On G-spaces*

Peter Hilton

0. Introduction.

In [4] the authors proved that, if the group G acts on the nilpotent group Q then G acts *nilpotently* on Q (in the sense of [1,5]) if and only if it acts nilpotently on Q_{ab}, the abelianization of Q. This result suggests a generalization to groups acting on nilpotent spaces. For if X is the Eilenberg-Maclane space $K(Q, 1)$ than X is a nilpotent space and we may interpret the result quoted as saying that G acts nilpotently on $\pi_1 X$ if and only if G acts nilpotently on $H_1 X$.

The present note provides then, the anticipated generalization to the case of a group G acting on a nilpotent space X. We may either suppose that G acts as a group of base-point-preserving homeomorphisms of X or as group of based homotopy classes of self-homotopy-equivalences of X. We prove that G acts nilpotently on $\pi_i X$, $i \leq n$, if and only if G acts nilpotently on $H_i X$, $i \leq n$ (Theorem 2.1.).

In Section 1 we establish the algebraic context for Theorem 2.1. It turns out that the usual action of the group $\pi_1 X$ on the groups $\pi_m X$ and $H_m \tilde{X}$, where $\tilde{X}$ is the universal cover of X, is enriched to an action of the G-group $\pi_1 X$ on the G-groups $\pi_m X$ and $H_m \tilde{X}$ which is compatible in precisely the sense of [4; (0,2)]. Thus the results on compatible G-actions proved in [4] are available to us and the appropriate algebraic context for our topological results is that of a G-group K acting compatibly on a commutative G-group N. This situation is studied in Section 1.

We remark that there is a small difference between the statement of Theorem 1.1 of [4] and the result quoted at the beginning of the Introduction. For if a group G acts nilpotently on a group Q, then Q is necessarily a nilpotent group. On the other hand a group G may act nilpotently on the homotopy groups of a space X without X being a nilpotent space. Indeed the trivial action (of any group G) on the homotopy groups X will be nilpotent provided $\pi_1 X$ is nilpotent. Thus it is necessary for us to postulate in our main theorem that X should be nilpotent.

*Recebido em Agosto de 1976.

Section 2 closes with the answer to a question raised by Joseph Roitberg and independently by David Singer, relating nilpotent fibrations [2] to quasi-nilpotent fibrations [3]. In the final section we generalize the argument used in proving Theorem 2.1 to study the case of a *quasi-nilpotent G-fibration* $F \longrightarrow E \longrightarrow B$. That is, we have a fibration which is quasi-nilpotent [3] in the sense that all spaces are connected and $\pi_1 B$ operates nilpotently on the homology of F; and we further assume that G acts on F, E, B, in either of the manners described above, in such a way that the maps of the fibration are G-maps. It is then possible, given that the actions of G on the homology groups of two of the spaces F, E, B are nilpotent (up to certain dimensions), to infer that the action of G on the homology group of the third space is also nilpotent (up to a contingent dimension).

This paper was written while the author was in Brazil at the invitation of the Pontifícia Universidade Católica of Rio de Janeiro. He is most grateful to PUC for the kind invitation and the opportunity to talk with colleagues and work under very delightful conditions. He is also grateful for the opportunity thereby provided to participate in ELAM III in Rio and in the Escola de Álgebra in São Paulo.

1. Group theoretical preliminaries.

In [4] the authors discussed the situation of a G-group K acting on a G-group N. Thus the group G acts on the groups K and N; and K acts on N in a manner *compatible* with the G-actions in the sense that

$$x(a \cdot b) = xa \cdot xb, x \in G, \ a \in K, \ b \in N. \tag{1.1}$$

We resume this theme here but we will insist that N is *commutative*. Since K acts on N, we may then form the homology groups $H_m(K; N)$. Now the condition (1.1) precisely asserts that for each $x \in G$, the map $x : N \longrightarrow N$, given by $b \longrightarrow xb, b \in N$, is a module-map with respect to the map $x : K \longrightarrow K$, given by $a \longrightarrow xa$, $a \in K$. Thus G acts on $H_m(K; N)$ and we will call this the induced *diagonal* action of G on $H_m(K; N)$. It is with this induced diagonal action that we will be concerned in this section. In preparation for our main result, we prove some preliminary lemmas.

