Abstract non-linear Hyperbolic Equation in Hilbert Spaces

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In Lion's paper [5] it was proved that the Cauchy Problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sum_{j=1}^{n} \left\{ a_j \frac{\partial u}{\partial x_j} + b_j \frac{\partial^2 u}{\partial x_j \partial t} \right\} = 0$$

is well posed in $L^2(\Omega)$, where Ω is an open and bounded subset of $\mathbb{R}^n$. In this paper we generalize the Lions paper and we will show that the abstract Cauchy problem for the equation $u'' + (A + P)u + Bu' = F(u, u')$ is well posed in Hilbert Spaces.

1. Introduction.

Given a non-linear function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, the abstract Cauchy problem for the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sum_{j=1}^{n} \left\{ a_j \frac{\partial u}{\partial x_j} + b_j \frac{\partial^2 u}{\partial x_j \partial t} \right\} = f\left(u, \frac{\partial u}{\partial t}\right),$$

consists in finding a mapping $u\colon R^+ \longrightarrow H$ such that

$$\text{(1.1)} \qquad \begin{cases} u''(t) + (A + P)\, u(t) + Bu'(t) = F(u(t), u'(t)), & (t \geq 0) \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where u_0 and u_1 are vectors of the Hilbert space H; A, P and B are linear operators of H, and F is a function from the domain $D(F) \subset H \times H$ into H.

In the equation (1.1), the case $F = 0$ is solved by Lions ([5]) in Hilbert space $L^2(\Omega)$; the case $F(u, u') = M(u)$, $P = B = 0$ is treated by Browder ([1]) in Hilbert space; Medeiros ([8]) generalized Browder's paper to the case $A = A(t)$, time's dependent; Goldstein ([2]) studied the case $B = 0$, A and P in the time dependent case. In Goldstein's paper ([3]) there is a brief survey of the literature on abstract hyperbolic Cauchy problems.

In the following we will consider a Hilbert space H with inner product $\langle\,|\,\rangle$ and norm $\|\ \|$. Let A be a linear operator of H with the domain $D(A)$

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dense in H, self-adjoint and positive (i.e.: there is $\varepsilon_0 > 0$ such that $\langle Au \mid u \rangle \geq \varepsilon_0 \|u\|^2$, for each $u \in D(A)$). Let us represent by $W = D(A^{1/2})$ the domain of $A^{1/2}$. In W we introduce the inner product

$$\langle u \mid v \rangle_W = \langle A^{1/2}u \mid A^{1/2}v \rangle, \qquad (u \text{ and } v \text{ in } W). \tag{1.2}$$

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2. The linear case.

The vector space consisting of all bounded linear operators from W into H will be denoted by $\mathscr{L}(W, H)$; in this vector space we introduce the norm $\|T\|_{\mathscr{L}(W,H)} = \sup \{\|Tu\|;\ u \in W,\ \|u\|_W = 1\}$.

We assume that B and P are bounded linear operators from W into H such that:

$$a = \max \{\|B\|_{\mathscr{L}(W,H)},\ \|P\|_{\mathscr{L}(W,H)}\} < 1 \tag{2.1}$$

$$Re \langle Bu \mid u \rangle \geq 0, \qquad \text{for all} \quad u \in W. \tag{2.2}$$

Suppose $E = W \times H$ eqquiped with the inner product

$$\langle w_0 \mid w_1 \rangle_E = \langle u_0 \mid u_1 \rangle_W + \langle v_0 \mid v_1 \rangle \tag{2.3}$$

for each $w_0 = (u_0, v_0)$ and $w_1 = (u_1, v_1)$ in $E = W \times H$.

