BOL. SOC. BRAS. MAT. 7(1976), 59–63 

# Coefficient rings of isomorphic group rings

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Let $R$, $S$ be commutative rings with 1 and let $\langle x \rangle$ be an infinite cyclic group. The problem of determining when the group rings $R\langle x \rangle$ and $S\langle x \rangle$ are isomorphic was previously studied in [3] and in [4]. In particular, it was shown in the latter paper that, in a very special case (namely when $R$ and $S$ have no idempotents except 0 and 1), $R\langle x \rangle \simeq S\langle x \rangle$ implies $R$ and $S$ are subisomorphic.

In this paper, we drop all conditions on $R$ and $S$ from the above theorem. Specifically, we prove that $R\langle x \rangle \simeq S\langle x \rangle$ always implies $R$ and $S$ are subisomorphic.

## 1. Preliminary Results.

In this section, we note some basic facts which will be required in the proof of the main theorem. The proofs of the first two lemmas previously appeared in [4] and will not be repeated here.

**Lemma 1.1.4.** *Let $R$ be a commutative ring with 1. If $\Sigma a_i x^i$ is a unit in $R\langle x \rangle$ and $(\Sigma a_i x^i)^{-1} = \Sigma b_i x^i$, then $\Sigma a_i b_{-i} = 1$ and $a_i b_j$ is nilpotent whenever $i + j \neq 0$.*

**Lemma 1.2.4.** *Let $R$ be a commutative ring with 1. Then $x \longrightarrow \Sigma a_i x^i$ induces an $R$-automorphism of $R\langle x \rangle$ if and only if the following two conditions hold:*

*(i) $\Sigma a_i x^i$ is a unit in $R\langle x \rangle$.*

*(ii) $a_i$ is nilpotent whenever $i \neq 1, -1$.*

However, the major result required in the proof of the main theorem is the following:

**Lemma 1.3.** *Let $R$ be a commutative ring with 1 and let $\Sigma a_i x^i$ be a unit in $R\langle x \rangle$ such that $a_0$ is nilpotent. Then:*

*(i) $R\langle \Sigma a_i x^i \rangle$ is a group ring.*

*(ii) If $r = g(x)\ ((\Sigma a_i x^i) - 1)$ where $r \in R$ and $g(x) \in R\langle x \rangle$, then $r = 0$.*

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*Recebido em Julho de 1976.
*This work supported in part by National Research Council of Canada grant A-8775.

**Remark.** Result (i) in the above Lemma simply means that powers of $\Sigma a_i x^i$ are independent over $R$.

*Proof.* Before beginning, we note that since $\Sigma a_i x^i$ is a unit in $R\langle x\rangle$, Lemma 1.1 says that $\Sigma a_i b_{-1} = 1$ and $a_i b_j$ is nilpotent whenever $i + j \neq 0$ (where $\Sigma b_i x^i = (\Sigma a_i x^i)^{-1}$). Note that if $P$ is a prime ideal of $R$ and $a_i \notin P$, then $b_j \in P$ whenever $i + j \neq 0$ since $a_i b_j$ is nilpotent. Also $b_{-i} \notin P$ since $\bar{a}_i \bar{b}_{-i} = \bar{1}$ in $R/P$. This then implies that $a_j \in P$ for all $j \neq i$ since $a_j b_{-i}$ is nilpotent. We conclude that each of the pairs $\{a_i, b_{-i}\}$ have associated disjoint sets of prime ideals $P$ for which $a_i \notin P$ and $b_{-i} \notin P$. In particular, we note that $a_i a_j$ and $b_i b_j$ are nilpotent if $i \neq j$.

Now let us prove (i). Assume to the contrary that $\Sigma c_j (\Sigma a_i x^i)^j = 0$ for $c_j \in R$. We will first show that each $c_j$ is nilpotent – if not, let $n$ be the largest integer such that $c_n$ is not nilpotent and let $P$ be a prime ideal of $R$ such that $c_n \notin P$. We know also that there is a unique $i$ such that $a_i \notin P$ and $b_{-i} \notin P$. In $R/P\langle x\rangle$, we have

$$\sum_{j>0} \bar{c}_j (\bar{a}_i x^i)^j + \bar{c}_0 + \sum_{j<0} \bar{c}_j (\bar{b}_{-i} x^{-i})^{-j} = 0.$$

Note that since $\bar{a}_0 = 0$, all powers of $x$ involved are different. If $n > 0$, we get $\bar{c}_n \bar{a}_i^n = 0$ which means $\bar{c}_n = 0$ or $\bar{a}_i = 0$ and contradicts the choice of $P$. If $n = 0$, we get $\bar{c}_0 = 0$ which is impossible. If $n < 0$, we get $\bar{c}_n \bar{b}_{-i}^{-n} = 0$ which is again a contradiction. Hence $c_j$ is nilpotent for all $j$.