Lemma 1.1. *Let A and B be abelian groups and let us form $A \otimes B$, $Tor(A, B)$. Let G act on A and B and hence diagonally on $A \otimes B$, $Tor(A, B)$. If G acts nilpotently on A and B, it acts nilpotently on $A \otimes B$ and on $Tor(A, B)$.*

Proof. The first assertion is Lemma 1.2 of [4]. As to the second assertion, we prove it by induction on $nil_G B$. If $nil_G B = 1$ the result is contained in

Corollary I.4.16 of [2]. If $nil_G B = c+1$, set $\Gamma = \Gamma_G^c B$. Then $\Gamma \rightarrowtail B \twoheadrightarrow B/\Gamma$ is a short exact sequence of G-modules and we may assume that G acts nilpotently on $Tor(A, \Gamma)$ and on $Tor(A, B/\Gamma)$. But $Tor(A, \Gamma) \longrightarrow Tor(A, B) \longrightarrow Tor(A, B/\Gamma)$ is an exact sequence of G-modules, so that, by Proposition I.4.3 of [2], G acts nilpotently on $Tor(A, B)$.

Lemma 1.2. *Let K be a G-group. Then G acts on the homology groups $H_m K$. If G acts nilpotently on K, then G acts nilpotently on the homology groups $H_m K$.*

Proof. We argue by induction on $nil_G K$. If $nil_G K = 1$, the action of G on K, and hence $H_m K$, is trivial. If $nil_G K = c + 1$, we set $\Gamma = \Gamma_G^c K$ and have a central extension $\Gamma \rightarrowtail K \twoheadrightarrow K/\Gamma$ with G acting trivially on Γ. In the Lyndon-Hochschild-Serre spectral sequence we have

$$E_{pq}^2 = H_p(K/\Gamma; H_q\Gamma) \tag{1.2}$$

and we may assume that G acts nilpotently on $H_p(K/\Gamma)$, for all p. Since, in (1.2), K/Γ acts trivially on $H_q\Gamma$, we have the short exact sequence of G-modules

$$H_p(K/\Gamma) \otimes H_q\Gamma \rightarrowtail E_{pq}^2 \twoheadrightarrow Tor(H_{p-1}(K/\Gamma), H_q\Gamma)$$

Appeal to Corollary I.4.16 of [2] (or to Lemma 1.1) shows that the G-actions on $H_p(K/\Gamma) \otimes H_q\Gamma$ and on $Tor(H_{p-1}(K/\Gamma), H_q\Gamma)$ are nilpotent and so therefore, by Proposition I.4.3 of [2], is the G-action on E_{pq}^2. Passage through the spectral sequence then shows that the G-action on E_{pq}^∞ is nilpotent and so finally by further applications of Proposition I.4.3 of [2] is the G-action on $H_m K$.

Remark. Lemma 1.2, in the case that K is commutative, was proved as Theorem I.4.17 of [2].

We are now ready to state and prove the main result of this section.

Theorem 1.3. *Let the G-group K act compatibly on the commutative G-group N, so that (1.1) is satisfied, and let G act on the homology groups $H_m(K; N)$ by the induced diagonal action. If G acts nilpotently on K and on N, and if K acts nilpotently on N, then G acts nilpotently on $H_m(K; N)$.*

Proof. We argue by induction on $nil_K N$. If $nil_K N = 1$, we have a short exact sequence of G-modules

$$H_m K \otimes H \rightarrowtail H_m(K; N) \twoheadrightarrow Tor(H_{m-1}K, N)$$

where the G-actions are, in all three cases diagonal. Since G acts nilpotently on K, it follows from Lemma 1.2 that G acts nilpotently on $H_m K$, $H_{m-1}K$.

