D.[\overline{S^C_i(\eta, \omega)}] \,\varepsilon\, H^*_C(\mathrm{Hom}_C(\eta, \omega); Z)$ and $P.D.[\overline{S_{2k}(\xi, \tau)}] \,\varepsilon\, H^*_C(\mathrm{Hom}(\xi, \tau); Z)$.

3.2. Let $X^n \overset{i_F}{\rightarrow} M \overset{p_F}{\rightarrow} X^n$ be the graph of F and let A: $TM \mid i_F(X) \longrightarrow T(i_F(X)) \oplus \nu(\bar{E}) \mid i_F(X)$ be the (canonical) vector bundle isomorphism defined in the previous section. Fixing this isomorphism, then, the tangent map of i_F gives $T(i_F)$: $TX^n \longrightarrow T(i_F(X^n)) \oplus \nu(\bar{E}) \mid i_F(X)$ and a second bundle map $\pi_2 \circ T(i_F)$: $TX^n \longrightarrow \nu(\bar{E}) \mid i_F(X)$, where π_2: $T(i_F(X)) \oplus \nu(\bar{E}) \mid i_F(X) \longrightarrow \nu(\bar{E}) \mid i_F(X)$ is given by the linear projection of the second factor. Let ϕ: $\nu(\bar{E}) \mid i_F(X) \longrightarrow \nu(F)$ be the (canonical) vector bundle isomorphism defined by the composition $\nu(\bar{E}) \mid i_F(X) \overset{\phi_1}{\rightarrow} i^*_F(\nu(\bar{E})) \overset{\phi_2}{\rightarrow} \nu(i_F^{-1}(\bar{E})) = \nu(F)$. (The map ϕ_2 is given in 1.3)-2.)

Putting these various bundle maps together yields a well-defined bundle map: $TX \longrightarrow \nu(F)$ which, by suppressing both ϕ_2 and π, shall again be called $T(i_F)$. Finally, define a section $\sigma_F \, \varepsilon \, \Gamma^\infty(\mathrm{Hom}(TX, \nu(F)))$, the *1-jet of F*, by $\sigma_F(x) = T(i_F)(x)\colon TX^n_x \longrightarrow \nu(F)_x$, $x \, \varepsilon \, X^n$.

Definition 3.2.1. [cf. 13, p. 372] Let $S_i(TX, \nu(F))$ be the sub-bundle of $\mathrm{Hom}(TX, \nu(F)) \longrightarrow X^n$ defined in (3.1). The section σ_F is said to be a *T-section* if it has transversal intersection with each $S_i(TX, \nu(F))$ for $0 \leq i \leq \min(n, q)$. If σ_F is a *T*-section, F is said to be a *T-Γ^∞_q-structure* on X^n.

3.3. Let X^n be, as usual a C^∞ compact, connected n-manifold and F a Γ^∞_q-structure on X^n.

Definition 3.3.1. 1) Let $\sigma^*_F\colon H^*_c(\mathrm{Hom}(TX, \nu F); Z_2 \rightarrow H^*(X^n; Z_2)$ be the map induced on cohomology by σ_F. Define $b_i(F) \, \varepsilon \, H^{i(n-q+i)}(X^n; Z_2)$, $0 \leq i \leq q$ by $b_i(F) = \sigma^*_F(P.D.[\overline{S_i(TX, \nu(F))}])$.

2) Make the further assumption that X^n is oriented. Let TX^C, $\nu(F)^C$ be the complexification of the bundles TX, $\nu(F)$ respectively. Let $\mathrm{Hom}_C(TX^C, \nu F^C)$ and $S^C_i(TX^C, \nu F^C)$ be as in 3.1. Define $b^C_i(F) \, \varepsilon \, H^{2i(n-q+i)}(X^n; Z)$ by $b^C_i(F) = \bar{\sigma}^*_F(P.D.[\overline{S^C_i(TX^C, \nu F^C)}])$. (Here one defines $\bar{\sigma}_F \, \varepsilon \, \Gamma(\mathrm{Hom}_C(TX^C, \nu F^C))$ by $\bar{\sigma}_F(x)(v, w) = (\sigma_F(x)(v), \sigma_F(x)(w))$. It is easy to check that with the definition of bundle complexification given in [15, p. 78], $\bar{\sigma}_F$ is complex-linear.)

3) Assume that X^n is oriented and F is transversally oriented. Then for $n-q=2r$, $j=2k$, define $r_j(F) \, \varepsilon \, H^{j(n-q+j)}(X^n; Z)$ to be $\sigma^*_F(P.D.[\overline{S_{2k}(TX, \nu F)}])$.

Set $B(F)$ (respectively $B^C(F)$, $R(F)$) equal to $b_0(F) \oplus \ldots \oplus b_q(F) \, \varepsilon \, H^*(X; Z_2)$ (respectively $b^C_0(F) \oplus \ldots \oplus b^C_q(F) \, \varepsilon \, H^*(X^n; Z)$; $r_0(F) \oplus \ldots \oplus r_q(F) \, \varepsilon \, H^*(X^n; Z)$). Thus, in each case there is defined a total cohomology class which measures the "first-order complexity" of $S(F)$, *the singular set of F*, defined to be those points in the manifold X^n over which σ_F has rank $< q$. (Note that for F given by a single global function $f\colon X^n \rightarrow R^q$, this singular set reduces to the usual set of singular (or critical points of f.)

Theorem 3.3.2. 1) *Each of the classes $B(F)$, $R(F)$, $B^C(F)$ is an invariant of the homotopy class of F, i.e., $F \simeq F' = > B(F) = B(F')$, $B^C(F) = B^C(F')$, and $R(F) = R(F')$.*

2) *If F is homotopic to a foliation E, $B(F) = 1 \, \varepsilon \, H^0(X; Z_2) \cong Z_2$; $R(F) = B^C(F) = 1 \, \varepsilon \, H^0(X^n; Z) \cong Z$.*

Proof. We will prove the Theorem for the class $B(F)$ (the proof for $R(F)$ is identical) and indicate what further statements are needed to extend the proof to the case of the complexified class $B^C(F)$.

1) What needs to be shown here is that there exists a vector bundle isomorphism $\lambda\colon \mathrm{Hom}(TX, \nu(F)) \longrightarrow \mathrm{Hom}(TX, \nu(F'))$ taking $S_i(TX, \nu(F))$ to $S_i(TX, \nu(F'))$ and such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(TX, \nu(F)) & \xrightarrow{\ \lambda\ } & \mathrm{Hom}(TX, \nu(F')) \\ {\scriptstyle \sigma_F}\nwarrow & & \nearrow{\scriptstyle \sigma_{F'}} \\ & X^n & \end{array}$$

To that end, let $\psi\colon m(F) \longrightarrow m(F')$ be the microbundle isomorphism given by Theorem 2.1-3. That is, if U and U' denote open neighborhoods of $i_F(X)$, $i_{F'}(X)$ in the total spaces of $m(F)$ and $m(F')$, respectively, there is a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \psi\ } & U' \\ {\scriptstyle i_F}\nwarrow & & \nearrow{\scriptstyle i_{F'}} \\ & X^n & \end{array}$$

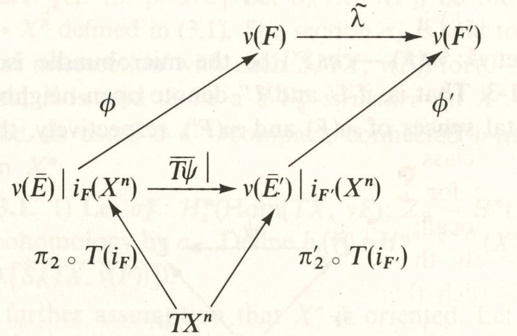
where ψ is a diffeomorphism. Lifting this commutative diagram to the level of tangent spaces, naturality yields a second commutative diagram:

$$\begin{array}{ccc} TM \mid i_F(X^n) & \xrightarrow{\ T\psi\ } & TM' \mid i_{F'}(X^n) \\ {\scriptstyle T(i_F)}\nwarrow & & \nearrow{\scriptstyle T(i_{F'})} \\ & TX^n & \end{array}$$