Let L be the linear operator defined in $D(L) = D(A) \times D(A^{1/2})$ by

$$Lw = (v, -(A + P)u - Bv), \qquad w = (u, v) \in D(L). \tag{2.4}$$

Proposition 2.1. *L is the infinitesimal generator of* a C_0 *semigroup on E, $\{G(t);\ t \geq 0\}$, such that*

$$\|G(t)\|_{\mathscr{L}(E)} \leq e^{5/2}t, \qquad \text{for all} \quad t \geq 0. \tag{2.5}$$

Remark. If I_E denotes the identity mapping of E, by the Hille-Yosida Theorem (see [11], page 249) we need show that for each $\lambda > 5/2$, the mapping $L_\lambda = I_E - \lambda^{-1}L$, from $D(L)$ into E, is bijective and L_λ^{-1} is a bounded linear operator such that $\|L_\lambda^{-1}\| \leq \lambda/(\lambda - 5/2)$.

For the proof of the Proposition 2.1, we need the following result:

Lemma 2.1. *For each $\lambda > 5/2$, the mapping $J_\lambda = I_H + \lambda^{-2}(A + P) + \lambda^{-1}B$: $D(A) \longrightarrow H$ is onto.*

Proof. We take $C_\lambda = \lambda^{-2}P + \lambda^{-1}B$ and $N_\lambda(u) = \|u + \lambda^{-2}Au\|^2$, $(u \in D(A))$. Since $I_H + \lambda^{-2}A$ is onto, so, for each $x \in H$ we can construct a sequence (u_m) of $D(A)$ such that

$$\begin{cases} u_0 + \lambda^{-2}Au_0 = x \\ u_{m+1} + \lambda^{-2}Au_{m+1} = -C_\lambda u_m, \end{cases} \qquad m \in \mathbb{N} = \{0, 1, \ldots\}. \tag{2.6}$$

If $\lambda > 5/2$, then $\lambda > (\sqrt{2} - 1)^{-1}$ and $1/\lambda^2 + 2/\lambda + 1 < 2$. Also $2/\lambda^2 \|u\|_W^2 \leq N_\lambda(u)$, for each $u \in D(A)$. Now, for $m = 0, 1, \ldots$ we have:

$$\begin{aligned} N_\lambda(u_{m+1}) &= \|u_{m+1}\|^2 + 2/\lambda^2 \langle Au_{m+1} \mid u_{m+1} \rangle + \|1/\lambda^2 Au_{m+1}\|^2 \\ &= \|u_{m+1}\|^2 + 2 \langle -u_{m+1} - C_\lambda u_m \mid u_{m+1} \rangle + \|u_{m+1} + C_\lambda u_m\|^2 \\ &= \|C_\lambda u_m\|^2 = 1/\lambda^4 \|Pu_m\|^2 + 2/\lambda^3 Re \langle Pu_m \mid Bu_m \rangle + 1/\lambda^2 \|Bu_m\|^2 \\ &\leq a/\lambda^2 \{1/\lambda^2 + 2/\lambda + 1\} \|u_m\|_W^2 \\ &\leq a^2\, 2/\lambda^2 \|u_m\|_W^2 \leq a^2 N_\lambda(u_m), \end{aligned}$$

where a is given in 2.1.

The last inequality implies:

$$N_\lambda(u_m) \leq a^{2m} \|x\|^2, \qquad (m = 0, 1, 2, \ldots). \tag{2.7}$$

Putting $v_m = \sum_{j=0}^{m} u_j$, then (2.1) and (2.7) imply that $(A^s v_m)_m$ is a Cauchy sequence of H, for $s = 0, 1/2, 1$, therefore, there is $v \in D(A)$ such that $(A^s v_m)_m$ converges to $A^s v$, $(s = 0, 1/2, 1)$ from (2.6) we obtain $J_\lambda v = x$.

Proof of Proposition 2.1. First, we note that L is a closed operator of E, because A and $A^{1/2}$ are closed operators of H, and B and P are bounded linear operators from W into H.

