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Minimal submanifolds of the bicylinder boundary

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In this paper we wish to describe a collection of closed 2-dimensional surfaces embedded or immersed in equilibrium in the (3-dimensional) boundary C^2 of the bicylinder $D^2 \times D^2 = \{(x, y, u, v) \mid x^2 + y^2 \leq 1,\ u^2 + v^2 \leq 1\}$ in $\mathbb{R}^4$. Several of these embeddings are analogous to a set of minimal embeddings of closed surfaces in the ordinary 3-sphere $= \{(x, y, u, v) \mid x^2 + y^2 + u^2 + v^2 = 1\}$ constructed independently recently by B. Lawson. In order to motivate the concept of a minimal submanifold of a non-differentiable surface such as the bicylinder boundary, we begin with an elementary treatment of the analogous problem in one lower dimension – that of finding closed geodesics on the boundary C^2 of an ordinary cylinder $\{(x, y, u) \mid x^2 + y^2 \leq 1, u^2 \leq 1\}$ in $\mathbb{R}^3$. This section indicates the basic advantage of working in cylinder boundaries as opposed to spheres – the cylinder boundary can be decomposed into disjoint pieces, each of which can be realized isometrically in Euclidean space as a convex cell with identification along its boundary, and therefore classical facts about geodesics and minimal surfaces in ordinary Euclidean space can be used to construct these examples in higher dimensional spaces. In this paper we shall present equilibrium embeddings of all orientable surfaces with Euler characteristic of the form $\chi = 2n(2 - n)$, and we also present a minimal immersion of the Klein bottle.

Lawson, in his thesis, was the first to construct closed minimal surfaces of higher genus in the standard 3-sphere [1]. In the final section of this paper, we sketch a theory of closed equilibrium surfaces in the 3-sphere with the metric given by taking two standard 3-discs in $\mathbb{R}^3$ and identifying them along their boundaries. In this 3-sphere, we obtain examples of embedded surfaces in equilibrium of arbitrary genus by a construction precisely parallel to that used in [1], but again using only classical facts about minimal surfaces in $\mathbb{R}^3$.

1. Closed Geodesics on the Cylinder Boundary

In this section, we describe the collection of all closed geodesics on the cylinder boundary C^2, i.e., curves which are in equilibrium on C^2 in the same way that great circles are in equilibrium on the ordinary sphere S^2. A useful

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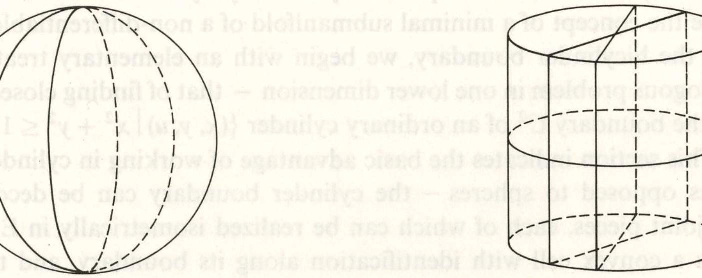
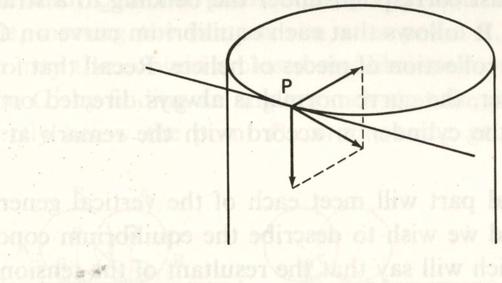
physical interpretation of a geodesic on a convex body is given by considering the curve as a string under tension and constrained to lie on the surface. The string will be in equilibrium on the surface if at each point there is no *preferred* direction in which to move which will make the curve shorter. With this definition, any curve of the surface which is pointwise fixed under a reflection of space through a plane which sends the surface to itself must automatically be in equilibrium since there cannot be any preferred direction in which any point of the curve would move under the influence of the tension of the string. Therefore the great circles on the sphere will be in equilibrium and we may find analogous equilibrium curves of symmetry on the cylinder boundary C^2. Through any point of the sphere S^2, there will be a



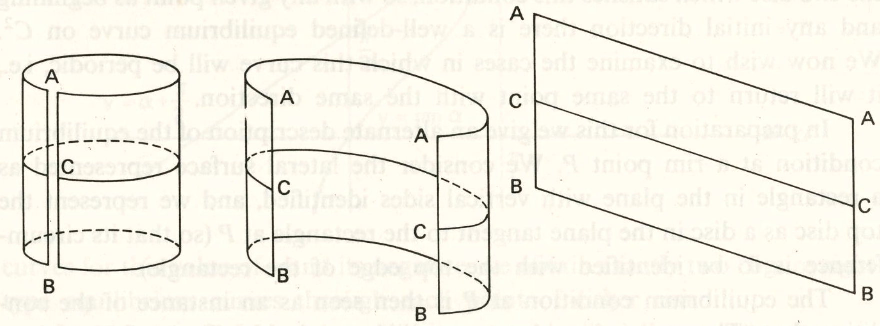
closed geodesic in every direction, but we shall show that the centers of the discs of C^2 are the only two points of C^2 with this property. In addition to the lateral curve of symmetry of C^2, there are embedded equilibrium curves on C^2 which are not curves of symmetry obtained by taking other horizontal curves $\{x^2+y^2=1, z=c$, a constant with $c^2<1, c \neq 0\}$. These two classes of equilibrium curves contain the only embedded equilibrium curves on C^2, but C^2 has equilibrium curves which are not embedded (and which are not merely multiple coverings of an embedded closed geodesic). In fact we shall show that through every point other than the centers of the end discs there pass (exactly) countably many closed equilibrium curves.

Before describing some non-embedded closed equilibrium curves, we discuss the nature of the equilibrium condition at a point P on one of the rim curves $\{x^2+y^2=1, z=\pm\}$ of C^2. For one of the geodesics already described passing through the centers of the discs, the curve meets the rim orthogonally from each side. The tension on the string may be represented by a pair of unit vectors at the point on the rim, each pointing in the direction of the tangent to a piece of the curve at P. The resultant tension vector is then directed orthogonal to the rim, and this is precisely the condition which insures that there is no force tending to move the curve along the rim at P.



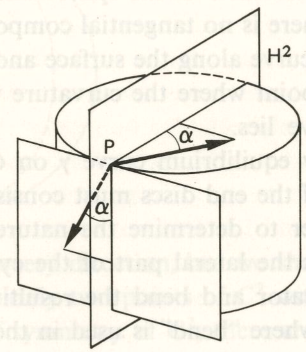
The situation is similar in the smooth case. There the tension vector at the point of a curve is a constant multiple of the curvature vector of the curve, i.e., the principal unit normal vector multiplied by the curvature at a point. In the case of a great circle on the sphere, the curvature vector at any point is directed toward the center of the sphere and orthogonal to the tangent space. Consequently there is no tangential component of the tension vector tending to displace the curve along the surface and the curve is therefore in a equilibrium at each point where the curvature vector is normal to the surface on which the curve lies.