Let $A(A')\colon TM \mid i_F(X^n) \rightarrow T(i_F(X^n)) \oplus \nu(\bar{E}) \mid i_F(X)$ $(\colon TM' \mid i_{F'}(X^n) \longrightarrow T(i_{F'}(X^n)) \oplus \nu(\bar{E}') \mid i_{F'}(X))$ be the isomorphisms of theorem (2.1)-2. Letting $\overline{T(\psi)} = A' \circ T(\psi) \circ A^{-1}$ yields

$$\begin{array}{ccc} T(i_F(X^n)) \oplus \nu(\bar{E}) \mid i_F(X) & \xrightarrow{\ \overline{T\psi}\ } & T(i_{F'}(X)) \oplus \nu(\bar{E}') \mid i_{F'}(X) \\ {\scriptstyle A}\uparrow & & \uparrow{\scriptstyle A'} \\ TM \mid i_F(X^n) & \xrightarrow{\ T\psi\ } & TM' \mid i_{F'}(X^n) \\ {\scriptstyle T(i_F)}\nwarrow & & \nearrow{\scriptstyle T(i_{F'})} \\ & TX^n & \end{array}$$

with $\overline{T\psi}$ of maximal rank. Now, since $T\psi$ preserves the tangent spaces of the zero sections, it is clear that $\overline{T\psi}\,|\,(\nu(\bar{E})\,|\,i_F(X))$ will be well-defined and of maximal rank as a map into $\nu(\bar{E}')|\,i_{F'}(X^n)$ and that there exists a maximal rank bundle map $\tilde{\lambda}$: $\nu(F) \longrightarrow \nu(F')$ so that



commutes. Recalling the definition of σ_F and $\sigma_{F'}$ and noting that a bundle isomorphism: Hom$(TX, \nu(F)) \to$ Hom$(TX, \nu(F'))$ must necessarily preserve the singularity decompositions, it will follow that the isomorphism λ defined by $\lambda_x(\beta_x)(v) = \lambda_x(\beta_x(v))$, for $\beta_x \,\varepsilon\, (\text{Hom}(TX, \nu(F)))_x$, $v \,\varepsilon\, TX^n_x$ establishes part (1).

The proof of (2) requires a lemma.

Lemma 1. *Let X^n be a compact, connected C^∞ manifold and F a $T-\Gamma^\infty_q$-structure on X^n. Set $S_i(F) = \sigma_F{}^{-1}(S_i(TX, \nu(F)))$. Then*

1) *$S_i(F)$ is a regular submanifold of X^n with codimension equal to $i(n-q+i)$.*

2) $\overline{S_i(F)} = \bigcup_{j=0}^{q-i} S_{i+j}(F) \overset{def.}{=} \sigma_F^{-1}(\overline{S_i(TX, (F))})$

3) *$\overline{S_i(F)}$ possesses a fundamental homology class (over Z_2). Further, if $[\overline{S_i(F)}]$ denotes the image of this class in $H_*(X^n; Z_2)$, then $b_i(F) \overset{def.}{=} \sigma_F^*(P.D.[\overline{S_i(TX, \nu F)}]) = P.D.[\overline{S_i(F)}] \,\varepsilon\, H^{i(n-q+i)}(X^n; Z_2)$.*

Proof. 1) This is a standard consequence of the transversality of σ_F. 2) This is a special case of [13, p. 373].

By a lemma of Haefliger-Kosinski cf. [7, p. 8-02] and (2), it suffices to show that $\overline{S_i(F)}$ is an ANR (absolute neighborhood retract). Moreover, by a theorem of S. T. Hu [cf. 8, Thm. (7.1), p. 168], this reduces to showing that $\overline{S_i(F)}$ is locally contractible, which one accomplishes as follows: by (1), the manifold topology of $S_i(F)$ agrees with the relative topology induced from the inclusion of $S_i(F)$ into X^n. Thus if $x \,\varepsilon\, \overline{S_i(F)}$ and V is any open set in $\overline{S_i(F)}$ contaning x, $V = V' \cap \overline{S_i(F)}$ where V' is open in X^n. Since X^n is a manifold it is locally contractible and so there is an open set U', $x \,\varepsilon\, U' \subset V'$ with U' contractible to x in V', i.e., there is a continuous map G': $U' \times I \longrightarrow V'$ with $G'_0 =$ inclusion : $U' \longrightarrow V'$, and $G'_1(U') = x$. Setting $U = U' \cap \overline{S_i(F)}$, and G: $U \times I \longrightarrow V$ the restriction of G' to $U \times I$, 3) follows.

Let E be a codimension-q foliation of X^n.

Assertion: Image $(\sigma_E) \cap S_i(TX, \nu E) = \phi$ for all $i > 0$. As $S_0(TX, \nu E)$ is open in Hom$(TX, \nu E)$ it follows that σ_E is a T-section. Indeed, let $x \,\varepsilon\, X^n$ and let $T(i_E) : TX \longrightarrow \nu E$ be the bundle map defined in (3.2). It suffices to show that rank $T(i_E)(x) : TX_x \longrightarrow \nu E_x$, equals q.

To that end, note that by [6, p. 188], $i_E(x) = (x, w) \,\varepsilon\, X^n \times R^q$ where w is an equivalence class of vectors in R^q, a typical representative of which is of the form $f_i(x)$, for f_i: $U_i \longrightarrow R^q$ a local projection of E containing x in its domain. Pick a locally trivializing subset U of U_i for M, which shall remain fixed. More precisely, there is an open subset V of M and a diffeomorphism h: $V \longrightarrow U \times R^q$ such that $h(i_E(x)) = (x, 0)$. In terms of this product representation, if $(x_1, \ldots, x_n)$ is a coordinate system for U, one sees that $T(i_E(x))$ is given by an equivalence class of matrices, whose representative corresponding to the choice of f_i as a representative (near x) of i_E, is given by the $n \times q$ matrix of first-order partials of f_i with respect to $(x_1, \ldots, x_n)$. As E is a foliation, this matrix does indeed have rank q. Further, if f_j: $U_j \longrightarrow R^q$ is any other local projection containing x in its domain, the matrix of partial derivatives $(\partial f^l_j/\partial x_k)(x)$ will also have rank q. Thus the rank of $T(i_E)(x)$ is independent of the choice of local representative for i_E, and is equal to q, as asserted. Hence, by lemma 1-3, $b_0(E) = \sigma_E^*(P.D.[\overline{S_0(TX, \nu E)}]) = P.D.[\overline{S_0(E)}] = P.D.[X^n]$ $= 1 \,\varepsilon\, H^0(X^n; Z_2)$. Further, for $i > 0$, $b_i(E) = \sigma_E^*(P.D.[\overline{S_i(TX, \nu E)}]) = P.D.[\overline{S_i(E)}]$ $= P.D.[\phi] = 0$, and so the theorem is proved for $B(F)$.

To complete the proof for the complexified classe $B^C(F)$, it only needs to be verified that for F a Γ^∞_q-structure on X^n, real rank $(\sigma_F(x)) =$ complex rank $(\overline{\sigma_F}(x))$. It will then follow, for example, that the sets $\sigma_F^{-1}(S_i(TX, \nu F))$ and $\overline{\sigma}_F^{-1}(S^C_i(TX^C, \nu F^C))$ will be identical and the results of Lemma (1) will be applicable. This lemma, coupled with the assertion analogous to the preceding one will extend the proof to this case. This desired fact follows from the

Lemma 2. *Let ϕ: $R^n \longrightarrow R^q$ be a linear transformation. Represent C^n, C^q by $C^n = \{(v_1, w_1)\,|\,v_1, w_1 \,\varepsilon\, R^n\}$ and $C^q = \{(v_2, w_2)\,|\,v_2, w_2 \,\varepsilon\, R^q\}$ with complex multiplication given by $i \cdot (v_i, w_i) = (-w_i, v_i)$, $i = 1, 2$. Let $\tilde{\phi}$: $C^n \longrightarrow C^q$ be given by $\tilde{\phi}(v, w) = (\phi(v), \phi(w))$. Then real rank $(\phi) =$ complex rank $(\tilde{\phi})$.*

Proof. This is trivial. In fact, if $(e_1, \ldots, e_n)$ is a basis for R^n, $(e'_1, \ldots, e'_q)$ a basis for R^q, and $((e_1, e_1), \ldots, (e_n, e_n))$, $((e'_1, e'_1), \ldots, (e'_q, e'_q))$ the corresponding bases for C^n and C^q, the matrices expressing ϕ and $\tilde{\phi}$, respectively, in terms of these two bases are identical.