Since G acts nilpotently on N, it follows from Lemma 1.1 that G acts nilpotently on $H_mK \otimes N$ and on $Tor(H_{m-1}K, N)$. We again invoke Proposition I.4.3 of [2] to infer that G acts nilpotently on $H_m(K; N)$. If $nil_K N = c + 1$, set $\Gamma = \Gamma_K^c N$, so that, by Lemma 1.11 of [4], $\Gamma \rightarrowtail N \twoheadrightarrow N/\Gamma$ is a short exact sequence of K-modules *and* of G-modules. It follows that the induced coefficient sequence

$$H_m(K; \Gamma) \longrightarrow H_m(K; N) \longrightarrow H_m(K; N/\Gamma)$$

is an exact sequence of G-modules. We may suppose inductively that G acts nilpotently on $H_m(K; N/\Gamma)$ and it certainly acts nilpotently on $H_m(K; \Gamma)$ since K acts trivially on Γ. Thus, once again, we infer that G acts nilpotently on $H_m(K; N)$ and the theorem is proved.

We require one further group-theoretical result, of a negative nature, before proceeding to the topological applications.

Proposition 1.4. *Let the G-group K act compatibly on the commutative G-group N. If G acts nilpotently on K and K acts nilpotently on N, then G acts nilpotently on N if and only if G acts nilpotently on* $H_0(K; N)$.

Proof. We already know from Theorem 1.3 that if G acts nilpotently on N then G acts nilpotently on $H_0(K; N)$. Conversely, suppose that G does not act nilpotently on N. We argue by induction on $nil_K N$. If $nil_K N = 1$ *then* $H_0(K; N) = N$ and the conclusion is obvious. If $nil_K N = c + 1$, set $\Gamma = \Gamma_K^c N$ so that $\Gamma \longrightarrow N \longrightarrow\!\!\!\!\!\rightarrow N/\Gamma$ is a short exact sequence of K-modules and of G-modules. It follows that the induced coefficient sequence

(1.3) $$H_1(K; N/\Gamma) \longrightarrow H_0(K; \Gamma) \rightarrowtail H_0(K; N) \twoheadrightarrow H_0(K; N/\Gamma)$$

is an exact sequence of G-modules. Since G does not act nilpotently on N, we have two possibilities: (i) G does not act nilpotently on N/Γ, and (ii) G acts nilpotently on N/Γ but G does not act nilpotently on Γ. In case (i) the inductive hypothesis tells us that G does not act nilpotently on $H_0(K; N/\Gamma)$ and hence, by (1.3), G does not act nilpotently on $H_0(K; N)$. In case (ii) we know that G does not act nilpotently on $H_0(K; \Gamma)$ and, by Theorem 1.3, G acts nilpotently on $H_1(K; N/\Gamma)$. It thus again follows from (1.3) that G does not act nilpotently on $H_0(K; N)$.

2. Nilpotent G-spaces.

Let X be a nilpotent space [2] on which the abstract group G acts as a group of base-point-preserving homeomorphisms.* If $\tilde{X}$ is the universal

*It would suffice that G be represented as a group of homotopy classes of self-homotopy-equivalences of X.

Recall that a space X is nilpotent if $\pi_1 X$ is nilpotent and acts nilpotently on the higher on the homology groups $H_i X, i \geq 1$, as well as on the homology groups $H_i X$ of the universal cover $\tilde{X}$ of X.

covering space of X then, of course, G also acts on $\tilde{X}$, in such a way that the projection $\tilde{X} \longrightarrow X$ is a G-map. Moreover there are induced actions of G on the homotopy and homology groups of X and $\tilde{X}$; and it is easy to verify that the G-group $\pi_1 X$ acts compatibly on the commutative G-groups $\pi_n X$, $H_n \tilde{X}$. Thus we are in a position to apply the results of preceding section and we prove.

Theorem 2.1. *Let the group G act as a group of base-point-preserving homeomorphisms of the nilpotent space X, and let* $n \geq 1$. *Then the following statements are equivalent*:

(i) *G acts nilpotently on* $\pi_i X$, $i \leq n$;
(ii) *G acts nilpotently on* $H_i X$, $i \leq n$;
(iii) *G acts nilpotently on* $\pi_1 X$ *and on* $H_i \tilde{X}$, $i \leq n$.