Next we observe that any equilibrium curve γ on C^2 is such that the part of γ which meets either of the end discs must consist of a collection of straight line segments. In order to determine the nature of the part of an equilibrium curve which meets the lateral part of the cylinder, we may cut this lateral part along a generator and bend the resulting piece of surface into a rectangle in the plane, where "bend" is used in the classical technical sense which requires that no lengths measured along the surface will be changed during the bending process. Under this deformation, generators are



mapped into vertical lines and horizontal circles into horizontal lines. (with end points identified) In fact since distances along the surface are unchanged under the bending, any part of an equilibrium curve which lies on the lateral section of the cylinder must correspond under the bending to a straight line segment on the rectangle. It follows that each equilibrium curve on C^2 meets the lateral part of C^2 in a collection of pieces of helices. Recall that for a helix on a right circular cylinder, the curve normal is always directed orthogonal to the tangent plane of the cylinder in accord with the remark at the end of the last paragraph.

A helix on the lateral part will meet each of the vertical generators at the same angle, say α, and we wish to describe the equilibrium condition at a point P on the rim, which will say that the resultant of the tension vectors from the two pieces of curve at P will lie in the plane H^2 orthogonal to the rim at P. But this condition is satisfied if and only if both tension vectors make the same angle α with H^2 and if the vectors lie on opposite sides of H^2 (since the components of the tension vectors orthogonal to H^2 must



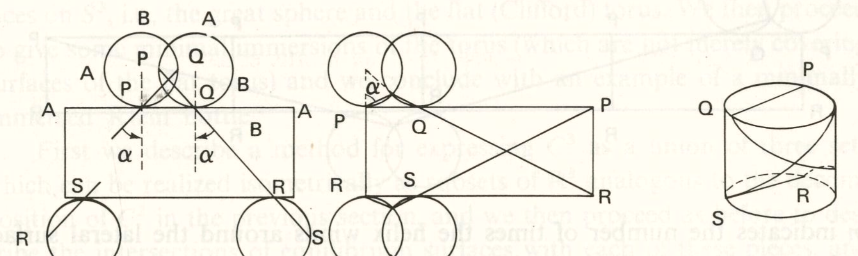
balance one another). There is precisely one continuation of the curve into the end disc which satisfies this condition, so with any given point as beginning and any initial direction there is a well-defined equilibrium curve on C^2. We now wish to examine the cases in which this curve will be periodic, i.e., it will return to the same point with the same direction.

In preparation for this we give an alternate description of the equilibrium condition at a rim point P. We consider the lateral surface represented as a rectangle in the plane with vertical sides identified, and we represent the top disc as a disc in the plane tangent to the rectangle at P (so that its circumference is to be identified with the top edge of the rectangle).

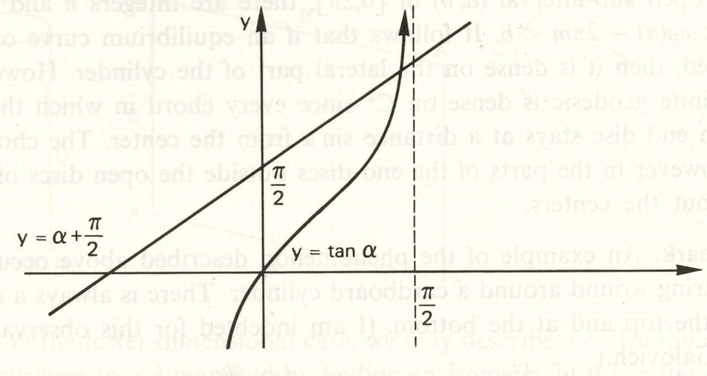
The equilibrium condition at P is then seen as an instance of the condition that "The angle of incidence equals the angle of reflection." If we continue the line from P on the top disc until it meets the rim again at Q, then the continuation into the lateral surface again will meet the generators at the angle $(-\alpha)$. (In the diagram, we "roll" the disc along the top edge of the

rectangle until Q is the point of tangency, and we then continue into the rectangle, meeting the bottom rim at a point R.)

We may then obtain a closed equilibrium curve in the form of a "figure eight" on C^2 by finding an angle α such that the point R in the figure below lies directly below the point P, on the same generator.



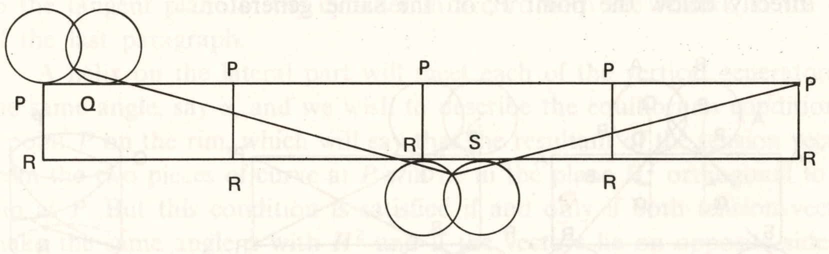
The condition on α is seen to be that the length $\widehat{PQ}$ of the arc plus the length of the segment $\overline{QP}$ (through B) on the top of the rectangle gives 2π. But $\widehat{PQ} = \pi - 2\alpha$ and $\overline{QP} = \overline{PR} \tan\alpha = 2\tan\alpha$, so the condition on α is that $\pi - 2\alpha + 2\tan\alpha = 2\pi$, or $\alpha + \pi/2 = \tan\alpha$, and this equation has at least one solution in the interval $[0, \pi/2]$ by the intermediate value theorem, and at most one solution by the mean value theorem. By taking the equilibrium



curves for this value of α and its negative, we obtain exactly two "figure eight" type equilibrium curves through each point of either rim.

In order to find closed equilibrium curves which have more than one self-intersection but which still meet each end disc in precisely one segment, we consider the lateral surface not as a rectangle with vertical edges identified, but as an infinite strip $\{(x, y) \mid y^2 \leq 1\}$ with identifications $(x, y) \sim$

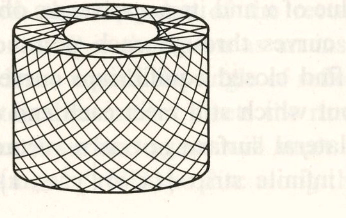
$(x + 2\pi, y)$. As before, in order to have a closed equilibrium curve at P which meets each end disc precisely once, we want the point R, the first point where the curve from Q meets the bottom rim, to lie directly below a point identified with P. The condition for this is that $\pi - 2\alpha + 2\tan\alpha = 2\pi m$, where the integer



m indicates the number of times the helix winds around the lateral surface before reaching the bottom rim. As before, we have precisely one solution to the condition $\alpha + [(2m + 1)/2]\,\pi = \tan\alpha$ in the domain $[0, \pi/2]$ for each non-negative integer m.