3.4. Examples. In order to calculate some examples of the classes $B(F)$, $B^C(F)$ and $R(F)$, and for use in the next section, we introduce the following theorem, due collectively to R. Thom, I. Porteous and F. Ronga. [cf. 18, Prop. 1.3, p. 298; 20, p. 314]

Theorem (TPR). 1) *Let X^n be a compact, connected C^∞ manifold and $\xi^m \longrightarrow X^n$, $\tau^q \longrightarrow X^n$ real vector bundles over X^n of rank m and q, respectively. Let $Hom(\xi, \tau) \xrightarrow{\pi} X^n$ be the bundle of linear maps between ξ and τ. Then, with notation as in (3.1), the cohomology class $\pi^{*-1}(P.D.[\overline{S_i(\xi, \tau)}])$ is given as a "universal" polynomial q_i, the so-called i-th Thom polynomial, in the Stiefel-Whitney classes of the fiber difference, $W_k(\xi - \tau)$. (Here universal means that the polynomial depends only on m, q, and i, and in no way on the particular bundles). This polynomial is equal to the determinant of the $i \times i$ matrix $A = (a_{s,t})$ with $a_{s,t} = W_{m-q+i-s+t}$.*

2) *Suppose that X^n is orientable and $n^m \longrightarrow X$, $\omega^q \longrightarrow X$ are complex vector bundles over X^n. Then $\pi^{*-1}(P.D.[\overline{S_i^C(\eta, \omega)}])$ is given by $q_i\ (c_k(\eta - \omega))$, the $c_k(\eta - \omega)$ being the Chern classes of the fibre difference. The polynomial q_i is equal to the determinant of the $i \times i$ matrix $B = (b_{s,t}) = c_{m-q+i-s+t}$.*

3) *Assume, in addition to the hypotheses of (1), that the bundle $Hom(\xi, \tau) \longrightarrow X^n$ is orientable, and that $j = 2k$, $m - q = 2r$. Then the class $\pi^{*-1}(P.D.[\overline{S_{2k}(\xi, \tau)}])$ in integral cohomology is determined by its reduction mod 2 and its rational reduction. The reduction mod 2 is the determinant of the matrix $(W_{j+2r-s+t}(\xi - \tau))$, $s, t = 1, \ldots, j$. The rational reduction is given as a universal polynomial $\underline{q}_j$ in the rational Pontryagin classes of the fibre difference $p_l'(\xi - \tau)$. This polynomial is equal to the determinant of the $k \times k$ matrix $C = (c_{s,t}) = p'_{k+r-s+t}$.*

Remark. Let $\sigma \varepsilon \Gamma^\infty(\text{Hom}(\xi, \tau))$ be an arbitrary section. Then since $\pi \circ \sigma$ is the identity map of X^n and $\sigma \circ \pi$ is homotopic to the identity on $\text{Hom}(\xi, \tau)$, $\pi^{*-1} \equiv \sigma^* : H^*(\text{Hom}(\xi, \tau)) \longrightarrow H^*(X^n)$. In particular, $\sigma_F^* : H^*(\text{Hom}(TX, \nu(F))) \longrightarrow H^*(X^n)$ is an inverse for π^*.

To compute $W_k(TX - \nu F)$, recall the Whitney Product Theorem [cf. 15, p. 6]. Namely if $\xi \oplus \tau$ is the Whitney sum of the bundles ξ and τ, then $W(\xi \oplus \tau) = W(\xi) \cup W(\tau)$, i.e. $W_k(\xi \oplus \tau) = \sum_{i+j=k} W_i(\xi) \cup W_j(\tau)$. Then $W(\xi - \tau)$ is defined to be $W(\xi \oplus \bar{\tau})$ for $\bar{\tau}$ any bundle over X^n inverse to τ. From the relation $W(\tau) \cup W(\bar{\tau}) = W(\varepsilon^N) = 1$, and the fact that $W_0(\tau) = W_0(\bar{\tau}) = 1$, it is easy to compute the classes $W_j(\bar{\tau})$.

We now exhibit specific examples. As manifolds we will use the various projective spaces $P^n(R)$, $P^m(C)$, and $P^m(K)$ (real, complex and quaternionic projective spaces, respectively) while the Γ_q^∞-structure F_q will be that given by a single global C^∞ function $f : X^n \longrightarrow R^q$, with $\nu(F) = f^*(TR^q) \cong \varepsilon^q$. (Note that any two such Γ_q^∞-structures are homotopic.)

Example 3.4.1. Let $X = P^n(R)$, and $q = 1$. Then $b_1(F_1) = W_{n-1+1}(TX - \nu F) = W_n(TX) \bigcup W_0(\overline{\nu F}) + W_{n-1}(TX) \cup W_1(\overline{\nu F}) = W_n(TX)$. Now, as is well known, $W_i(P^n(R)) = \binom{n+1}{i}_2 \alpha^i$, where $\binom{n+1}{i}_2$ is the mod (2) binomial coefficient and $\alpha \varepsilon H^1(P^n(R); Z_2)$ is the canonical generator. So, $b_1(F_1) = W_n(TX) = (n+1)_2 \alpha^n$, which is non-zero if n is even. All higher classes $b_i(F_1)$ are zero for dimensional reasons.

Example 3.4.2. Here we produce an example for which $b_i(F)$, $i > 1$ do not all vanish. Let $X^8 = P^2(K)$ and $q = 6$. Then $b_2(F_6) \varepsilon H^8(P^2(K); Z_2) \cong Z_2$ is given by the determinant of the matrix

$$\begin{pmatrix} W_4(TX - \nu F) & W_5(TX - \nu F) \\ W_3(TX - \nu F) & W_4(TX - \nu F) \end{pmatrix} = W_4^2(TX - \nu F) - W_5(TX - \nu F) \cup W_3(TX - \nu F)$$
$$= W_4^2(P^2(K) - W_5 W_3(P^2(K)).$$

In this case, if $\alpha \varepsilon H^4(P^m(K); Z_2)$ denotes the generator, $W_{4r}(P^m(K)) = \binom{m+1}{r}_2 \alpha^r$ and all the other Stiefel-Whitney classes are zero. Then $b_2(F) = W_4^2(P^2(K)) = (\binom{3}{1}_2 \alpha)^2$ and $B(F) = 1 + b_2(F) = 1 + \alpha^2 \varepsilon H^*(P^2(K); Z_2)$.

Example 3.4.3. In this final example we turn our attention to the classes $b_i^C(F_q)$, $r_j(F_q)$ with $X^n = P^{n/2}(C)$. Specifically, consider $X^8 = P^4(C)$ and F of codimension 6. Then $r_2(F_6) \varepsilon H^8(P^4(C); Z) \cong Z$ is defined. By Theorem $(TPR) - (3)$ one has

$$r_2(F_6)(2) = \begin{vmatrix} W_4(X) & W_3(X) \\ W_5(X) & W_4(X) \end{vmatrix} = W_4^2(P^4(C)) - W_5 W_3(P^4(C)).$$

Thus $r_2(F_6)(2) = W_4^2(P^4(C))$ ($P^4(C)$ has no odd-dimensional cohomology) $= (\binom{5}{2}_2 \alpha_1)^2$, α_1 the generator of $H^2(P^4(C); Z_2) \cong Z_2$ and so is zero. Also, $r_2(F_6)(Q) = p_2'(P^4(C)) = \binom{5}{2}(\alpha_2)^4 = 30\alpha_2^4$, where α_2 generates $H^*(P^4(C);\ Q)$. It then follows that $r_2(F_6)$ in integral cohomology is given by $30\alpha_3^4$, where α_3 generates $H^*(P^4(C); Z\cdot$ and so $R(F_6)$ is non-trivial. On the other hand, $b_1^C(F_6) = c_{n-q+i}(TX^C - \nu F^C) = c_3(TX^C - \nu F^C) = 0$, since all odd Chern classes of complexified bundles vanish.