Let $I$ be the (nilpotent) ideal of $R$ generated by the $c_j$, the nilpotent $a_i$ and $b_i$, all terms $a_i b_j$ where $i + j \neq 0$, and all terms $a_i a_j$ and $b_i b_j$ where $i \neq j$. Choose (if possible) $k$ so that all $c_j$ lie in $I^k$ but some $c_j$ does not lie in $I^{k+1}$. In $R/I^{k+1}\langle x\rangle$, $\Sigma c_j (\Sigma a_i x^i)^j = 0$ becomes

$$\sum_{j>0} \bar{c}_j (\Sigma \bar{a}_i^j x^{ij}) + \bar{c}_0 + \sum_{j<0} \bar{c}_j (\Sigma \bar{b}_i^{-j} x^{-ij}) = 0$$

since $\bar{c}_j \bar{a}_i \bar{a}_l = 0$ and $\bar{c}_j \bar{b}_i \bar{b}_l = 0$ if $i \neq l$. Let $m$ be an integer such that $a_m$ is not nilpotent- note that $m$ exists since $\Sigma a_i b_{-i} = 1$ and $m \neq 0$ by assumption. Multiplying by $\bar{a}_m$ yields

$$\sum_{j>0} \bar{c}_j \bar{a}_m^{j+1} x^{mj} + \bar{c}_0 \bar{a}_m + \sum_{j<0} \bar{c}_j \bar{a}_m \bar{b}_{-m}^{-j} x^{mj} = 0$$

since $\bar{c}_j \bar{a}_m \bar{a}_i = 0$ if $m \neq i$ and $\bar{c}_j \bar{a}_m \bar{b}_i = 0$ if $m + i \neq 0$. Since $m \neq 0$, all powers of $x$ involved are different. Thus we have $\bar{c}_j \bar{a}_m^{j+1} = 0$ if $j \geq 0$ and $\bar{c}_j \bar{a}_m \bar{b}_{-m}^{-j} = 0$ if $j < 0$. In particular, we conclude that there exists $l > 0$ such that $\bar{c}_j \bar{a}_m^l \bar{b}_{-m}^l = 0$ for all $j$ and for all $m$ such that $a_m$ is not nilpotent.



However, $\Sigma \bar{a}_i \bar{b}_{-i} = \bar{1}$ in $R/I^{k+1}$. Thus $(\Sigma \bar{a}_i \bar{b}_{-i})^l = \bar{1}$ and $\bar{c}_j (\Sigma \bar{a}_j \bar{b}_{-i})^l = \bar{c}_j$. But this becomes $\bar{c}_j (\Sigma \bar{a}_i^l \bar{b}_{-i}^l) = \bar{c}_j$ since $\bar{c}_j \bar{a}_i \bar{a}_s = 0$ if $i \neq s$. This means, by the previous paragraph, that $\bar{c}_j = 0$. Thus we have shown that $c_j \in I^{k+1}$ for all $j$. Since $I$ is nilpotent, $c_j = 0$ for all $j$ and $R\langle \Sigma a_i x^i\rangle$ is a group ring.

Next we proceed to prove (ii). Thus we have $r = g(x)((\Sigma a_i x^i) - 1)$ where $r \in R$ and $g(x) \in R\langle x\rangle$. If $P$ is a prime ideal of $R$, this becomes $\bar{r} = \bar{g}(x)(\bar{a}_i x^i) - 1)$ in $R/P\langle x\rangle$ where $i$ is the unique integer such that $a_i \notin P$ and $\bar{g}(x)$ is simply $g(x)$ reduced mod $P$. Since $R/P\langle x\rangle$ is an integral domain and $i \neq 0$ by assumption, this implies that $g(x) \in P\langle x\rangle$. Since this is true for all $P$, we conclude that the coefficients of $g(x)$ are nilpotent.

Let $J$ be the (nilpotent) ideal of $R$ generated by all nilpotent $a_i$ and $b_i$, the coefficients of $g(x)$, and all $a_i a_j$ and $b_i b_j$ where $i \neq j$. If $g(x) = \Sigma g_i x^i$, assume (if possible)that all $g_i \in J^s$ but that some $g_i \notin J^{s+1}$.

Choose $n$ so that $a_n$ is not nilpotent and pass to $R/J^{s+1}\langle x\rangle$. Multiplying $\bar{r} = \bar{g}(x)((\Sigma \bar{a}_i x^i) - \bar{1})$ by $\bar{a}_n$ we obtain $\bar{r}\bar{a}_n = \bar{g}(x)(\bar{a}_n^2 x^n - \bar{a}_n)$ since $\bar{g}(x)\bar{a}_n \bar{a}_i = 0$ if $n \neq i$.