To find the most general closed equilibrium curve which meets each end disc in precisely n segments, we find solutions to the equation $\pi - 2\alpha + 2\tan\alpha = 2\pi(m/n)$ where m/n is a non-negative rational number in lowest terms. This will give all closed equilibrium curves, because if the number $g(\alpha) = \pi - 2\alpha + 2\tan\alpha$ is not a rational multiple of 2π, then the numbers $\{ng(\alpha)(\mathrm{mod}\ 2\pi)\}$ for positive integers n are dense on the interval $[0,2\pi]$, i.e., for any open sub-interval (a, b) of $[0,2\pi]$, there are integers n and m such that $a < ng(\alpha) - 2\pi m < b$. It follows that if an equilibrium curve on C^2 is not closed, then it is dense on the lateral part of the cylinder. However no such infinite geodesic is dense on C^2 since every chord in which the curve meets an end disc stays at a distance $\sin\alpha$ from the center. The chords are dense however in the parts of the end discs outside the open discs of radius $\sin\alpha$ about the centers.

Remark. An example of the phenomenon described above occurs in a ball of string wound around a cardboard cylinder. There is always a circular hole at the top and at the bottom. (I am indebted for this observation to Steven Galovich.)



2. Closed Minimal Sufaces on the Bicylinder Boundary – The Orthogonal Case.

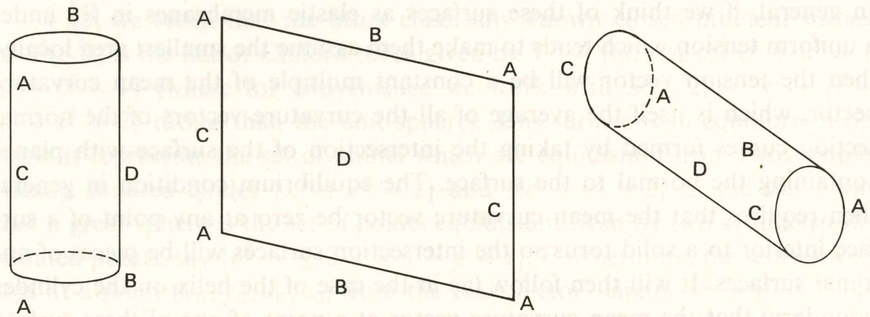
In this section we describe a collection of examples of closed surfaces in equilibrium in C^3, the boundary of the bicylinder $D^2 \times D^2 = \{(x, y, u, v) \mid x^2 + y^2 \leq 1, u^2 + v^2 \leq 1\}$, analogous to the "classical" closed minimal surfaces on S^3, i.e., the great sphere and the flat (Clifford) torus. We then proceed to give some minimal immersions of the torus (which are not merely covering surfaces of the flat torus) and we conclude with an example of a minimally immersed Klein bottle.

First we describe a method for expressing C^3 as a union of three sets which can be realized isometrically as subsets of $\mathbb{R}^3$ analogous to the decomposition of C^2 in the previous section, and we then proceed as before to describe the intersections of equilibrium surfaces with each of these pieces, and to describe the way these intersection surfaces fit together at the boundary.

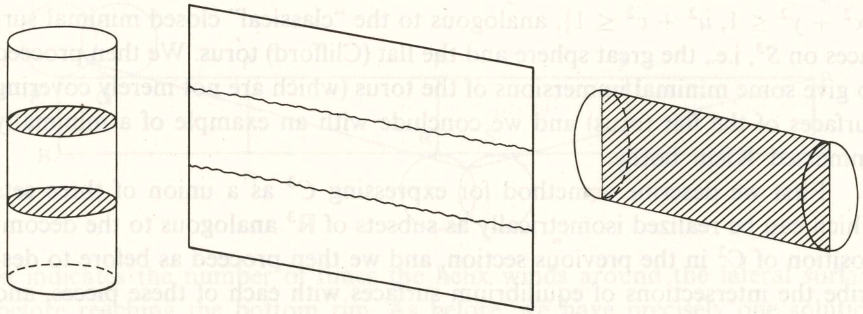
We may express $C^3 = (\text{boundary of } D^2 \times D^2)$ as

$$\{(x, y, u, v) \mid x^2 + y^2 = 1, u^2 + v^2 < 1\} \cup \{x^2 + y^2 = 1, u^2 + v^2 = 1\}$$
$$\cup \{x^2 + y^2 < 1, u^2 + v^2 = 1\}.$$

The second set is simply a torus obtained by identifying opposite edges of a square of side length 2π, and each of the other sets is a solid torus which can be expressed as a solid cylinder in $\mathbb{R}^3$ with end discs identified.



As in the lower dimensional case, we may describe a collection of equilibrium surfaces in a 3-manifold by finding an isometry of the entire space $\mathbb{R}^4$ which leaves the surface pointwise fixed and which sends the 3-manifold into itself. For example, a great 2-sphere in S^3 may be described as the fixed set under a reflection in a 3-dimensional linear subspace through the origin, e.g., $(x, y, u, v) \to (x, y, u, -v)$. This last-mentioned transformation also sends C^3 into itself and the fixed surface consists of two lines $\{x^2 + y^2 = 1, v = 0, u = \pm 1\}$ in the torus, two discs $\{x^2 + y^2 < 1, v = 0, u = \pm 1\}$ in one of the solid tori, and a rectangle with two sides identified $\{x^2+y^2=1, v=0, u^2<1\}$ in the other solid torus. The union of these pieces gives a 2-dimensional sphere isometric to the cylinder boundary C^2, in fact C^2 is precisely the intersection of C^3 with the hyperplane $\{v=0\}$.

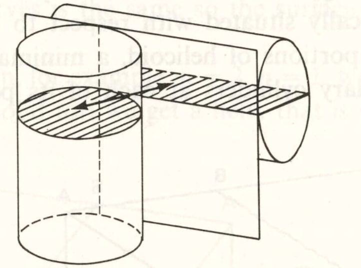


In a similar way we obtain a C^2 in equilibrium in C^3 whenever we take the intersection of C^3 with a hyperplane which is orthogonal to a vector either in the $x-y$-plane or in the $u-v$-plane, so any point of C^3 lies in at least two of these closed surfaces isometric to C^2 in equilibrium in C^3.