4. Embedding Vector Bundles in $\mathbf{Tx}^n$.

It is the purpose of Section 4 to show that theorem (TPR) together with the techniques of 3.3 may be used to yield a general theorem, extending known results, concerning the following question: Given a compact, connected C^∞

manifold X^n, when does X^n possess a q-plane field (with specified characteristic classes)?

We begin with the following definition:

Definition 4.1. Let $\nu^q \longrightarrow X^n$ be a real vector bundle over X^n of fibre dimension q. With notation and orientability assumptions as in theorem (TPR), set $b_i(\nu^q) = q_i(W_k(TX - \nu^q)$; $b_i^C(\nu^q) = q_i(c_k(TX^C - \nu^C)$; $r_j(\nu^q)(2) = q_j(W_k(TX - \nu^q)$; and $r_j(\nu^q)(Q) = \bar{q}_j(p_i'(TX - \nu^q)$, for $0 \leq i, j \leq q$. These classes are elements of $H^*(X^n; G)$, $G = Z_2, Z, Z_2, Q$, respectively.

We can now state the announced result:

Theorem 4.2. *Let $\nu^q \longrightarrow X^n$ be as above. Then*

1) *The classes $b_i(\nu^q)$, $b_i^C(\nu^q)$, $r_j(\nu^q)(2)$ and $r_j(\nu^q)(Q)$ are invariants of the isomorphism class of ν^q; and*
2) *Suppose that ν^q embeds in TX^n, i.e. is vector bundle isomorphic to a q-dimensional sub-bundle of TX^n. Then for $1 \leq i, j \leq q$, the above cohomology classes vanish.*

Proof. 1) is immediate from the definition and the Whitney Product Theorem since characteristic classes are isomorphism invariants. To establish (2), observe that if ν^q embeds in TX^n, there is induced a v.b. isomorphism $\psi\colon TX^n \longrightarrow \nu^q \oplus \xi^{n-q}$, where ξ^{n-q} is the ($n - q$-sub-bundle of TX^n orthogonal to (the isomorphic image of) ν^q in TX^n via a Riemannian metric on TX^n. Define a bundle map $\psi\colon TX^n \longrightarrow \nu^q$ by $\phi_x = (p_1)_x \circ \psi_x$ where $\psi_x = \psi \mid TX^n_x$ and $(p_1)_x$ is the linear projection: $\nu^q_x \oplus \xi_x \longrightarrow \nu_x$. It is immediate that ϕ is a bundle epimorphism, i.e. $\forall x \,\varepsilon\, X^n$, $\phi_x\colon TX^n_x \longrightarrow \nu_x$ has a maximal rank q. Let $\sigma_\phi \,\varepsilon\, \Gamma^\infty(\mathrm{Hom}(TX, \nu^q))$ be the section induced by ϕ, i.e. $\sigma_\phi(x) = \phi_x$.

By theorem (TPR) (and the remark following it), one has that $b_i(\nu^q) = \sigma_\phi^*(P.D.[\overline{S_i(TX, \nu^q)}])$. Since σ_ϕ meets only $S_0(TX, \nu^q)$ it is transverse to all the $S_i(TX, \nu^q)$ and one can then show, exactly as in the proof of theorem 3.3.2, that $b_i(\nu^q) = \sigma_\phi^*(P.D.[\overline{S_i(TX, \nu^q)}]) = P.D.[\sigma_\phi^{-1}(\overline{S_i(TX, \nu^q)})] = P.D.[\varnothing] = 0$. Similar statements hold for the other singularity type classes, and so the theorem is established.

Remark. It is possible to view theorem 3.3.2 as a corollary to this result, since if F deforms to a foliation E, $\nu(F) \cong \nu(E) \cong Q(E) \subset TX^n$. The material on Γ_q^∞-structures was presented separately, however, so as to emphasize the relation of the singularities of F to the associated cohomology classes and further to make possible Corollary 5.2.3. We also remark that for general bundles there is no distinguished section of $\mathrm{Hom}(TX^n, \nu^q)$ available and thus the singularity classes for ν^q were defined by means of theorem (TPR) in order to facilitate the proof of 4.2-1.

We now restrict our attention to the case $\nu^q \cong \varepsilon^q$ in order to derive a typical application of theorem 4.2. In this case the embedding $\nu^q \subset TX^n$ is equivalent to the existence of q linearly-independent vector fields on X^n (a q-frame field on X^n) and the above result may thus be used to give an upper bound for $\mathrm{Span}(X^n)$ [cf. 24, p. 649]. As an example, one can compare the neccessary condition on $b_2(\nu^q)$ to the "first obstruction" result of Stiefel-Whitney [cf. 23, p. 199] to show:

Corollary 4.3. *For each $q \geq 9$, there is a compact, orientable manifold X^n with vanishing Euler characteristic and signature such that X^n has a q-frame field over its 3-skeleton but no global q-frame field.*

Proof. For $q = 9$, let $X^n = X^{11} = P^{11}(R)$. Then the first obstruction is $\delta^* W_2(P^{11}(R)) \,\varepsilon\, H^3(P^{11}(R); Z)$ which is zero since the cohomology group is zero. Thus $P^{11}(R)$ has a nine-frame field over the 3-skeleton.

Consider $b_2(\varepsilon^9)$, $\varepsilon^9 \longrightarrow P^{11}(R)$. As in (3.4), $b_2(\varepsilon^9) = \det(W_{4+s-t}(TX - \varepsilon^9)) = W_4^2(P^{11}(R)) - W_5 W_3(P^{11}(R))$. Now $W_4^2(P^{11}(R)) = \alpha^8 \neq 0 \,\varepsilon\, H^8(P^{11}(R); Z_2)$, while $W_3(P^{11}(R)) = 0$ and thus $b_2(\varepsilon^9)$ is non-trivial. For $q \geq 9$, let $X^n = X^{q+2} = \bar{X}^{-q+2-11} \times P^{11}(R)$ where $X^{-q+2-11}$ is any compact orientable $q+2-11$ manifold with $W(\bar{X}) = 1$, e.g. S^{q+2-11} or T^{q+2-11}, and let $p_1\colon X \longrightarrow \bar{X}$, $p_2\colon X \longrightarrow P^{11}(R)$ be the projections. Then $TX^n \cong p_1^*\, T\bar{X} \oplus p_2^*\, TP^{11}(R)$ and by the Whitney Product Theorem, $W(X^n) = W(p_1^*\, T\bar{X}) \cup W(p_2^*\, TP^{11}(R)) = 1 \cup W(p_2^* TP^{11}(R))$. Further, by the functoriality of $S - W$ classes, $W(p_2^* TP^{11}(R)) = p_2^\# W(P^{11}(R))$, where $p_2^\#\colon H^*(P^{11}(R); Z_2) \longrightarrow H^*(X^n; Z_2)$ is the induced map on cohomology. By the Kunneth Theorem, if α' denotes $p_2^\# \alpha$, $(\alpha')^k \neq 0$ in $H^*(X^n; Z_2)$ for $1 \leq k \leq 11$ and so $b_2(\varepsilon^q) = 1 \cup (\alpha)^8 \neq 0 \,\varepsilon\, H^8(X^{q+2}; Z_2)$. The corollary now follows from the fact that the Bockstein is a ring map, i.e., $\delta^*(W_2(X^n)) = \delta^*(1 \cup W_2(P^{11}(R)) = 1 \cup \delta^* W_2(P^{11}(R)) = 0$.

5. T- Γ_q^∞-structures.

5.1. In 5.1 we complete the set of obstructions for Γ_q^∞-structures by considering in more detail the pair (X^n, F) where X^n is a connected, closed (compact, without boundary) orientable C^∞ manifold and F is a transversally orientable $T - \Gamma_1^\infty$-structure on X^n, in order to prove a proposition (prop. 5.1.3) relating the singularities of F to $\chi(X^n)$, the Euler characteristic of X^n. The corollary of interest here (Corollary 5.1.4) is that $\chi(X^n)$ fits quite naturally into the collection of classes defined in Section 3.3, i.e., $\chi(X^n) = \langle P.D.[\overline{S_1(F)}], [X^n] \rangle$, where $[X^n]$ is the fundamental homology class $\varepsilon\, H_n(X^n; Z)$ and $\langle , \rangle$ is the Kronecker Index. (Notice that while this fact is certainly to be anticipated, it is not a consequence of the general framework of Section 3. Thus, for the sake of completeness, a proof of the proposition is included.)