Proof. We first establish the equivalence of (i) and (iii). To do this it suffices (replacing X by $\tilde{X}$) to suppose X simplyconnected and to show that then (i) and (ii) are equivalent. Thus let X be simply-connected and suppose that G acts nilpotently on $\pi_i X$, $i \leq n$. Form the Postnikov tower, for $m \leq n$,

(2.1) $$\begin{array}{c} K(\pi_m X, m) \subseteq X_m \\ \downarrow \\ X_{m-1} \end{array}$$

Then G acts on the fibration (2.1). We assume inductively that G acts nilpotently on the homology of X_{m-1}, certainly true if $m = 2$. By Lemma II.2.17 of [2], G acts nilpotently on the homology of $K(\pi_m X, m)$. In the Serre spectral sequence we have $E^2_{pq} = H_p(X_{m-1}; H_q(\pi_m X, m))$. Thus by exactly the same argument as that of the first step in the proof of Theorem 1.3, G acts nilpotently on E^2_{pq}. Passage through the spectral sequence shows that G acts nilpotently on the homology of X_m.

We conclude that G acts nilpotently on the homology of X_n. But the map $X \longrightarrow X_n$ induces homotopy isomophisms up to dimension n and a surjection of π_{n+1} (since $\pi_{n+1} X_n = 0$). Thus $X \longrightarrow X_n$ induces homology isomorphisms up to dimension n, so that G acts nilpotently on $H_i X$, $i \leq n$.

Now suppose that G acts nilpotently on $H_i X$, $i \leq n$; recall that we are supposing X simply-connected. Form the Cartan-Serre-Whitehead tower*, for $m \leq n$,

*Recall that $X_{(m)}$ is obtained from X by killing the first $(m-1)$ homotopy groups of X. Since X is simply-connected, $X_{(2)} = X$.

$$(2.2)\qquad \begin{array}{c} K(\pi_{m-1}, m-2) \subseteq X_{(m)} \\ \downarrow \\ X_{(m-1)} \end{array}$$

Then G acts on the fibration (2.2). We assume inductively that G acts nilpotently on $H_i X_{(m-1)}$, $i \leq n$. This is certainly true by hypothesis if $m = 3$, since $X_{(2)} = X$. Since $H_{m-1}X_{(m-1)} = \pi_{m-1}X$, G acts nilpotently on $\pi_{m-1}X$ and hence on the homology of $K(\pi_{m-1}X, m-2)$. In the Serre spectral sequence we have $E^2_{pq} = H_p(X_{(m-1)}; H_q(\pi_{m-1}, m-2)$. Thus G acts nilpotently on E^2_{pq} if $p + q \leq n$. Passage through the spectral sequence shows that G acts nilpotently on $H_i X_{(m)}$, $i \leq n$. Thus G acts nilpotently on $H_i X_{(m)}$, $i \leq n$, $m \leq n$. Since $H_m X_{(m)} = \pi_m X$, it follows that G acts nilpotently on $\pi_i X$, $i \leq n$.

It ramains to prove the equivalence of (ii) and (iii); now, of course, we merely assume X nilpotent. We exploit the Cartan spectral sequence for the covering $\tilde{X} \longrightarrow X$ which may be regarded as the Serre spectral sequence of the 'fibration'.

$$(2.3)\qquad \tilde{X} \longrightarrow X \longrightarrow K(\pi_1 X, 1)$$

Then G acts on the fibration (2.3). We suppose that G acts nilpotently on $\pi_1 X$ and on $H_i\tilde{X}$, $i \leq n$. It then follows from Theorem 1.3 that G acts nilpotently on $E^2_{pq} = H_p(\pi_1 X; H_q\tilde{X})$, provided that $p + q \leq n$. (Recall that, since X is nilpotent, $\pi_1 X$ operates nilpotently on $H_q\tilde{X}$). Passage through the spectral sequence shows that G acts nilpotently on $H_i X$, $i \leq n$.