Since the surface is in equilibrium, the intersection of the surface with each solid torus must be a collection of surfaces which themselves are in equilibrium, and in the case given above these intersection surfaces are planar. In general, if we think of these surfaces as elastic membranes in $\mathbb{R}^3$ under a uniform tension which tends to make them assume the smallest area locally, then the tension vector will be a constant multiple of the mean curvature vector, which is itself the average of all the curvature vectors of the normal section curves formed by taking the intersection of the surface with planes containing the normal to the surface. The equilibrium condition in general then requires that the mean curvature vector be zero at any point of a surface interior to a solid torus so the intersection surfaces will be pieces of minimal surfaces. It will then follow (as in the case of the helix on the cylinder boundary) that the mean curvature vector at a point of one of these surface pieces will be directed orthogonal to the solid torus at each point when the 3-manifold C^3 is "reassembled" in $\mathbb{R}^4$.

We now consider the nature of the equilibrium condition where the pieces of minimal surfaces fit together along the torus "rim." In the examples of the "great C^2" on C^3, at each boundary curve of a surface piece in one of the solid tori, the tension may be represented as a vector tangent to the surface and orthogonal to the curve. If the example at hand, the intersection curves with the torus rim are geodesics on the torus so there is no tension

vector arising from this curve on the surface. The equilibrium condition then requires that the resultant of the tension vector directed into the two solid tori be orthogonal in R^4 to the tangent plane to the torus at the point in question. In order to check this condition in R^3, we bring the three pieces of C^3 together at the point p of the torus rim. If the intersection curve with this rim at p is a geodesic, then the equilibrium condition at p in the two solid tori, which are orthogonal to the intersection curve at p, will be negatives of one another, i.e., they will lie on the same straight line. In our example, the condition is satisfied easily since the tension vectors from the surface pieces are all orthogonal to the torus rim.

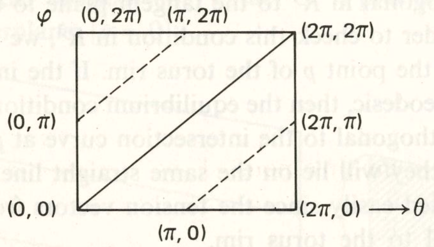


We now turn to construct some examples of greater interest.

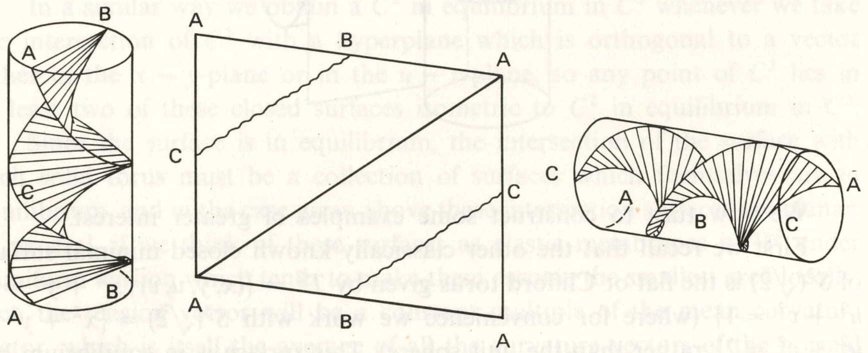
First we recall that the other classically known closed minimal surface of $S^3(\sqrt{2})$ is the flat or Clifford torus given by $T^2 = \{(x, y, u, v) \mid x^2 + y^2 = 1, u^2 + v^2 = 1\}$ (where for convenience we work with $S^3(\sqrt{2}) = \{x^2 + y^2 + u^2 + v^2 = 2\}$ rather than the unit sphere). This surface is in equilibrium because it represents the set of points which are equidistant from two symmetrically situated circles $\{x^2 + y^2 = 2\}$ and $\{u^2 + v^2 = 2\}$ in the same way that a great sphere is the set of points equidistant from by two symmetrically situated points, say $\{v^2 = \pm 2, s = 0 = y = z\}$. Furthermore, the tension vectors at a point (x_0, y_0, u_0, v_0) from the two "factor" circles $x^2 + y^2 = 1$ and $u^2 + v^2 = 1$ are just given by the vectors $(-x_0, -y_0, 0, 0)$ and $(0, 0, -u_0, -v_0)$ which have as resultant tension vector the negative of the position vector to the point on the 3-sphere, so each point is in equilibrium.

For these same reasons the torus rim, which is identical with this flat Glifford torus, is in equilibrium on C^3, but we wish to describe a class of less apparent examples.

In describing curves on the flat torus, it is convenient to give a parametrization of this surface by setting $x = \cos\theta$, $y = \sin\theta$, $u = \cos\phi$, $v = \sin\phi$ for $0 \le \phi < 2\pi$, $0 \le \phi < 2\pi$. The intersection curves of the C^2 surface first described above would then be $\phi = 0$ and $\phi = \pi$. We now consider the pair of curves given by the condition $\theta = \phi$ and $\theta = \phi + \pi$.



The corresponding curves on the boundary of each solid torus are pairs of helixes symmetrically situated with respect to the axes. We fill each of these pairs in with portions of helicoid, a minimal surface which is orthogonal to the boundary cylinders at each of its points.

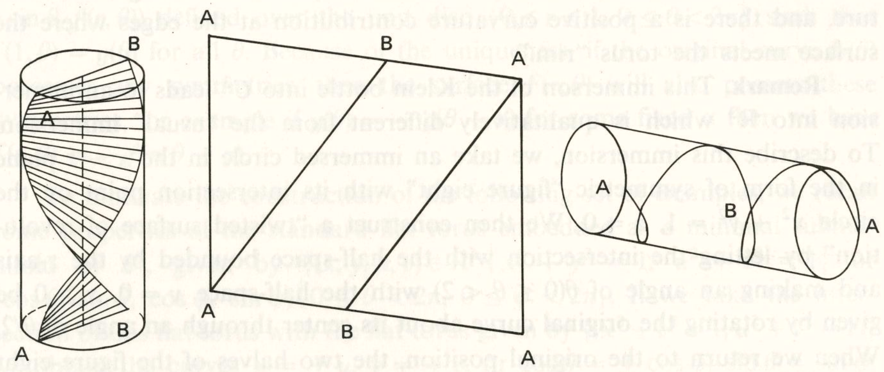


Because the helicoids come into the boundary cylinders of the solid tori at right angles, and because the intersection curves are geodesics on each boundary cylinder, the equilibrium condition is satisfied and the surface obtained by identifying the two pieces of helicoid along the boundaries is a closed surface in equilibrium on C^3. Since the surface is embedded in the boundary of a convex body in $\mathbb{R}^4$, it must be orientable (by central projection we get an embedded surface on S^3, and by stereographic projection from a point not on the image we get an embedded surface in $\mathbb{R}^3$, so the original surface must be orientable). Moreover, the surface is the union of two topological cylinders with boundary circles identified, so the equilibrium surface is a torus.