We begin with some neccessary generalities concerning codimension-one singularities.

Lemma 5.1.1. *Let (X^n, F) be a compact, connected manifold together with a $T-\Gamma_1^\infty$-structure. Then each local projection of F, $F_i\colon U_i \longrightarrow R$, is a Morse function.*

Proof. It suffices to show that each critical point p for f_i is nondegenerate in the sense that $\det\left(\dfrac{\partial^2 f_i}{\partial x_j\,\partial x_k}\right)(p) \neq 0$ where in terms of the coordinate system $(x_1, \ldots, x_n)$ for U_i, $\partial f_i/\partial x_j(p) = 0$, $j = 1, \ldots n$. To that end, recall that p a critical point for f_i means that $\sigma_F(p)\,\varepsilon\, S_1(TX, vF)$, i.e., $\sigma_F(p) = (p, 0)\,\varepsilon\,\mathrm{Hom}(TX, vF)_p$, (the fibre over p) since $S_1(TX, vF)$ is just the image of the zero-section σ_0. Let $\pi_2\colon T(\mathrm{Hom}(TX,\ vF))_{(p,0)} \longrightarrow T(\mathrm{Hom}(TX,\ vF))_{(p,0)}/T(S_1(TX,\ vF))_{(p,0)}$ denote the linear projection. Then, by the definition of transversality, (*) $\pi_2 \circ T(\sigma_F)(p)\colon TX_p^n \longrightarrow T(\mathrm{Hom}(TX,\ vF))_{(p,0)}/T(S_1(TX,\ vF))_{(p,0)}$ is onto. Let U be an open set containing p over which $\mathrm{Hom}(TX, vF)$ has a product representation, and let $(x_1, \ldots, x_n)$ be local coordinates for U. In terms of this representation, (*) says that the linear map whose matrix is

$$A_{kj} = \left(\frac{\partial^2 f_i}{\partial x_k\,\partial x_j}\right)(p)$$

is surjective on the tangent space to the fiber over p. As the dimension of the fiber is n, this matrix is non-singular, and the lemma is proved.

Lemma 5.1.2. *Let (X^n, F) be as above. Then*

1) *$S(F)$ is a finite collection of points $p_1, \ldots, p_s$; and*

2) *It may be assumed that there are no singular points of F in $U_i \cap U_j$, $j \neq i$. (It follows that there is a well-defined leaf through each $x\,\varepsilon\,X^n$, which is a (connected) smooth manifold away from $S(F)$).*

Proof. 1) follows immediately from the lemma of Morse [cf. 14, p. 59] and the compactness of X^n. To prove 2), suppose F is given by the cocycle $(\{U_i\}, f_i, \gamma_{ij}^\infty)_{i,\,j\varepsilon J}$. Since X^n is compact, J can be chosen to be finite and so $\bar{U} = \{U_i\} = U_1, \ldots, U_r$. Since $S(F)$ is finite, there exists an open set $V_1 \subset \bar{V}_1 \subset U_1$, where $\bar{V}_1$ is a compact manifold with boundary and $S(F) \cap U_1 \subset \bar{V}_1$. Consider U_2. Choose $V_2 \subset \bar{V}_2 \subset U_2$ with $\bar{V}_2$ a compact manifold with boundary such that $\bar{V}_2 \cap \bar{V}_1 = \varnothing$ and $(S(F) \cap U_2) - (S(F) \cap \bar{V}_1) \subset \bar{V}_2$. Choose $\bar{V}_3$ in U_3 such that $\bar{V}_3 \cap (\bar{V}_2 \cup \bar{V}_1) = \varnothing$, and $(S(F) \cap U_3) - ((S(F) \cap \bar{V}_1) \cup (S(F) \cap \bar{V}_2)) \subset \bar{V}_3$. Continuing in this manner a finite number of times, one defines a collection of compact sub-manifolds with boundary $\bar{V}_1, \ldots, \bar{V}_r$. Set $U'_j = U_j \bigcup\limits_{j \neq i} \bar{V}_i$. Claim: $\bar{U}' = \{U'_j\}_{j=1,\cdots r}$ is an open cover for X^n. Indeed, as each U'_j is obviously open, it suffices to show that the union of the sets U'_j cover X^n. To that end, let $x\,\varepsilon\,X^n$ and U_i any element of $\bar{U}$ containing x. If $x \notin \bar{V}_j$ for all $j \neq i$, we are done as then $x\,\varepsilon\,U'_i$. On the other hand, if $x\,\varepsilon\,\bar{V}_j$ for some $j\,\varepsilon\,J$, then $x \notin \bar{V}_l$ for all $l \neq j$ and so $x\,\varepsilon\,U'_j$. One now defines a new cocycle representing F by restricting the original cocycle to the cover $\bar{U}' \subset \bar{U}$. It follows easily from the construction that this new cocycle (and hence F) satisfies (2).

For the remainder of 5.1, let X^n be connected, closed and oriented and F a transversally oriented $T-\Gamma_1^\infty$-structure on X^n. Set $I(F) = $ *index of* $F = \sum\limits_{i=1}^{r} (\sum\limits_{\lambda} (-1)^\lambda C_\lambda(f_i))$ where $C_\lambda(f_i)$ is the number of singular points of f_i of index λ. (By the above lemma, each $p\,\varepsilon\,S(F)$ is counted exactly once in this sum). To state the announced proposition one also defines *the intersection number of σ_F and σ_0, $\#(\sigma_F,\ \sigma_0)$,* as follows: $\#(\sigma_F,\ \sigma_0) = \sum\limits_{p\varepsilon S(F)} \#(\sigma_F,\ \sigma_0)(p)$ where $\#(\sigma_F,\ \sigma_0)(p)$ is $+1$ or -1 according to whether the orientation class of the frame for $T(\mathrm{Hom}(TX,\ vF))_{(p,0)}$ given by $\xi^* = (T(\sigma_0)(p)(\xi_1), \ldots, T(\sigma_0)(p)(\xi_n), T(\sigma_F)(p)(\xi_1), \ldots, T(\sigma_F)(p)(\xi_n))$, $(\xi_1, \ldots \xi_n)$ a positively oriented frame for TX_p^n, agrees with a (fixed) pre-assigned orientation for $T(\mathrm{Hom}(TX, vF))_{(p,0)}$. (Here one assumes that the orientation for $\mathrm{Hom}(TX, vF)$ is that induced from orientations of TX^n and $v(F)$.) Finally, let $\chi(\mathrm{Hom}(TX, vF))\,\varepsilon\,H^n(X^n; Z)$ denote the Euler class of the bundle [cf. 15, p. 34]. Then there is the

Proposition 5.1.3. $I(F) = \#(\sigma_F,\ \sigma_0) = \langle \chi(\mathrm{Hom}(TX, vF)), [X^n] \rangle = \chi(X^n)$.
(1) (2) (3) (4)

Remarks. 1. This proposition is a generalization of Morse's result on the alternating sum of the critical indices of a single non-degenerate mapping $f^P\ X^n \longrightarrow R$.

2. The first three numbers are defined in "decreasing genericity." $I(F)$ is defined only for F a $T-\Gamma_1^\infty$-structure, $\#(\sigma_F,\ \sigma_0)$ is defined whenever the singular points for the local projections are isolated (although perhaps degenerate) and $\langle \chi(\mathrm{Hom}(TX,\ vF)), [X] \rangle$ is defined for any F.

Proof. (1) = (2) Let $p\,\varepsilon\,X^n$ be a singular point for F, i.e., a zero of σ_F, and let f_i be the unique (Lemma 5.1.2)) local projection of F containing x in its domain. It suffices to show that $\#(\sigma_F,\ \sigma_0)(p) = (-1)^\lambda$, where λ is the index of p as a singular point for f_i.