Finally suppose that G acts nilpotently on $H_i X$, $i \leq n$. Since G acts nilpotently on $H_1 X$, it follows from Theorem 1.1 of [4] that G acts nilpotently on $\pi_1 X$. If there exists $i \leq n$, such that does not act nilpotently on $H_i\tilde{X}$, let q be the smallest such i. By Proposition 1.4, G does not act nilpotently on $E^2_{0q} = H_0(\pi_1 X; H_q\tilde{X})$ and, by Theorem 1.3, G acts nilpotently on $E^2_{rs} = H_0(\pi_1 X; H_s\tilde{X})$ for all r, s with $r + s = q + 1$, $r \geq 2$. Consider the diagram

$$(2.4)\qquad \begin{array}{ccccccc} E^2_{0q} & \longrightarrow > & E^3_{0q} & \longrightarrow > \ \dots \longrightarrow > & E^{q+1}_{0q} & \longrightarrow > & E^\infty_{0q} \subseteq H_q X \\ \uparrow d^2 & & \uparrow d^3 & & \uparrow d^{q+1} & & \\ E^2_{2,q-1} & & E^3_{3,q-2} & & E^{q+1}_{q+1,0} & & \end{array}$$

Since G acts nilpotently on $E^2_{2,q-1}$ but does not act nilpotently on E^2_{0q}, it follows that G does not act nilpotently on E^3_{0q}. Continuing the same line of argument, we infer that G does not act nilpotently on $E^4_{0q}, \dots, E^{q+1}_{0q}, E^\infty_{0q}$. But E^∞_{0q} is a G-subgroup of $H_q X$ and G acts nilpotently on $H_q X$. Thus we have arrived at a contradiction, showing that no $i \leq n$ exists such that G does not act nilpotently on $H_i\tilde{X}$. The proof of the theorem is complete.

We may now answer a question raised by Joseph Roitberg and independently by David Singer. Let $F \longrightarrow E \longrightarrow B$ be a fibration with all spaces

connected. Then the fibration is *nilpotent* [2] if $\pi_1 E$ operates nilpotently on the homotopy groups of F (including $\pi_1 F$), and *quasi-nilpotent* [3] *if* $\pi_1 B$ operates nilpotently on the homology groups of F. We then have

Corollary 2.2. *Let $F \longrightarrow E \longrightarrow B$ be a fibration with all spaces connected. Then the fibration is nilpotent if and only if it is quasi-nilpotent and F is nilpotent.*

Proof. If the fibration is nilpotent, then, since $\pi_1 F$ operates on the homotopy groups of F through $\pi_1 F \longrightarrow \pi_1 E$, it follows that F is itself nilpotent. Thus, by Theorem 2.1, $\pi_1 E$ operates nilpotently on the homology group of F. Since $\pi_1 E \longrightarrow \pi_1 B$ is surjective, $\pi_1 B$ also operates nilpotently on the homology groups of F. Conversely if $\pi_1 B$ operates nilpotently on the homology groups of F so does $\pi_1 E$ and so, by Theorem 2.1, F being nilpotent, $\pi_1 E$ operates nilpotently on the homotopy groups of F.

3. A generalization.

The argument which completes the proof of Theorem 2.1 relates to the fibration (2.3). Essentially what we prove is that, since G acts nilpotently on the homology of the base $K(\pi_1 X, 1)$ it will act nilpotently on the homology of the fibre $\tilde{X}$, up to dimension n, if and only if it acts nilpotently on the homology of the total space X, up to dimension n.

We now generalize this conclusion. We are concerned with a fibration

$$(3.1)\qquad F \longrightarrow E \longrightarrow B$$

on which the group G acts. Moreover the fibration is to be *quasi-nilpotent* in the sense already described, meaning that all spaces are connected and $\pi_1 B$ operates nilpotently on the homology groups of F. It is then easy to verify that the G-group $\pi_1 B$ operates compatibly on the G-groups $H_q F$. We first prove a lemma.

Lemma 3.1. *Let* (3.1) *be a quasi-nilpotent fibration on which G acts. Then G acts diagonally on the homology groups $H_p(B; H_q F)$. Moreover (i) if G acts nilpotently on $H_p B$, $p \leq P$, and on $H_q F$, $q \leq Q$, then G acts nilpotently on $H_p(B; H_q F)$, $p \leq P$, $q \leq Q$; and (ii) if G acts nilpotently on $H_1 B$ and does not act nilpotently on $H_q F$, then G does not act nilpotently on $H_0(B; H_q F)$.*

Proof. The arguments exactly parallel the proof of Theorem 1.3 and Proposition 1.4. Thus, in particular, (i) is proved by induction on the $\pi_1 B$-nilpotency class of $H_q F$.