Choose $t$ maximal so that $g_+ \notin J^{s+1}$. Assume that $n \gtrless 0$ for the moment. If $t \neq -n$, we have $\bar{g}_+ \bar{a}_n^2 = 0$ since $n > 0$. If $t = -n$, we have $\bar{g}_{+-n} \bar{a}_n^2 - \bar{g}_+ \bar{a}_n = 0$. Continuing, we obtain:

$$\bar{g}_+ \bar{a}_n = \bar{g}_{+-n} \bar{a}_n^2 = \bar{g}_{+-2n} \bar{a}_n^3 = \ldots = \bar{g}_{+-kn} \bar{a}_n^{k+1}.$$

Since $g_{+-kn} = 0$ for some $k$ (note that $n \neq 0$) we have $\bar{g}_+ \bar{a}_n = 0$.

Now assume $n < 0$. If $t \neq 0$, we have $\bar{g}_+ \bar{a}_n = 0$. If $t = 0$, we have $\bar{g}_{++n} \bar{a}_n = \bar{g}_+ \bar{a}_n^2$. Also $\bar{g}_{++2n} \bar{a}_n = \bar{g}_{++n} \bar{a}_2^2$. Since $g_{++kn} = 0$ for some $k$, we have $\bar{g}_+ \bar{a}_n^p = 0$ for some $p$.

Thus we have $\bar{g}_+ \bar{a}_n^p = 0$ for some $p$ in all cases, and $p$ can be chosen so this true for any non-nilpotent $a_n$. As before, $(\Sigma \bar{a}_i \bar{b}_{-i})^p = \bar{1}$ becomes $\bar{g}_+ = \bar{g}_+ (\Sigma \bar{a}_i^p \bar{b}_{-i}^p) = 0$. Thus, we have $g_+ \in J^{s+1}$ which is a contradiction.

## 2. Main Theorem

We now prove the main result of this paper. Recall that two rings $R$ and $S$ are subisomorphic if $R$ can be embedded in $S$ and $S$ can be embedded in $R$.

**Theorem 2.1.** *Let $R$, $S$ be commutative rings with 1. Then $R\langle x\rangle \simeq S\langle x\rangle$ implies $R$ and $S$ are subisomorphic.*

*Proof.* We may as well assume $R\langle x\rangle = T\langle \Sigma a_i x^i\rangle$ where $\Sigma a_i x^i$ is a unit in $R\langle x\rangle$. Hence, Lemma 1.1 tells us that $(\Sigma a_i x^i)^{-1} = \Sigma b_i x^i$ where $\Sigma a_i b_{-i} = 1$ and $a_i b_j$ is nilpotent if $i + j \neq 0$. As in the proof of Lemma 1.3, we know as well that $a_i a_j$ and $b_i b_j$ are nilpotent whenever $i \neq j$.

First, let us consider the case where $a_0$ is nilpotent (possibly even $a_0 = 0$). In this case, part (i) of Lemma 1.3 says that $R\langle\Sigma a_i x^i\rangle$ is a group ring. Hence $R\langle\Sigma a_i x^i\rangle \subset R\langle x\rangle = T\langle\Sigma a_i x^i\rangle$. Define $\alpha: R\langle\Sigma a_i x^i\rangle \longrightarrow T\langle\Sigma a_i x^i\rangle$ to be the inclusion map and let $\rho: T\langle\Sigma a_i x^i\rangle \longrightarrow T$ be the augmentation homomorphism obtained by factoring out the augmentation ideal $\Delta_T$ of $T\langle\Sigma a_i x^i\rangle$. Clearly, the augmentation ideal $\Delta_R$ of $R\langle\Sigma a_i x^i\rangle$ is contained in ker $\rho\circ\alpha$. Conversely, if $r \in R \cap \ker \rho\circ\alpha$, then $r = g(x)((\Sigma a_i x^i) - 1)$ for some $g(x) \in R\langle x\rangle$. Part (ii) of Lemma 1.3 then tells us that $r = 0$.

Thus ker $\rho\circ\alpha$ is exactly the augmentation ideal $\Delta_R$ of $R\langle\Sigma a_i x^i\rangle$ and $R \simeq R\langle\Sigma a_i x^i\rangle/\Delta_R$ is embedded in $T$ as required.