Let $(y_1, \ldots, y_n;\ U)$, $U \subset U_i$ be the coordinate system given by the lemma of Morse, i.e., $f_i \equiv f_i(p) - \sum\limits_{j=1}^{\lambda} y_j^2 + \sum\limits_{m=\lambda+1}^{n} y_m^2$ throughout U. Without loss of generality, it may be assumed that $\partial/\partial y_1|_p, \ldots, \partial/\partial y_n|_p$ gives a positively oriented frame for TX^n_p. Let $\pi_X\colon TX \longrightarrow X$ be the bundle projection. Then a positively oriented frame for $T(TX)_{(p,0)}$ is given by $\tilde{\zeta} = (\partial/\partial y_1|_{(p,0)}, \ldots, \partial/\partial y_n|_{(p,0)}, e_1(0), \ldots, e_n(0))$. Here $e_1, \ldots, e_n$ is the basis corresponding to the standard basis of $\rho_2^*(TR^n)$ under the isomorphism $T(\pi_X^{-1}(U)) \cong \rho_1^*(TU) \oplus \rho_2^*(TR^n)$, ρ_1, ρ_2 the projections: $U \times R^n \longrightarrow U$, $U \times R^n \longrightarrow R^n$, resp. Now, since $v(F)$ is an oriented line bundle it is trivial, and hence under the bundle isomorphism $A\colon \mathrm{Hom}(TX, vF) \longrightarrow v(F) \oplus T^*(X) \longrightarrow v(F) \oplus TX \cong TX$, the frame ζ for $\mathrm{Hom}(TX, v(F))$ at $(p, 0)$ corresponding to $\tilde{\zeta}$ under A is given by $\zeta = (a \cdot \partial/\partial y_1|_{(p,0)}, \ldots, a \cdot \partial/\partial y_n|_{(p,0)}, a \cdot e_1(0), \ldots, a \cdot e_n(0))$ for $a > 0 \,\varepsilon\, R$. To identify the frame ξ^*, consider $T(\sigma_0)(p)(TX_p)$. Since this space has no component in the fibre direction, one has

(*) $$T\sigma_0(p)(\partial/\partial y_j|_p) = \partial/\partial y_j|_{(p,0)},\ j = 1, \ldots, n.$$

(Here σ_0 is regarded, via A, as $\varepsilon\Gamma^\infty(TX^n)$).

Further, we are given that σ_F and σ_0 intersect transversally, and so the image of the collection of vectors $\{\partial/\partial y_j|_p\}$, $j = 1, \ldots, n$ under $T(\sigma_F)(p)$ fills up the subspace of the tangent space of $\mathrm{Hom}(TX, vF)$ at $(p, 0)$ spanned by $\{a \cdot e_j(0)\}$, $j = 1, \ldots, n$. Thus it suffices to compute the coefficients a_{kj} in the expression $T(\sigma_F)(p)(\partial/\partial y_k|_p) = \sum_{j=1}^{n} a_{kj} \cdot e_j(0)$. But this is exactly the information given by the lemma of Morse. Indeed, in terms of the local product structure for TX^n over U, since p is contained in $U_i - \bigcup_{j=i} U_j$, σ_F can be thought of as a map taking $U \longrightarrow R^n$, $\sigma_F \equiv (\partial f_i/\partial y_1, \ldots, \partial f_i/\partial y_n)$ and thus one has:

(**) $$T(\sigma_F)(p)\left(\frac{\partial}{\partial y_k}\bigg|_p\right) = \sum_{j=1}^{n} \frac{\partial^2 f_i}{\partial y_k\, \partial y_j}(p) \cdot e_j(0)$$

which in turn implies that

$$a_{kj} = \begin{cases} 0 & k \neq j \\ -2 & k = j \quad 1 \leq j \leq \lambda \\ 2 & k = j \quad \lambda + 1 \leq j \leq n \end{cases}$$

Combining (*) and (**) one sees that with respect to the basis ζ, the matrix of ξ^* is diagonal with $\det(\text{matrix } (\xi^*)) = (-2)^\lambda \cdot 2^{n-\lambda}$ which is positive or negative as λ is even or odd.

(2) = (3). Since X^n is compact, there is an $r > 0$ such that $\mathrm{Im}(\sigma_F) \subset (D(\mathrm{Hom}(TX, vF)))$, the disc bundle of radius r associated to $\mathrm{Hom}(TX, vF)$ by a choice of fibre metric. Let $S(\mathrm{Hom}(TX, vF))$ be the associated sphere bundle of radius r. To vaoid using cohomology with compact supports, we consider the Thom class U as an element of the group $H^n(D(\mathrm{Hom}(TX, vF)), S(\mathrm{Hom}(TX, vF)); Z)$; it is the unique class which when restricted to the fibre corresponds to the given orientation generator $U_b \,\varepsilon\, H^n(D(\mathrm{Hom}(TX, vF))_b, S(\mathrm{Hom}(TX, vF))_b; Z) \cong H^n(D^n; S^{n-1}; Z) \cong Z$. Let $[\sigma_0] \,\varepsilon\, H_n(D(\mathrm{Hom}(TX, vF)); Z) = (\sigma_0)_*[X^n]$, $[\sigma_F] = (\sigma_F)_*[X^n]$, and let $\zeta' \,\varepsilon\, H_{2n}(D(\mathrm{Hom}(TX, vF)), S(\mathrm{Hom}(TX, vF)); Z)$ be the fundamental homology class.

Assertion. Under the (inverse to the) Lefschetz duality isomorphism for manifolds with boundary,

$$H^n(D(\mathrm{Hom}(TX, vF)), S(\mathrm{Hom}(TX, vF)); Z) \xrightarrow{\cap \zeta'} H_n(D(\mathrm{Hom}(TX, vF)); Z),$$

$U| \longrightarrow [\sigma_0]$, U maps into $[\sigma_0]$, where the isomorphism is given by capping with the fundamental class. Indeed, it follows directly from a theorem of Leray-Hirsch [cf. 22, Thm. # 9, p. 258] that if μ_b denotes the homology generator dual to U_b, then $[X^n] \oplus \mu_b = \pi_*(U \cap \zeta') \oplus \mu_b \,\varepsilon\, Z \oplus {}_Z Z$. This in turn implies $[X^n] = \pi_*(U \cap \zeta') => \sigma_{0*}[X^n] = \sigma_{0*}\pi_*(U \cap \zeta') = U \cap \zeta'$, as claimed. To establish the desired equality, one now observes that according to [4, (13.5), p. 337, see also (13.26), p. 343], $\#(\sigma_F, \sigma_0) = \langle L.D.[\sigma_F] \cup L.D.[\sigma_0], \zeta' \rangle$. Since σ_F is homotopic to σ_0, $[\sigma_F] = (\sigma_F)_*[X^n] = (\sigma_0)_*[X^n] \equiv [\sigma_0]$. Thus int $\#(\sigma_F, \sigma_0) = \langle L.D.[\sigma_0] \cup L.D.[\sigma_0], \zeta' \rangle = \langle U \cup U, \zeta' \rangle$ by the Assertion.

A consideration of the following diagram

$$\begin{array}{ccc} H^n(X^n; Z) & \xrightarrow{\ \phi\ } & H^{2n}(D(\mathrm{Hom}(TX, vF)),\ S(\mathrm{Hom}(TX, vF)); Z) \\ \uparrow \psi_1 & & \uparrow \psi_2 \\ H_n(X^n; Z) & \xrightarrow{\ \tilde{\phi}\ } & H_{2n}(D(\mathrm{Hom}(TX, vF)),\ S(\mathrm{Hom}(TX, vF)); Z) \end{array}$$

where ϕ is the Thom isomorphism, the vertical isomorphisms correspond to the choice of a dual generator, and $\tilde{\phi} = \psi_2^{-1}\phi\psi_1$ then shows that int $\#(\sigma_F, \sigma_0) = \langle U \cup U, \zeta' \rangle = \langle \phi^{-1}(U \cup U), \tilde{\phi}^{-1}\zeta' \rangle = \langle \chi(\mathrm{Hom}(TX, vF)), [X] \rangle$

To prove that (3) = (4) (and thus conclude the proof of the Proposition) is immediate. As previously remarked, $\mathrm{Hom}(TX, vF) \cong TX^n$ and so $\langle \chi(\mathrm{Hom}(TX, vF)), [X^n] \rangle = \langle \chi(TX), [X^n] \rangle = \chi(X^n)$ by [15, Thm. 17, p. 52].