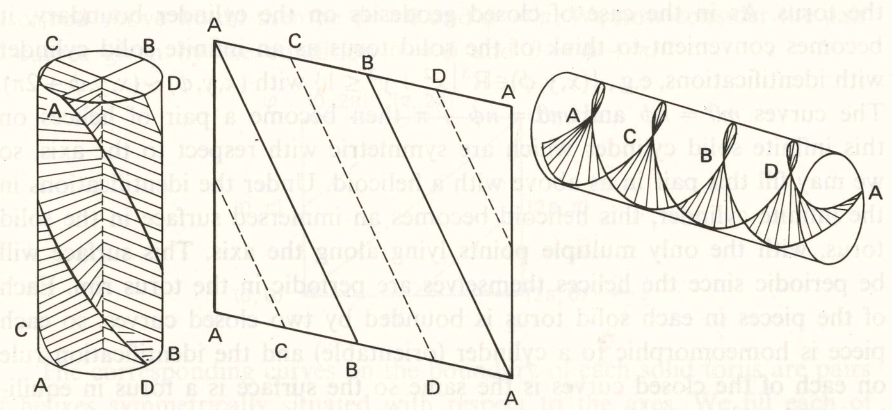
By taking other curves on the rim torus it is possible to find a family of immersed tori in equilibrium on C^3. Let m and n be two relatively prime odd integers and consider the two curves $m\theta = n\phi$ and $m\theta = n\phi + \pi$ on

the torus. As in the case of closed geodesics on the cylinder boundary, it becomes convenient to think of the solid torus as an infinite solid cylinder with identifications, e.g., $\{(x, y, \phi) \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq 1\}$ with $(x, y, \phi) \sim (x, y, \phi + 2\pi)$. The curves $m\theta = n\phi$ and $m\theta = n\phi + \pi$ then become a pair of helices on this infinite solid cylinder which are symmetric with respect to the axis, so we may fill this pair in as above with a helicoid. Under the identifications in the infinite cylinder, this helicoid becomes an immersed surface in the solid torus, with the only multiple points lying along the axis. This surface will be periodic since the helices themselves are periodic in the torus rim. Each of the pieces in each solid torus is bounded by two closed curves, so each piece is homeomorphic to a cylinder (orientable) and the identification rule on each of the closed curves is the same so the surface is a torus in equilibrium on C^3.

If we take m or n even, for example $m = 2$, $n = 1$, we get a different phenomenon. In one of the solid tori we get a helix that is symmetrical to itself

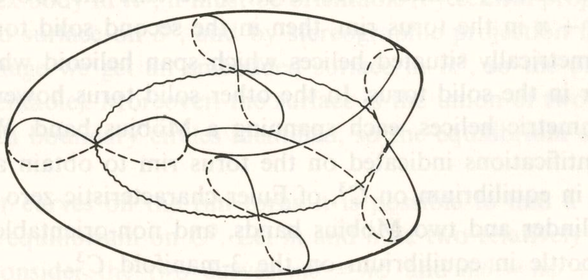


under reflection through the axis, so we can fill this in with a piece of helicoid which forms a Möbius band. In the other solid torus however we get a single curve which does not span a minimal surface which comes in orthogonal to the boundary cylinder. However, if we take the two curves $2\theta = \phi$ and $2\theta = \phi + \pi$ in the torus rim, then in the second solid torus we have a pair of symmetrically situated helices which span helicoid which is an ordinary cylinder in the solid torus. In the other solid torus however, we obtain two self-symmetric helices, each spanning a Möbius band. We may still make the identifications indicated on the torus rim to obtain a closed connected surface in equilibrium on C^3, of Euler characteristic zero since it is a union of a cylinder and two Möbius bands, and non-orientable. Thus we have a Klein bottle in equilibrium on the 3-manifold C^3.



Remark. The minimal immersions obtained in this way for the torus and the Klein bottle are not flat – the helicoid pieces are of negative curvature, and there is a positive curvature contribution at the edges where the surface meets the torus "rim."

Remark. This immerson of the Klein bottle into C^3 leads to an immersion into $\mathbb{R}^3$ which is qualitatively different from the "usual" immersion. To describe this immersion, we take an immersed circle in the $x - z$ plane in the form of symmetric "figure eight" with its intersection point on the circle $x^2 + y^2 = 1$, $z = 0$. We then construct a "twisted surface of revolution" by letting the intersection with the half-space bounded by the z-axis and making an angle of $\theta(0 \leq \theta < 2)$ with the half-space $y = 0$, $x \geq 0$ be given by rotating the original curve about its center through an angle of $\theta/2$. When we return to the original position, the two halves of the figure eight have been interchanged so we obtain a Klein bottle. The top and bottom points of the eight together describe a single closed curve and the compliment of this curve is the union of two Mobius bands intersecting along the circle $x^2 + y^2 = 1$, $z = 0$.



This immersion differs from that of the "usual" immersion of the Klein bottle in which the curve of intersection possesses two orientable cylinder neighborhoods.

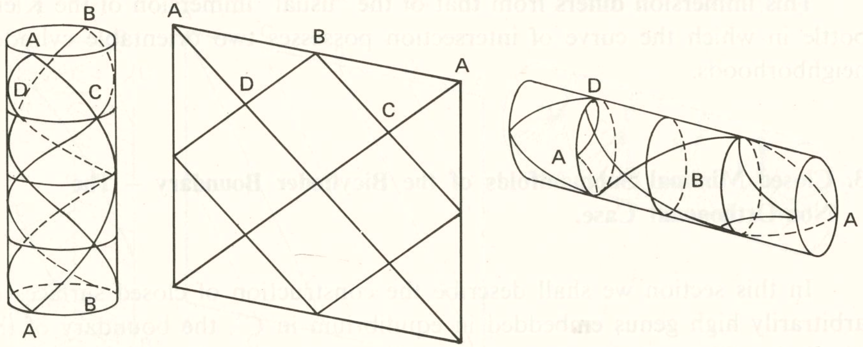
3. Closed Minimal Submanifolds of the Bicylinder Boundary – The Non-Orthogonal Case.

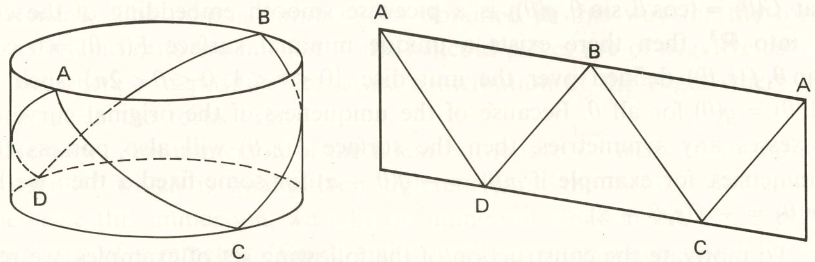
In this section we shall describe the construction of closed surfaces of arbitrarily high genus embedded in equilibrium in C^3, the boundary of the bicylinder. The main property used in the construction is a basic existence and uniqueness theorem in the classical theory of minimal surfaces, i.e., if $g: S^1 \longrightarrow R^1$ is a piecewise smooth map of the unit circle into the line, so that $G(\theta) = (\cos\theta, \sin\theta, g(\theta))$ is a piecewise smooth embedding of the circle S^1 into $\mathbb{R}^3$, then there exists a unique minimal surface $F(r, \theta) = (r\cos\theta, r\sin\theta, f(r, \theta))$ defined over the unit disc $\{0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$ such that $f(1, \theta) = g(\theta)$ for all θ. Because of the uniqueness, if the original curve $G(\theta)$ possesses any symmetries, then the surface $F(r, \theta)$ will also possess these symmetries, for example if $g(\theta) = -g(\theta + \alpha)$ for some fixed α then we have $f(r, \theta) = -f(r, \theta + \alpha)$.