Take $\lambda > 5/2$ and put $L_\lambda = I_E - \lambda^{-1}L$. For each $w \in D(L)$ we have

$$\|L_\lambda w\|_E^2 \geq \|w\|_E^2 + 2\lambda^{-1} Re\langle v \mid Pu \rangle \geq (1 - 5/2\lambda)^2 \|w\|_E^2. \tag{2.8}$$

Also, $D(A) \times H \subset \text{Range}(L_\lambda)$; in fact, if $(x, y) \in D(A) \times H$, by lemma 3.1 there is $v \in D(A)$ such that $J_\lambda v = y - \lambda^{-1}(A + P)x$; taking $u = x + \lambda^{-1}v$, we have $(u, v) \in D(L)$ and $L_\lambda(u, v) = (x, y)$. This result shows that L_λ is a mapping from $D(L)$ onto $E = W \times H$, because $D(A)$ is dense in W.

From (2.8) we have $\|L_\lambda^{-1}w\|_E \leq \lambda/(\lambda - 5/2) \|w\|_E$, for each $w \in E$, and Proposition 2.1 is a consequence of the Hille-Yosida Theorem.

Proposition 2.2. *For each $\omega_0 = (u_0, u_1) \in D(A) \times D(A^{1/2})$ there is a unique mapping u: $\mathbb{R}^+ \longrightarrow H$ such that:*

(1) *Range$(u) \subset D(A)$, $u \in C^1(\mathbb{R}^+, W) \cap C^2(\mathbb{R}^+, H)$*

(2) *$u''(t) + (A + P)\ u(t) + Bu'(t) = 0 \quad (t \geq 0)$*

(3) *$u(0) = u_0, \quad u'(0) = u_1$.*

Also if u and v are mappings from R^+ into H such that (1) and (2) are true for u and v then:

(4) $\|u(t)-v(t)\|_W^2+\|u'(t)-v'(t)\|^2 \le \{\|u(0)-v(0)\|_W^2+\|u'(0)-v'(0)\|^2\}\, e^{5t}$, $(t \ge 0)$.

Proof. We note that the conditions (1), (2) and (3) are equivalents to find a mapping $\omega \in C^1(\mathbb{R}^+, E)$ such that

$$\begin{cases}\omega'(t) = L\omega(t), & t \ge 0\\ \omega(0) = \omega_0,\end{cases} \tag{2.9}$$

and Proposition 2.2 is a consequence of the Phillip's Theorem (see [4], page 622] and Proposition 2.1.

3. Existence of local solutions in the non-linear case.

Let F be a non-linear operator from $D(F)$ into H with the following properties:

(3.1) $$D(L) \subset D(F) \subset E, \quad F(0,0) = 0$$

(3.2) For each $c > 0$, we have that

$$\alpha_c = \sup\left\{\frac{\|F(\omega_1) - F(\omega_2\|}{\|\omega_1 - \omega_2\|_E} \mid \omega_1 \in D_c,\ \omega_2 \in D_c,\ \omega_1 \ne \omega_2\right\} < +\infty$$

where $D_c = \{\omega \in D(L) \mid \|\omega\|_E \le c\}$ (i.e.: F is Lipschitzian in D_c, for each $c > 0$).

(3.3) For each $T > 0$ and $\omega \in C^1([0, T), E)$ such that range $(\omega) \subset D(L)$ the mapping $\omega_0(t) = F(\omega(t))$, $0 \le t \le T$, belongs to the space $C^1([0, T), H)$.

(3.4) There exist $r > 0$ and $p > 0$ such that

$$\mathrm{Re} \int_0^t \langle F(u(s),\ u'(s)) \mid u'(s)\rangle\, ds \le r\|u(0)\|_W^p + r\int_0^t \{\|u(s)\|_W^2 + \|u'(s)\|^2\}\, ds,$$

for each $u \in C^1([0, T), W)$ with the range in $D(A)$ and $0 \le t < T$.