Theorem 3.2. *Let* (3.1) *be a quasi-nilpotent fibration on which G acts. Then* (*a*) *if G acts nilpotently on* H_pB, $p \leq P$, *and on* H_qF, $q \leq Q$, *then G acts nilpotently on* H_nE, $n \leq N = \min(P, Q)$;

(*b*) *if G acts nilpotently on* H_pB, $p \leq P$, *and on* H_nE, $n \leq N$, *then G acts nilpotently on* H_qF, $q \leq Q = \min(P - 1; N)$;

(*c*) *if G acts nilpotently on* H_qF, $q \leq Q$, *and on* H_nE, $n \leq N$, *then G acts nilpotently on* H_pB, $p \leq P = \min(Q + 1, N)$.

Proof. (a) follows immediately from Lemma 3.1 (i). For G acts nilpotently on E^2_{pq}, $p + q \leq N$, in the Serre spectral sequence and passage through the spectral sequence yields the result. The proof of (b) closely resembles the last step in the proof of Theorem 2.1. Were the conclusion false we take $q \leq Q$ minimal for the property that G does not act nilpotently on H_qF. Since $P \geq 1$ (otherwise the conclusion is trivially true!) we may apply Lemma 3.1 (ii) to infer that G does not act nilpotently on E^2_{0q} while, by Lemma 3.1 (i) it does act nilpotently on E^2_{rs}, $r + s = q + 1$, $r \geq 2$. This last holds because $q + 1 \leq P$, and $s \leq q - 1$. Arguing as in the proof of Theorem 2.1 (see diagram (2.4)) we infer that G does not act nilpotently on $E^\infty_{0q} \subseteq H_qE$, which is contradiction because $q \leq N$.

The proof of (c) proceeds similarly. Were the conclusion false we take $p \leq P$ minimal for the property that G does not act nilpotently on H_pB. Then, by Lemma 3.1 (i), G acts nilpotently on E^2_{rs}, $r + s = p - 1$, $s \geq 1$. This last holds because $r \leq p - 2$ and $p - 1 \leq Q$. Consider the diagram

$$\begin{array}{ccccccc} H_pB = E^2_{p0} & \supseteq & E^3_{p0} & \supseteq \ldots \supseteq & E^p_{p0} & \subseteq E^\infty_{p0} & \twoheadleftarrow H_pE \\ \downarrow d^2 & & \downarrow d^3 & & \downarrow d^p & & \\ E^2_{p-2,1} & & E^3_{p-3,2} & & E^p_{0,p-1} & & \end{array}$$

Since G acts nilpotently on $E^2_{p-2,1}$ but not on E^2_{p0}, it cannot act nilpotently on E^3_{p0} (by Proposition I.4.3 of [2]). Continuing in this way, we eventually infer that G does not act nilpotently on E^∞_{p0}. But this yields a contradiction since, with $p \leq N$, G does act nilpotently on H_pE.

Remark. Note that, as was to be expected, the values of N, P, Q which we get in Theorem 3.2 are precisely those of the generalized (and strengthened) Zeeman comparison theorem of [3].

References

[1] Peter Hilton, *Nilpotent actions on nilpotent groups*, Proc. Logic and Math. Conference, Springer Lecture Notes 450 (1975), 174-196.

[2] Peter Hilton, Guido Mislin and Joseph Rotberg, *Localization of Nilpotent Groups and Spaces*, Mathematics Studies 15, North Holland (1975).

[3] Peter Hilton and Joseph Roitberg, *On the Zeeman comparison theorem for the homology of quasi-nilpotent fibrations*, Oxford Quart. Journ. of Math. (1976) (to appear).

[4] Peter Hilton and David Singer, *On G-nilpotency* (to appear).

[5] Peter Hilton and Urs Stammbach, *On group actions on groups and associated series*, Math. Proc. Cam. Phil. Society 80 (1976) 43-55.

Battelle Research Center, Seattle;
Case Western Reserve University, Cleveland;
Pontifícia Universidade Católica, Rio de Janeiro.