Corollary 5.1.4. *Let (X^n, F) be as above. Then $S_1(TX, vF)$ has a fundamental homology class over Z, and if P.D.$[S_1(F)]$ denotes the Poincaré dual of the corresponding fundamental class for $S_1(F) = \sigma_F^{-1}(S_1(TX, vF))$, then $\chi(X^n) = I(F) = \langle P.D.[S_1(F)], [X^n] \rangle$*

Proof. Define $[S_1(TX, vF)]$ to be $(\sigma_0)_*[X^n] = [\sigma_0]$. Since $\overline{S_1(TX, vF)} = S_1(TX, vF)$ (cf. (3.1)), the conclusion that this construction yields a fundamental homology class in the sense of [7] follows directly from the usual construction of $[X^n]$ [Cf. 22, p. 301] and the fact that σ_0 is a diffeomorphism onto its image. Consider $\chi(\mathrm{Hom}(TX, \nu F))$. According to [15, Thm. 1.2, p. 41] this class is equal to $g^* i^* U$ where $g \in \Gamma^\infty(\mathrm{Hom}(TX, \nu F))$ and $i^* : H^*(D(\mathrm{Hom}(TX, \nu F)); Z) \to H^*(D(\mathrm{Hom}(TX, \nu F)), S(\mathrm{Hom}(TX, \nu F)); Z)$ is induced by inclusion. In particular, then, $\chi(\mathrm{Hom}(TX, \nu F)) = \sigma_F^*(i^* L.D.[S_1(TX, \nu F)])$ by the assertion in the previous proof, which is in turn equal $\sigma_F^*(P.D.[S_1(TX, \nu F)])$ [cf. 22, p. 297-298]. The result now follows from Proposition 5.1.3 and Lemma 1 of Section 3, asserting that since σ_F is a T-map, $P.D.[S_1(F)]$ exists and is equal to $\sigma_F^*(P.D. [S_1(TX, \nu F)])$.

It is easy to show that the equality $I(F) = \chi(X^n)$ does not require that X^n be orientable, since for F as above, σ_F defines a vector field whose index sum is $I(F)$. On the other hand, if F is the $T-\Gamma_1^\infty$-structure on S^{2k+1} given by $f(x_1, \ldots, x_{2k+2}) = x_{2k+2}$ then F is invariant under the identification of antipodal points of S^{2k+1} and so define a $T-\Gamma_1^\infty$-structure F_0 on RP^{2k+1} with $S(F_0)$ a single point. It follows that the equality requires that F be transversally oriented. It should also be noted that since $\# S(F_0) \bmod(2) = \langle b_1(F_0), [RP^n]_2\rangle$, $b_1(F_0) \neq 0$ and so the homotopy class of F_0 contains neither a Morse function nor a foliation.

5.2. Let X^n be a connected, compact (with or without boundary) C^∞ n-manifold, and F a Γ_q^∞-structure on X^n, $1 \leq q \leq n$. It is the goal of this final section to prove a modified transversality theorem for such pairs (X^n, F) in order to show that the global homotopy invariants of F, $B(F)$ and $R(F)$ (when the latter is defined) can always be "realized" as the classes dual to the local singularities in X^n of some F' in the homotopy class of F. Recall that if $F \simeq$ foliation E, then E provides the desired realization (cf. the proof of thm. (3.3.2)-2).

To be precise, one makes the following

Definition 5.2.1. Let (X^n, F) be as above. Then the class $B(F)$ (resp. $R(F)$) can be *realized geometrically in* X^n if there is an F' in the homotopy class of F such that $B(F')$ (and so $B(F)$) is equal to $P.D.[\overline{S(F')}]$, i.e. $b_i(F') = P.D. [\overline{S_i(F')}]$ (resp. $r_j(F') = P.D.[\overline{S_{2k}(F')}]$, $0 \leq i, j \leq q$.

The result is then the following theorem:

Theorem 5.2.2. *Let X^n be a C^∞, compact, connected n-manifold and F a Γ_q^∞-structure on X^n. Then F is homotopic to a $T-\Gamma_q^\infty$-structure F'.*

Corollary 5.2.3. *Let (X^n, F) be as in theorem 5.2.2. Then the classes $B(F)$ and $R(F)$, when defined, can always be realized geometrically. In particular, for $1 \leq q \leq n$ the classes $q_i(W_k(X^n - \varepsilon^q))$ (resp. $q_{2k}(p'_m(X^n - \varepsilon^q))$ are always (resp.*

when X^n is orientable and $n - q$ is even) dual to the homology class defined by the singularities of Γ_q^∞-structure. [*Compare* 7, *p.* 8-02, *Cor.* 2].

Proof. The first statement is a consequence of Theorem 5.2.2 and Lemma 1 of 3.3. The second follows by taking $F = \{U_i, f_i, \gamma_{ij}^x\}_{i,j \in J}$ with $\{U_i\}$ any open cover of X^n, $f_i \equiv f \mid U_i$ for $f: X^n \longrightarrow R^q$, and $\gamma_{ij}^x =$ identity, $\forall\, i, j, x$; for which $\nu(F) \cong \varepsilon^q$.

Remark. The reason that the complexified class $B^C(F)$ does not appear in this result is that since $\mathrm{codim}(S_i^C(TX^C, \nu F^C)) = 2i(n - q + i)$ and not $i(n - q + i)$ the transversality of σ_F does not imply that of $\overline{\sigma_F}$ except in the case where F is a foliation and $i = 0$ is the only relevant sub-bundle.

Corollary 5.2.4. *Any Γ_1^∞-structure F on X^n is homotopic to a Γ_1^∞-structure F' such that $S(F')$ is a finite collection of points.*

Proof. This follows from lemma 5.1.2.

Proof of the theorem. The idea of the proof is to first use the transversality theorem of Thom (in a form applicable to the space of section of the fibre bundle $m_1(F)$ (cf. Section 2) to produce an element $i' \in \Gamma^\infty(m_1(F))$ satisfying certain first-order transversality conditions similar to those in the definition of $T-\Gamma_q^\infty$-structures. It is then shown that the Γ_q^∞-structure F' defined by $F' = i'^{-1}(\bar{E})$ ($\bar{E}$ the horizontal foliation on the graph of F) is indeed a $T-\Gamma_q^\infty$-structure, utilizing the properties of the graphs of F and F'.

Let $m(F)$ be the graph of F as in Theorem 2.1 and $m_1(F)$ the R^q bundle associated to it. That is, $m_1(F)$ is given by a smooth fibre bundle

$$\begin{array}{ccc} R^q & \longrightarrow & M_1 \\ & & \Big\downarrow p_F \\ & & X^n, \end{array}$$

together with the zero-section $i_F : X \longrightarrow M_1$. Recall (Thm. 2.2) that $M_1(F)$ is defined up to fibre bundle isomorphisms within M, the total space of $m(F)$, which leave the (image of the) zero-section pointwise fixed.

Consider $J^1(m_1(F))$, the associated 1-jet bundle of $m_1(F)$ [cf. 17, p. 64]. This is a fibre bundle with fibre over $x \in X^n$, $J^1(m_1(F))_x = R^q \times R^{n \cdot q}$ and structural group G equal to $G_1 \times G_2 \times G_1^1$ where G_1 is the group of the bundle $m_1(F)$, G_2 denotes the group of the vector bundle TX, and G_1^1 denotes the group of non-singular linear transformations of R^q of the form $T(g)$ $g \in G_1$, i.e., the group of "1-jets of elements of G_1." If $(v, B_{ij}) \varepsilon R^q \times R^{nq}$, and $(g, h, T(g')) \varepsilon G$, the action is given by $(v, B_{ij}, g, h, T(g')) \longmapsto (g(v), h \cdot B_{ij} \cdot T(g'))$, where h, $T(g')$ are considered as $n \times n$, $q \times q$ matrices, respectively, and the dot signifies matrix multiplication.