Given $T > 0$ and $\omega_0 \in D(L)$, let X be the set of mappings ω: $[0, T) \longrightarrow E$ such that ω is continuous and bounded in $[0, T)$, with the norm

(3.5) $$\|\omega\|_X = \sup_{0 \le t < T} \|\omega(t)\|_E \quad (\omega \in X,\ 0 \le t < T).$$

Given $\omega_0 \in D(L)$, we consider $X_{\omega_0} = \{\omega \in X \mid \omega(0) = \omega_0,$ range $(\omega) \subset D(L)\}$, and the mapping S: $X_{\omega_0} \longrightarrow X$ defined by

$$S\omega(t) = G(t)\omega_0 + \int_0^t G(t-s) f(\omega(s))\, ds \quad (\omega \in X_{\omega_0},\ 0 \le t < T),$$

where $f(z) = (0, F(z))$, when $z \in D(L)$.

From the hypothesis (3.2) we obtain the following results:

Lemma 3.1. *For each $c > 0$ there is $T = T(c) > 0$ such that, if $\omega_0 \in D(L)$ and $\|\omega_0\|_E \le c$, then*:

(1) $\|S\omega\|_X \le 3c$, *when* $\omega \in X_{\omega_0}$ *and* $\|\omega\|_X \le 3c$

(2) $\|S\omega_1 - S\omega_2\|_X \le 1/3\|\omega_1 - \omega_2\|_X$, *when* $\omega_i \in X_{\omega 0}$ *and* $\|\omega_i\|_X \le 3c$ $(i = 1, 2)$

Proof. We choose $T = T_c = \min\{2/5\alpha_{3c},\ 2/5\ ln\ 2\}$, where α_{3c} is given in (3.2).

Remark. We observe that X_{ω_0} is not a complete metric space and we cannot apply directly the fixed point theorem for contractions.

Lemma 3.2. *Let C be a self-adjoint linear operator in H. If $(u_n)_n$ is a sequence in $D(C)$ such that*:

(1) $(u_n)_n$ *converges to* $u \in H$

(2) $(Cu_n)_n$ *is a bounded sequence of H, then* $u \in D(C)$.

Proof. When f: $[a, b] \longrightarrow \mathbb{R}$ is continuous and α: $[a, b] \longrightarrow \mathbb{R}$ is of bounded variation, we have the following formula for integration by parts

$$\int_a^b f\, d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\, df \quad \text{(see [9], page 118).}$$

If $C = \int_{-\infty}^{+\infty} \lambda\, dE_\lambda$ (see [9], page 320), from the last formula, Lebesgue's Theorem and hypothesis (1) we have

$$\lim_{n\to\infty} \int_a^b \lambda^2\, d\|E_\lambda u_n\|^2 = \int_a^b \lambda^2\, d\|E_\lambda u\|^2, \quad \text{for all} \quad a < b$$

so,

$$\int_{-\infty}^{+\infty} \lambda^2\, d\|E_\lambda u\|^2 \le \sup\{\|Cu_n\|^2;\ n = 1, 2, \ldots\} < +\infty$$

and $u \in D(C)$

Proposition 3.1. *For each $c > 0$, $u_0 \in D(A)$, $v_0 \in D(A^{1/2})$, with $\|(u_0, v_0)\|_E \le$ $\le c$, there is a mapping u: $[0, T_c) \longrightarrow H$ such that*

(1) *range*$(u) \subset D(A)$

(2) *u is once differentiable in the norm of W and twice differentiable in the norm of H*

(3) $u''(t) + (A + P)\, u(t) + Bu'(t) = F(u(t), u'(t)), \quad 0 \le t < T_c$

(4) $u(0) = u_0, \quad u'(0) = u_1$.

Proof. We consider $z_0 = (u_0, v_0) \in D(L)$ and $(\omega_n)_{n\in\mathbb{N}}$ the sequence of X_{z0} defined by induction in the following way: $\omega_0(t) = G(t)z_0$, $\omega_{n+1} = S\omega_n$ $(0 \le t < T_c,\ n = 0, 1, 2, \ldots)$. By Phillip's Theorem ([4] page 622), the sequence $(\omega_n)_{n\in\mathbb{N}}$ is well defined, ω_n is once differentiable in the norm of E and

$$\begin{aligned}\omega'_{n+1}(t) &= L\omega_{n+1}(t) + f(\omega_n(t)), \quad &(0 \le t < T_c,\quad n \in \mathbb{N})\\ w'_0(t) &= L\omega_0(t) &(0 \le t < T_c)\end{aligned}$$

By Lemma 3.1, $(\omega_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of X, so, there is $\omega\in X$ such that such that $(\omega_n(t))_{n\in\mathbb{N}}$ converges to $\omega(t)$, uniformly for $0\le t<T_c$.