To motivate the construction of the following set of examples, we recall some properties of the standard flat torus embedded as a minimal submanifold of S^3, given by $\{(x, y, u, v) \in \mathbb{R}^4 \mid x^2 + y^2 = 1,\ u^2 + v^2 = 1\}$ or $\{\cos\theta, \sin\theta, \cos\phi, \sin\phi) \mid 0 \leq \theta < 2\pi,\ 0 \leq \phi < 2\pi\}$. If we take the intersection of this flat torus with the flat torus given by $\{(x^2 + v^2 = 1, u^2 + y^2 = 1)\}$ we obtain the curves $x = \pm u$, $y = \pm v$, or $\cos\theta = \pm\cos\phi$, $\sin\theta = \sin\phi$, which yields the curves $\theta = \pm\phi$, $\theta = \pm\phi + \pi$. These four curves represent circles on the torus and they divide the torus up into eight regions bounded by spherical quadrilaterals. In the flat torus itself each of these quadrilaterals is contained in a hemisphere, and the regions which they bound are portions of minimal surfaces in the sphere S^3, i.e., they represent the surfaces on S^3 with the smallest area bounded by the given quadrilateral.

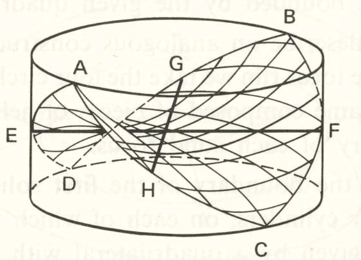
We now wish to describe an analogous construction on the boundary of the bicylinder. On the torus rim we take the four circles $\theta = \pm\phi$, $\theta = \pm\phi + \pi$ and we consider the frame composed of pieces of helix obtained from these curves on the boundary of each solid torus.

We may think of the boundary of the first solid torus as a union of four horizontal circular cylinders, on each of which the framework formed by the four curves is given by a quadrilateral with four helical sides.





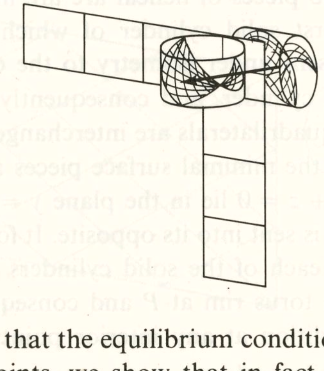
This quadrilateral is piecewise smooth curve given by a function $G(\theta) = (\cos\theta, \sin\theta, g(\theta))$ where $g(\theta+\pi/2) = -g(\theta)$ and where $g(-\theta) = g(\theta) = g(\pi/2-\theta)$, since the quadrilateral is invariant under the symmetries of $\mathbb{R}^3$ given by $(x, y, u) \longrightarrow (-x, y, u)$, $(x, y, u) \longrightarrow (x, -y, u)$. The unique minimal surface spanned by this quadrilateral will have all the same symmetries, and in particular, the surface will have as lines of symmetry the segments EF and GF joining the midpoints of opposite helical arcs on the quadrilateral. In fact this piece of minimal surface can be considered as a union of four minimal



surface pieces, obtained by reflecting the minimal surface spanned by the quadrilateral $\mathcal{O}GBF$ with center at the origin $\mathcal{O}$.

We may then fill the frame on the first solid torus with four pieces of minimal surface, and then do the same for the second solid torus by considering its boundary as the union of four vertical cylinders and filling in a minimal surface on each of the quadrilaterals on these cylinders. In this way we obtain an embedded closed surface on C^3 with 8 vertices, 16 edges, and 8 faces, so the surface is an orientable surface with Euler characteristic $= 0$, i.e., a torus. We claim that this surface is in equilibrium on C^3.

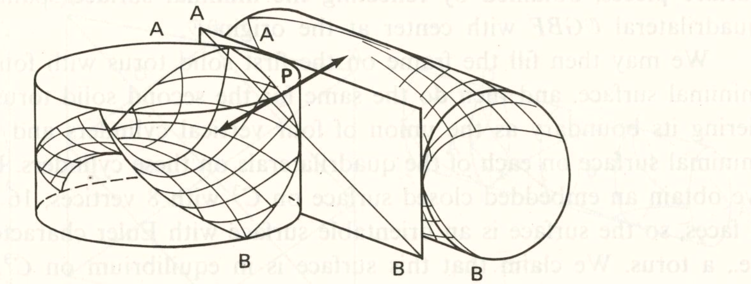
In order to show that the surface is in equilibrium, we must show that at each point of a vertex or edge of a piece of minimal surface, the resultant tension vector will be orthogonal to the torus rim, and as in the previous section, to do this we must show that if the solid cylinders representing the isometric images of the pieces of C^3 in $\mathbb{R}^3$ are brought together at a point P of the boundary, then the tension vectors which point into each of the solid cylinders should be opposite. This condition is certainly satisfied at a midpoint of a helical arc since the tension vector tangent to the surface and orthogonal to the helical arc will be directed orthogonal to the cylinder.



In order to show that the equilibrium condition also holds at edge points other than the midpoints, we show that in fact there is an isometry of all of $\mathbb{R}^3$ which fixes the point of contact and the plane which represents the torus rim and which interchanges the two pieces of minimal surface. Specifically, we may introduce coordinates of the first cylinder by taking $(x+1)^2 + z^2 = 1$ and the second cylinder by $(x-1)^2 + y^2 = 1$, where the common tangent plane at the origin is $x = 0$.

The straight line in the plane which is tangent to each helical arc is given by $x = 0$ and $y + z = 0$.