As in Section 3.1, let $S_k(n, q) = \{n \times q$ matrices B_{ij} with rank $q - k\}$, $0 \leq k \leq q$. Set $S_k(m_1(F))_x \subset J^1(m_1(F))_x$ equal to $R^q \times S_k(n, q)$ and let $S_k(m_1(F)) = \bigcup_x S_k(m_1(F))_x$. Then, as $S_k(m_1(F))_x$ is invariant under the action of the group G, $S_k(m_1(F))$ is a sub-bundle of $J^1(m_1(F))$ and so a submanifold of $J^1(m_1(F))$. For $\sigma \varepsilon \Gamma^\infty(m_1(F))$, let $j^1(\sigma)$ be the 1-jet extension of σ. Recall [cf. 17, p. 64] $j^1(\sigma)(x) = \{x, \sigma(x), D^1\sigma(x)\}$ where $D^1\sigma(x)$ is the collection of all first order partials of σ (considered locally as a map into R^q) at x. The fact needed here, which is a consequence of [1, Thm. (19.1), p. 48, see also Thm. 12.3, p. 31, Thm. 12.4, p. 32], is that there exists a dense set $A \subset \Gamma^\infty(m_1(F))$ such that $\sigma \varepsilon A \Rightarrow j^1(\sigma)$ has transversal intersection with each $S_k(m_1(F))$.

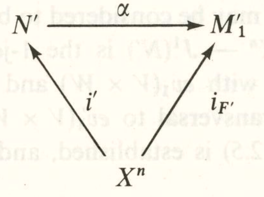
Now, let U be any open set in $\Gamma^\infty(m_1(F))$ with $i_F \varepsilon U$. As A is dense in $\Gamma^\infty(m_1(F))$, $U \cap A \neq \emptyset$; let $i' \varepsilon U \cap A$ and set $F' = i'^{-1}(\bar{E}_1)$. Recall that $\bar{E}_1$ is the foliation defined by the restriction of the "horizontal" foliation of $\bar{E}$ of M to the open set $M_1 \subset M$. Thus F' is a Γ_q^∞-structure on X, and the proof of Theorem (5.2.2) is reduced to:

Lemma 5.2.5. *Let X, F, F' be as above. Then*
1) *F' is homotopic to F; and*
2) *F' is a $T - \Gamma_q^\infty$-structure on X^n.*

Proof. 1) Claim: i' is isotopic to i_F, considered as embeddings of X^n into M_1. Indeed, define H: $X^n \times I \to M_1$ by $H(x, t) = ti'(x) + (1 - t)\, i_F(x)$. Since i', i_F are global section, the points $i'(x)$, $i_F(x)$ are well defined in $(M_1)_x$, independent of the choice of local product representations. As such, H is a well defined C^∞ function. Further, since each H_t: $X^n \longrightarrow M_1$ is a section, it is an embedding. (1) now follows from 1.4 − 3.

2) The proof of (2) follows from the fact, to be shown subsequently, that $j^1(i_{F'})$ has transversal intersection with $S(m_1(F')) = \bigcup_{k=0}^{q} S_k(m_1(F') \subset J^1(m_1(F'))$. Indeed, it is then trivial to show that under the canonical identification of $J^1(m_1(F'))$ with the bundle $m_1(F') \oplus \mathrm{Hom}(TX, \nu F')$ (they have the same fibre and group) and the induced identification of the corresponding spaces of sections, that $j^1(i_{F'})$ corresponds to $i_{F'} \oplus \sigma_{F'}$ and thus $\sigma_{F'}$ is a T-section. Thus we restrict our attention to $j^1(i_{F'})$.

Assertion: There exists a tubular neighborhood N' of $i'(X^n)$ in M_1, such that, considered as an R^q bundle over X^n (make the canonical identification of $i'(X^n)$ with X^n under p_F), N' is isomorphic to $m_1(F')$. Further if α: $N' \longrightarrow M'_1$ (= total space of $m_1(F')$) is the isomorphism, the following diagram is commutative:



Indeed, as i_F, i' are isotopic embeddings of X^n in M_1, there exists a vector bundle isomorphism $\bar{\psi}$: $\nu(i_F(X^n)) \longrightarrow \nu(i'(X^n))$ covering the diffeomorphism $i' \circ i_F^{-1}$: $i_F(X^n) \longrightarrow i'(X^n)$, where $\nu(i_F)$ (resp. $\nu(i')$) is the normal bundle to the embedding i_F (resp. i'). Hence, by the Tubular Neighborhood Theorem [11, Theorem 9, p. 73] there exist open neighborhoods N of $i_F(X^n)$, N' of $i'(X^n)$ and a diffeomorphism ψ: $N' \longrightarrow N$ such that $\psi \circ i' = i_F$. Further, with respect to the R^q-bundle structures on N', N inherited from $\nu(i')$, $\nu(i_F)$, respectively, ψ is actually a bundle map and hence a bundle equivalence. Note that the bundle structure on N is exactly that induced by $m_1(F)$, i.e., the projection map is just $p_F \mid N$.

Since F is homotopic to F', by Theorem 2.1 − (3), $m(F) \overset{h}{\approx} m(F')$ and restricting h to $M_1 \subset M$ defines a zero-section preserving fibre bundle isomorphism $\bar{h}$: $m_1(F) \longrightarrow m_1(F')$. As (up to the equivalence given by Theorem (2.2)), N represents $m_1(F)$, the composition $\alpha = \bar{h} \mid N \circ \psi : N' \longrightarrow m_1(F')$ is a well-defined isomorphism, and certainly satisfies the property claimed. This proves the assertion.

Define $j^1(\alpha)$: $J^1(N') \longrightarrow J^1(m_1(F'))$ (N' considered as an R^q-bundle over X^n) by $j^1(\alpha)(j^1(i)(x)) = j^1(\alpha \circ i(x))$, $x \varepsilon X^n$, $i \varepsilon \Gamma^\infty(N')$. Since α is maximal rank on each fibre, $j^1(\alpha \circ i) \varepsilon S_k(m_1(F'))$ if and only if $j^1 i \varepsilon S_k(N')$ for $0 \leq k \leq q$. Thus $j^1(\alpha)$ restricts to give a well-defined diffeomorphism on each $S_k(N')$. Since $j^1(\alpha)$ is a global diffeomorphism, it is trivial to check that $j^1(i')$ transversal to $S(N') > j^1(\alpha)(j^1(i')) = j^1(\alpha \circ i')$ is transversal to $S(m_1(F'))$. In view of the Assertion, then, the transversality of $j^1(i_{F'})$ is equivalent to proving

(*) $\quad j^1(i')$ has transversal intersection with $S(N')$.

To that end, let V be a neighborhood of i' in $\Gamma^\infty(m_1(F))$ sufficiently small so as to satisfy $\sigma \varepsilon V = > \mathrm{Im}(\sigma) \subset N'$. Let $ev_1 : \Gamma^\infty(m_1(F)) \times X^n \longrightarrow J^1(m_1(F))$ be the map $ev_1(\sigma, x) = j^1\sigma(x)$ and let $x \varepsilon X^n$. According to [1, Thm. 12.4, p. 32] ev_1 is a submersion (and hence an open mapping) and so if W is any neighborhood of x, $ev_1(V \times W) = \{j^1(i(y)) \mid i \varepsilon V, y \varepsilon W\}$ is open in $J^1(m_1(F))$. As a result, $T(J^1(m_1(F)))_{j^1 i'(x)}$ can be identified with $T(ev_1(V \times W))_{j^1 i'(x)}$ and the transversal intersection of $j^1 i'$ with $S(m_1(F))$ at $j^1 i'(x) \Rightarrow j^1 i'$ is transversal to $ev_1(V \times W) \cap S(m_1(F))$ there. On the other hand, it follows from the construction of V and N' that V may be considered to be an open subset of $\Gamma^\infty(N')$. Hence, if $ev_1' : \Gamma^\infty(N') \times X^n \longrightarrow J^1(N')$ is the 1-jet evaluation map for N', by identifying $ev_1'(V \times W)$ with $ev_1(V \times W)$ and viewing i' as a section of N', it follows that $j^1 i'$ is transversal to $ev_1'(V \times W) \cap S(N')$ at $j^1 i'(x)$, which proves (*). Thus lemma (5.2.5) is established, and the proof of the theorem is complete.

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Wayne State University,
Detroit, Michigan