Now, we show that $\omega(t)\in D(L)$, $(0\le t<T_c)$. If $\varphi_n(t)=\|\omega_n'(t)\|_E^2$ we have $\varphi_n(t)\le 8\varphi_n(0)+8(\alpha_{3c})^2\, t\int_0^t\varphi_{n-1}(s)\,ds$, for $0\le t<T_c$ and $n\in N$.

We put $c_0=\max\{4(1+\|L\omega_0\|^2), 8\|L\omega_0+f(\omega_0)\|^2\}$, $c_1=8(\alpha_{3c})^2$, then $\|\omega_n'(t)\|_E\le c_0\,e^{c_1T_c}=R$ $(0\le t<T_c,\ n\in N)$. If $u_n(t)=P_W\,\omega_n(t)$, $u(t)=P_W\omega(t)$, $v_n(t)=P_H\,\omega_n(t)=u_n'(t)$ and $v(t)=P_H\,\omega(t)$, from the last inequality we find that $(Au_n(t))_{n\in\mathbb{N}}$ and $(A^{1/2}\,u_n(t))_{n\in\mathbb{N}}$ are bounded sequences of H; also: $(u_n(t))_{n\in\mathbb{N}}$ converges to $u(t)$ and $(v_n(t))_{n\in\mathbb{N}}$ is convergent to $v(t)$, therefore $\omega(t)=(u(t),v(t))\in\in D(A)\times D(A^{1/2})=D(L)$, by Lemma 3.2.

Because $\omega\in X_{\omega_0}$ and $\|\omega\|_X\le 3c$, by Lemma 3.1 we have

$$\omega=\lim_{n\to\infty}\omega_{n+1}=\lim_{n\to\infty}S\omega_n=S\omega$$

or

$$\omega(t)=G(t)\,\omega_0+\int_0^tG(t-s)\,f(\omega(s))\,ds,\qquad(0\le t<T_c).$$

If $u(t)=P_W\,\omega(t)$, $(0\le t<T_c)$, then u is the solution of the problem (1) to (4).

4. Uniqueness, continuous dependence and continuation.

In the following, we say that the mapping $u\colon[0,T)\longrightarrow H$ is a solution of the Cauchy problem (1.1) in $[0,T)$ $(0<T\le+\infty)$, when $\mathrm{Range}(u)\subset D(A)$, u is differentiable in the norm of W, twice differentiable in the norm of H and satisfy (1.1) in $[0,T)$.

Now, we will show that the Cauchy problem (1.1) is well posed in $D(A)\times\times D(A^{1/2})$, in the following sense:

(i) For each $(u_0,u_1)\in D(A)\times D(A^{1/2})$ there is a unique solution u of (1.1).

(ii) The solutions of the differential equation in (1.1) are continuously dependent of the initial conditions $u(0)$ and $u'(0)$.

In Proposition 4.1 we will show (ii) and this result implies the unique ness of the solution u. Also, the existence of local solutions for (1.1) shown in Proposition 3.1, the uniqueness and a standard argument imply the existence of solutions for $t\ge 0$.