We wish to consider two pieces of minimal surface, one in each solid torus, which arc identified at a boundary point P, other than the midpoint of a helical edge. We may then translate in $\mathbb{R}^3$ so that the point P becomes

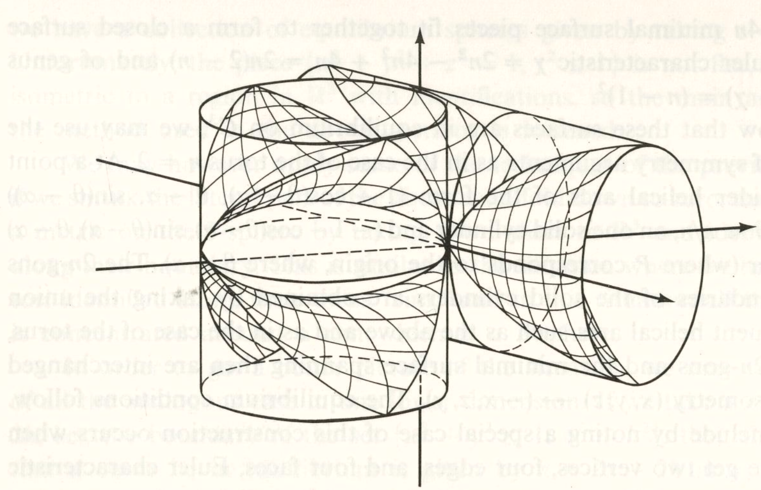


the origin, and we may parametrize the helical segments at P by $(1+\cos(\theta-\alpha),\ \theta-\alpha,\ \sin(\theta-\alpha))$ on the boundary of the solid cylinder with equation $(x-1)^2+z^2=1$ and by $(-1\ \cos(\theta-\alpha),\ \sin(\theta\ \alpha),\ \theta+\alpha)$ to get an oppositely oriented helix with the same pitch on the cylinder $(x+1)^2+y^2=1$. The point P then corresponds to the point of the helix with value $\theta=\alpha$.

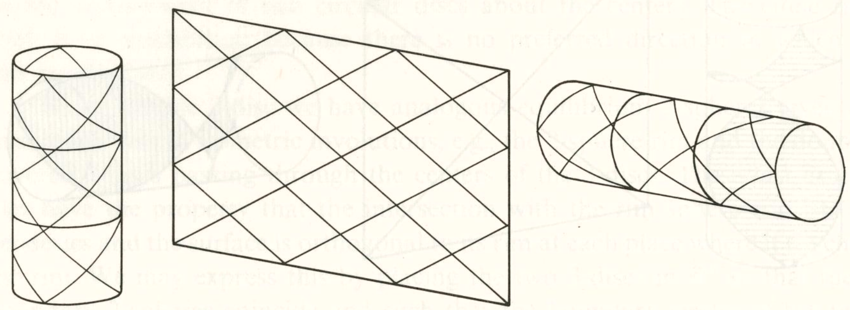
Under the isometry of $\mathbb{R}^3$ given by $(x, y, z) \longrightarrow (-x, z, y)$ the point $P=(0,0,0)$, the line $x=0$, $y+x=0$, and the plane $x=0$ are all mapped into themselves and the two pieces of helical arc are interchanged. Morever, the quadrilateral in the first solid cylinder of which the helical arc with $0 \le \theta < \pi/2$ is an edge is sent under isometry to the corresponding quadrilateral in the second solid cylinder, and consequently, the unique minimal surfaces spanned by these quadrilaterals are interchanged under the isometry. But the tangent vectors to the minimal surface pieces at P which are orthogonal to the line $x=0$, $y+z=0$ lie in the plane $y=z$, and under the isometry each of these vectors is sent into its opposite. It follows that the tension vectors at P directed into each of the solid cylinders have resultant vector which is orthogonal to the torus rim at P and consequently the surface satisfies the equilibrium condition at any interior point of a helical segment on the torus rim.

The only remaining points at which the condition must be checked are the "corner" points where the circles $\theta=\pm\phi$ and $\theta=\pm\phi+\pi$ intersect on the flat torus. At each of these points there are four pieces of minimal surface, two from each solid cylinder. Again, as above, the portions inside the solid cylinders may be interchanged by an isometry of $\mathbb{R}^3$ so the equilibrium condition is satisfied – there is no preferred direction toward which the point is displaced.

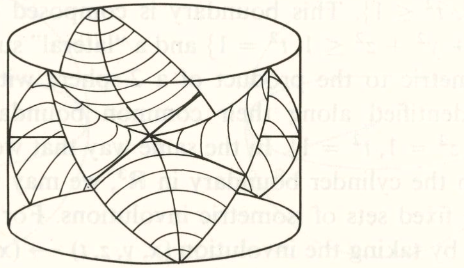
For the construction of surfaces of arbitrarily high genus in equilibrium in C^3, we follow the same procedure as above using $2n$ circles on the flat torus given by $\theta=\pm\phi+2\pi/n$. These circles intersect in $2n^2$ vertices, each



of which has four edges emanating from it. The frame on the boundary of each solid torus may be considered as a union of $2n$ "semi-regular" $2n$-gons



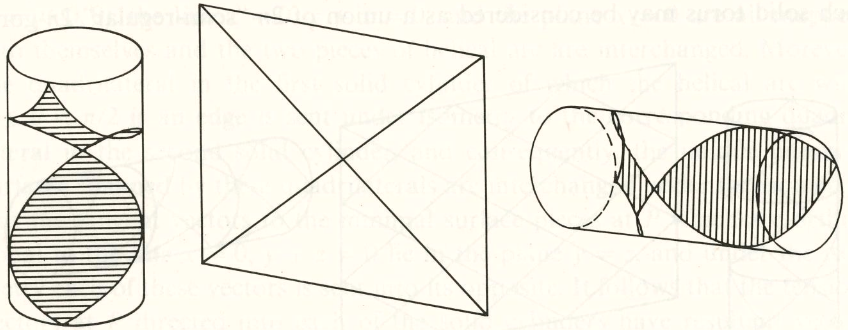
composed of helical segments on a circular cylinder and each spanning a unique minimal surface possessing all the symmetries of the $2n$-gon.



These $4n$ minimal surface pieces fit together to form a closed surface in C^3 of Euler characteristic $\chi = 2n^2 - 4n^2 + 4n = 2n(2-n)$ and of genus $g = 1/2(2-\chi) = (n-1)^2$.

To show that these surfaces are in equilibrium on C^3, we may use the same sort of symmetry arguments as in the case of the torus $n = 2$. At a point P, we consider helical arcs of the form $(1 + \cos(\theta - \alpha),\ \theta - \alpha,\ \sin(\theta - \alpha))$ where $0 \leq \theta < \pi/n$, on one solid cylinder and $(-1 - \cos(\theta - \alpha), \sin(\theta - \alpha), \theta - \alpha)$ on the other (where P corresponds to the origin, where $\theta = \alpha$). The $2n$-gons on the boundaries of the solid cylinders are obtained by taking the union of $2n$ congruent helical arcs such as the above and as in the case of the torus, $n = 2$, the $2n$-gons and the minimal surface spanning then are interchanged under the isometry $(x, y, z) \longrightarrow (-x, z, y)$. The equilibrium conditions follow.