Next, we must consider the case where $a_0$ is not nilpotent. Since $\sum a_i b_{-i} = 1$ and $a_i b_j$ is nilpotent whenever $i + j \neq 0$, we know that $a_0 b_0 a_0 - a_0$ is nilpotent and $(a_0 b_0)^2 = a_0 b_0 + n$ where $n$ is nilpotent. Hence there exists an idempotent $e$ in $R$ such that $e - a_0 b_0$ is nilpotent ([2] p. 72). Since central idempotents in a group ring have finite support group [1], e also lies in $T$. Since $e \in R \cap T$, $e(R\langle x\rangle) = (eR)\langle x\rangle = (eR)\langle ex\rangle$ and $e(T\langle\Sigma a_i x^i\rangle) = (eT)\langle\Sigma a_i x^i\rangle = (eT)\langle e(\Sigma a_i x^i)\rangle$. Similarly for the idempotent $1 - e$.

But $(1-e)(\Sigma a_i x^i) - (1 - a_0 b_0)(\Sigma a_i x^i)$ is nilpotent. Hence $(1-e)(\Sigma a_i x^i)$ has constant coefficient $a_0 - a_0 b_0 a_0 + m$ where $m$ is nilpotent. Since $a_0 - a_0 b_0 a_0$ is nilpotent, the equality $(1 - e)R\langle x\rangle = (1 - e)T\langle\Sigma a_i x^i\rangle$ belongs to the initial case of our proof. Hence we know that $(1 - e)R$ can be embedded in $(1 - e)T$.

Also $ea_i$ is nilpotent if $i \neq 0$, so in $(eR)\langle x\rangle = (eT)\langle\Sigma a_i x^i\rangle$, we may assume that $a_i$ is nilpotent whenever $i \neq 0$. If we could conclude that $eR \subset eT$, then we would have $R$ embedded in $T$ since $R \simeq eR + (1 - e)R$. Thus we have reduced the problem to the following case: $R\langle x\rangle = T\langle\Sigma a_i x^i\rangle$ where $a_0$ is a unit and $a_i$ is nilpotent if $i \neq 0$.

From now on, we will deal solely with the above case. Since $x$ is a unit in $T\langle\Sigma a_i x^i\rangle$, Lemma 1.1 says that $x = \sum c_j(\Sigma a_i x^i)^j$ for some $c_j \in T$ such that there exist $d_j \in T$ with $\Sigma c_j d_{-j} = 1$ and such that $c_j d_k$ is nilpotent whenever $j + k \neq 0$. As before, we conclude that $c_j c_k$ and $d_j d_k$ are nilpotent whenever $j \neq k$.

Choose $c_k \neq 0$. The $T$-homomorphism of $T\langle\Sigma a_i x^i\rangle$ defined by $\Sigma a_i x^i \longrightarrow c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}$ is an automorphism of $T\langle\Sigma a_i x^i\rangle$ by Lemma 1.2 since

$$[c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}][(\sum_{l\neq -k} d_l)(\Sigma a_i x^i) + d_{-k}(\Sigma a_i x^i)^{-1}] = 1 + n_0$$

where $n_0$ is nilpotent, so $c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}$ is a unit in $T\langle\Sigma a_i x^i\rangle$.



Hence $R\langle x\rangle = T\langle c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}\rangle$. We will now show that $c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}$ satisfies the condition of Lemma 1.3, namely, that it has a nilpotent constant term (in $R\langle x\rangle$). Once this is done, the Lemma can be used exactly as before to conclude that $R$ can be embedded in $T$.

Recall that $x = \Sigma c_j(\Sigma a_i x^i)^j$ and that $a_i$ is nilpotent if $i \neq 0$. Let $P$ be a prime ideal of $R$. In $R/P\langle x\rangle$, we obtain $x = \Sigma \bar{c}_j \bar{a}_0^j$ since all other $a_i$ are nilpotent. In fact, since $P\langle x\rangle$ is a prime ideal of $R\langle x\rangle$, we have $x = \bar{c}_s \bar{a}_0^s$ for some particular $s$. Since $a_0$ is a unit, it follows that $c_s = m_s x + \sum_{i\neq 1} m_{i,s} x^i$ where $m_{i,s}$ is nilpotent for all $i$. This is true for any non-nilpotent $c_s$ since, in that case, a prime ideal $P$ of $R$ can be found such that $\bar{c}_s \neq 0$ in $R/P\langle x\rangle$. Therefore, the constant term of $c_k(\Sigma a_i x^i) + (\sum_{l\neq k} c_l)(\Sigma a_i x^i)^{-1}$ is nilpotent, since the only non-nilpotent entries in the $c_i$'s are coefficients of $x$ and the only non-nilpotent $a_i$ is $a_0$. Thus, Lemma 1.3 is applicable and we have $R$ embedded in $T$.

The identical argument in the other direction shows that $T$ can be embedded in $R$. Hence $R$ and $T$ (or $R$ and $S$) are subisomorphic.

## References


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