Lemma 4.1. *For each $c>0$ and $T>0$, there is a constant $\beta=\beta(c,T)>0$ such that, if $u_0\in D(A)$, $u_1\in D(A^{1/2})$, $\|u_0\|_W^2+\|v_0\|^2\le c^2$ and u is a solution of the Cauchy problem (1.1) in $[0,T)$, then:*

$$\|u(t)\|_W^2+\|u'(t)\|^2\le\beta,\qquad(0\le t<T).$$

Proof. If $g(t)=\|u(t)\|_W^2+\|u'(t)\|^2$, $(0\le t<T)$, by hypothesis (3.4) we have

$$g(t)-g(0)=\int_0^tg'(s)\,ds=2\int_0^tRe\langle u''(s)+Au(s)\mid u'(s)\rangle\,ds$$
$$\le 2r\|u(0)\|_W^p+(2r+\|P\|)\int_0^tg(s)\,ds$$

and $\beta=\beta(c,T)=(c^2+2rc^p)\exp\{(2r+\|P\|)T\}$ is the constant.

Proposition 4.1. *For each $c>0$ and $T>0$ there is $\gamma=\gamma(c,T)>0$ such that, if u and v are solutions of (1.1) in $[0,T)$, with $\|u(0)\|_W^2+\|u'(0)\|^3\le c^2$, $\|v(0)\|_W^2+\|v'(0)\|^2\le c^2$, then*

$$\|u(t)-v(t)\|_W^2+\|u'(t)-v'(t)\|^2\le e^{\gamma t}\{\|u(0)-v(0)\|_W^2+\|u'(0)-v'(0)\|^2\},(0\le t<T).$$

Proof. We put $z(t)=u(t)-v(t)$, $h(t)=\|z(t)\|_W^2+\|z'(t)\|^2$, $(0\le t<T)$. From Lemma 4.1 and hypothesis (3.2) we obtain a constant $\alpha=\alpha_\beta=\alpha(c,T)>0$ such that

$$\|F(u(t),u'(t))-F(v(t),v'(t))\|\le\alpha\{\|z(t)\|_W^2+\|z'(t)\|^2\}^{1/2},\quad(0\le t<T).$$

Also:

$$\begin{aligned}h'(t)&=2\,Re\langle z''(t)+Az(t)\mid z'(t)\rangle\\&=-2\,Re\langle Bz'(t)+Pz(t)\mid z'(t)\rangle\\&\quad+2\,Re\langle F(u(t),u'(t))-F(v(t),v'(t))\mid z'(t)\rangle\\&\le(\|P\|+1+\alpha)\,h(t),\qquad(0\le t<T)\end{aligned}$$

then: $h(t)\le e^{\gamma t}h(0)$ $(0\le t<T)$, where $\gamma=\gamma(c,T)=1+\|P\|+\alpha(c,T)$.

Theorem 4.1. *For each $u_0\in D(A)$ and $u_1\in D(A^{1/2})$ there is a unique mapping $u\colon\mathbb{R}^+\longrightarrow H$, solution of the Cauchy problem (1.1) in $\mathbb{R}^+$.*

Also, for each $c>0$ and $T>0$ there is $\Gamma=\Gamma(c,T)>0$ such that

$$\sup_{0\le t<T}\{\|u(t)-v(t)\|_W^2+\|u'(t)-v'(t)\|^2\}\le\Gamma\{\|u(0)-v(0)\|_W^2+\|u'(0)-v'(0)\|^2\},$$

when u and v are solutions of the differential equation (1.1) and $\|u(0)\|_W^2++\|u'(0)\|^2+\|v'(0)\|^2\le c$.

Proof. The uniqueness and the second part are a direct consequence of Proposition 4.

Let us fix $(u_0,u_1)\in D(A)\times D(A^{1/2})$, and consider the set S of the numbers $0<T\le+\infty$ such that (1.1) has a solution in $[0,T)$; by Proposition 3, S is not empty. Putting $T_0=\max S$; if T_0 is finite, let $u\colon[0,T_0)\longrightarrow H$ be a solution of (1.1) in $[0,T_0)$, by the Lemma 5