We conclude by noting a special case of this construction occurs when $n = 1$, so we get two vertices, four edges, and four faces, Euler characteristic two, and genus zero. Thus we get another embedding of the sphere as a surface in equilibrium in C^3.



4. Minimal Submanifolds of 3-Cylinders and Double Discs.

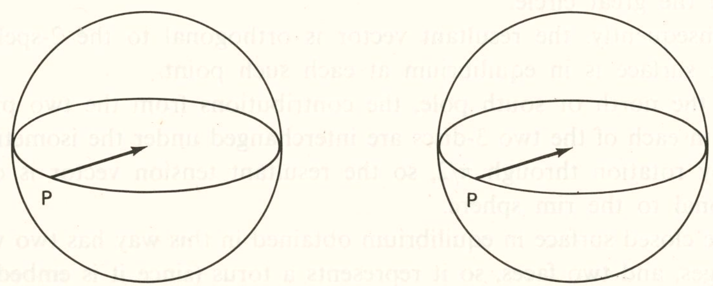
There are other metrics on the 3-sphere which are analogous to that given by the bicylinder, for example, that given by $\partial\{(x, y, x, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 \leq 1, t^2 \leq 1\}$. This boundary is composed of three parts, a pair of 3-discs $\{x^2 + y^2 + z^2 \leq 1, t^2 = 1\}$ and a "lateral" surface $\{x^2 + y^2 + z^2 = 1, t^2 \leq 1\}$ isometric to the product of a 2-sphere with an interval, and these parts are identified along their common boundary, a pair of 2-spheres $\{x^2 + y^2 + z^2 = 1, t^2 = 1\}$. In the same way that we found embedded closed geodesics on the cylinder boundary in $\mathbb{R}^3$, we may find equilibrium surfaces given as the fixed sets of isometric involutions. For example, the set C^2 can be obtained by taking the involution $(x, y, z, t) \longrightarrow (x, y, -z, t)$, and as before,

we have a collection of equilibrium spheres given by taking $t = c$, $c^2 < 1$. Unfortunately, the piece $\{x^2 + y^2 + z^2 = 1, t^2 \leq 1\}$ is not flat, so it is not isometric to a region in $\mathbb{R}^3$ with identifications, so the main advantage of the method used in the above sections is lessened.

We may however apply some of the techniques of the previous sections if we shrink the lateral part of the above manifold down to zero, thus obtaining a metric on the 3-sphere by taking two 3-discs in $\mathbb{R}^3$ and identifying them along their boundary. This is similar to the situation where we had two solid tori identified along their torus "rim," and in this situation all the curvature is concentrated around a rim which is a 2-sphere.

In order to get a better picture of the situation, we may consider first of all the analogous case in one lower dimension. If we take a can of height $2h$, i.e., the boundary of the set $\{x^2 + y^2 \leq 1, u^2 \leq h^2\}$, then the condition that a curve be in equilibrium is given by $\pi - 2\alpha + 2h \tan \alpha = p/q\, 2\pi$ for integers p and q. In particular, if $h = 0$ we have a double disc and we have a closed geodesic on this double disc if and only if α is a rational multiple of π. The only such curves which pass through the centers of the discs are doubly covered diameters, and if a geodesic is not closed, then it is dense on the complement of two circular discs about the centers. The circle rim itself is in equilibrium because there is no preferred direction in which it can be displaced.

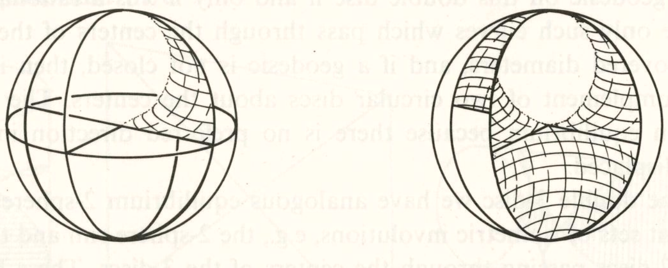
In the double 3-disc we have analogous equilibrium 2-spheres given as fixed point sets of isometric involutions, e.g., the 2-sphere rim and the doubly covered 2-discs passing through the centers of the 3-discs. These last examples have the property that the intersection with the rim sphere consists of geodesics and the surface is orthogonal to its rim at each place where it touches the rim. We may express this by placing the two 3-discs in $\mathbb{R}^3$ so that their boundary 2-spheres coincide and such that the boundary circles of the two 2-discs in the 3-discs are identified.



The two tension vectors directed into the 3-discs then represent two copies of the same vector, so their resultant vector is also pointing directly to the center of the sphere, orthogonal to the boundary 2-sphere. This is the equilibrium condition at a point of the rim sphere where two pieces of minimal surface meet along a great circle arc.

We may make use of this equilibrium condition to construct surfaces of arbitrary genus embedded in equilibrium in the double 3-disc by making use of the construction used by Lawson in his thesis [1] to obtain closed minimal submanifolds of S^3.

In the preliminary construction, Lawson described a framework on S^3 consisting of two great circles through the poles at right angles to one another, together with their horizontal diameters. One of the quadrilaterals composed of two great quarter circles and two radii is filled in with a (unique) minimal surface of the topological type of a disc, and this is reflected several times to give a surface in the 3-disc bounded by the four great semicircles.



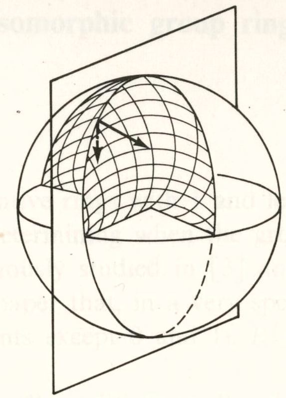
To obtain the part of the surface lying in the second 3-disc, we take the image of this surface under the antipodal involution $(x, y, z) \to (-x, -y, -z)$, and we then identify the two pieces along the four great semicircles along the boundary.

If we take any point P on the interior of one of these edges, then the tension vectors tangent to the surfaces at the point P in the two 3-discs correspond to one another under the isometric involution given by reflecting through the great circle.

Consequently, the resultant vector is orthogonal to the 2-spehre rim and the surface is in equilibrium at each such point.

At the north or south pole, the contributions from the two pieces of surface in each of the two 3-discs are interchanged under the isometric map given by rotation through $\pi/2$, so the resultant tension vector is directed orthogonal to the rim sphere.

The closed surface in equilibrium obtained in this way has two vertices, four edges, and two faces, so it represents a torus (since it is embedded).



Similarly if we take n great circles, we obtain an embedded closed surface in equilibrium with two vertices, $2n$ edges, and two faces, so we have Euler characteristic $2(2-n)$. Note that the case $n = 1$ leads to the doubly covered disc as an equilibrium 2-sphere in the double disc.

Bibliography

[1] Lawson, H. B. *Codimension-one Foliations of Spheres.* Ann. of Math., 94 (1971), 494-503.

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