$$c=\sup_{0\le t<T_1}\{\|u(t)\|_W^2+\|u'(t)\|^2\}^{1/2}<\infty$$

and $T_c\in S$ (where T_c is given in Lemma 3.1), we take $s_0=T_0-T_c/2$, then $0<s_0<T_0$ and there is a mapping $v\colon[0,T_c)\longrightarrow H$, solution of the differential

equation (1.1), with the initial conditions $v(0) = u(s_0)$, $v'(0) = u'(s_0)$. By the uniqueness shown in Proposition 4.1, we find that the mapping

$$\omega(t) = \begin{cases} u(t), & 0 \le t < T_0 \\ v(t - s_0), & s_0 < t < T_c + s_0 = T_0 + T_c/2, \end{cases}$$

is well defined, and ω is the solution of (1.1) in $[0, T_0 + T_c/2)$; this fact is a contradiction, because $T_0 = \max S$; then $T_0 = +\infty$, and the solution u of (1.1) is defined in $\mathbb{R}^+$.

5. Aplication.

We represented by Ω an open, bounded subset on $\mathbb{R}^n$. Let H be the real vector space $L^2(\Omega)$, with the inner product

$$\langle u \mid v \rangle_H = \int_\Omega u(x)\, v(x)\, dx, \qquad (u, v \quad \text{in} \quad H).$$

The vector space consisting of all mappings $u \in C^\infty(\Omega)$ such that $u(x) = 0$, for x outside a compact set in Ω, will be denoted by $C_0^\infty(\Omega)$; in this vector space we introduce the inner product

$$\langle u \mid v \rangle_{H_0^1(\Omega)} = \sum_{j=1}^{n} \langle D_j u \mid D_j v \rangle_H, \qquad (u, v \quad \text{in} \quad C_0^\infty(\Omega)).$$

The completion of $C_0^\infty(\Omega)$ will be denoted by $H_0^1(\Omega)$.

If $D(A) = \{u \in H_0^1(\Omega) \mid \Delta u \in H\}$, then the operator A: $D(A) \longrightarrow H$ defined by

$$Au = -\Delta u = -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \qquad (u \in D(A))$$

is self-adjoint and positive in H; also $W = D(A^{1/2}) = H_0^1(\Omega)$.

Given the continuous and bounded functions $f = (f_1, \ldots, f_n)$, and $g = (g_1, \ldots, g_n)$ from Ω into $\mathbb{R}^n$, let P and B be the operators from W into H defined by

$$Pu = f \cdot \nabla u = \sum_{j=1}^{n} f_j D_j u$$

$$Bu = g \cdot \nabla u = \sum_{j=1}^{n} g_j D_j u, \qquad u \in W = V,$$

then B and P are bounded linear operators and

$$\|P\| \le \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2}, \qquad \|B\| \le \left(\sum_{j=1}^{n} b_j^2 \right)^{1/2},$$

where $a_j = \sup_{x \in \Omega} |f_j(x)|$, $b_j = \sup_{x \in \Omega} |g_j(x)| \qquad (j = 1, \ldots, b)$.

If $g \in \hat{C}^1(\Omega, \mathbb{R}^n)$, $\nabla \cdot g(x) = \sum_{j=1}^{n} D_j g_j(x) \le 0$ $(x \in \Omega)$ and $D_j g_j$ are bounded functions from Ω into $\mathbb{R}$, then $2\langle Bu \mid u \rangle = -\int_\Omega (\nabla \cdot g)(x) \,|\, u(x) \,|^2\, dx \ge 0$, for each $u \in W$.

Now, we consider an example for the non-linear part F. Let $p > 2$ be a real number such that $W = H_0^1(\Omega) \subset L^p(\Omega)$ with continuous injection, if $r = p/2 - 1$ and φ: $\mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable with the derivative φ' a bounded function and $\varphi(0) = 0$, then the mapping.

$$f(u, v) = -c\,|u|^r u - d\varphi(v) \qquad (u \in L^p(\Omega),\ v \in L^2(\Omega))$$

satisfies hypothesis (3.1), (3.2), (3.3) and (3.4), when $c > 0$ and $d > 0$.

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