On the Structure of the Pareto Set of Generic Mappings

W. de Melo

We consider the problem of optimizing functions defined on a compact manifold.

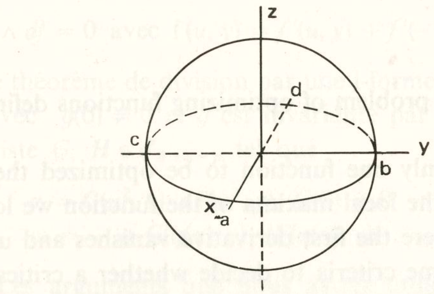
When there is only one function to be optimized the problem is well understood. To find the local maxima of the function we look at the critical points, i.e., points where the first derivative vanishes and using higher order derivatives we get some criteria to decide whether a critical point is a local maximum or not. In fact for almost all functions these criteria allways work and we have only to examine the second derivative at the critical points. Furthermore the set of critical points and local maxima is very simple. There is only a finite number of critical points and they are related to the topology of the manifold. These global results are studied in Morse-Theory.

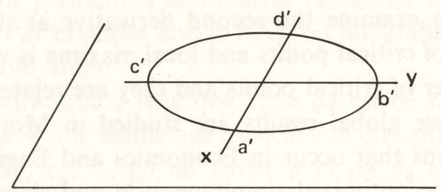
In many situations that occur in Economics and Engineering there are several functions to be optimized simultaneously and since in general there is conflict among them it is impossible to find a point which is a local maximum for all the functions. Therefore it is necessary to define another notion of optimum. This was done by an Economist called Pareto at the end of the last century. More recently Smale introduced methods of Global Analysis in the study of this problem. He defined the set of critical Pareto points which contains the set of optima. The purpose of this paper is to study the structure of this set for generic mappings.

1. We recall some definitions which can be found in [4]. Let M be a compact C^∞ manifold without boundary and let $f_1, \ldots, f_c : M \to \mathbb{R}$ be C^∞ functions. We consider the f_i's as the coordinate functions of a mapping $f: M \to \mathbb{R}^c$. A point $p \in M$ is a local Pareto optimum of f if there is a neighborhood V of p such that there is no $q \in V$ with $f_i(q) \geq f_i(p)$ for all $i = 1, \ldots, c$ and $f_j(q) > f_j(p)$ for some j. We denote by θ_{op} or $\theta_{op}(f)$ the set of local Pareto Optima of f. In order to study those points using differential calculus we define a larger set called the set of critical Pareto points of f. Denote by $\mathbb{R}^c_+$ the positive cone

of $\mathbb{R}^c$, i.e., the set of points of $\mathbb{R}^c$ whose coordinates are positive. We say that $p \in M$ is a critical Pareto point of f if the image of the derivative of f at p does not intersect the positive cone of $\mathbb{R}^c$: $\operatorname{Im} df_p \cap \mathbb{R}^c_+ = \phi$. Denote by θ or $\theta(f)$ the set of critical Pareto points of f. Clearly θ is contained in the set Σ of critical points of f which are the points where the derivative is not surjective.

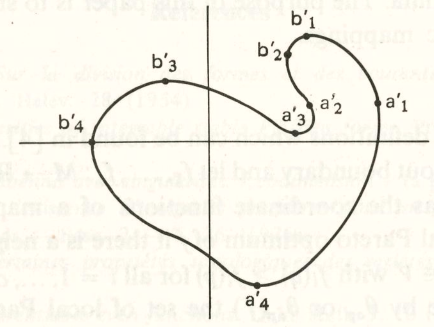
Example 1. Let $f: S^2 \to \mathbb{R}^2$ be the projection as in the pictures below





Here $\theta(f)$ is the union of the closed intervals $[a, b]$ and $[c, d]$ and θ_{op} is the segment $[a, b]$.

Example 2 Let $f: S^2 \to \mathbb{R}^2$ be the mapping whose image is the following:



Let $a_i = f^{-1}(a_i')$ and $b_i = f^{-1}(b_i')$. Then

$$\theta(f) = \bigcup_{i=1}^{4} [a_i, b_i]$$

$$\theta_{op} = [a_1, b_1] \cup (a_3, b_3)$$

Example 3 Consider the sphere $S^n = \left\{ x \in \mathbb{R}^{n+1};\ \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$ and let $f: S^n \to \mathbb{R}^k$ be defined by

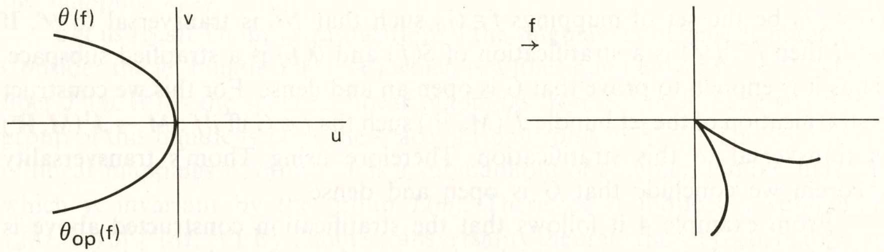
$$f(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_k).$$

Consider the closed $k-1$ simplecies

$$\Delta_1 = \{x \in S^n;\ x_i \geq 0 \text{ for all } i \text{ and } x_j = 0 \text{ for } j > k\},$$
$$\Delta_2 = \{x \in S^n;\ x_i \leq 0 \text{ for all } i \text{ and } x_j = 0 \text{ for } j > k\}.$$

Clearly $\theta = \Delta_1 \cup \Delta_2$ and $\theta_{op} = \Delta_1$.

Example 4 Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(u, v) = (u, uv + v^3 - u)$. Here $\Sigma(f) = \theta(f) = \{(u, v);\ u = -3v^2\}$ and $\theta_{op}(f) = \{(u, v);\ u = -3v^2 \text{ and } v < 0\}$.



In the examples above the critical Pareto set is a union of submanifolds of M and so is its image. To describe the structure of the critical Pareto set for generic mappings we need another definiton.

Let $A \subset M$ be a closed subset. A stratification $\mathscr{S}$ of A is finite collection of connected submanifolds of M satisfying the following properties:

1) $\bigcup_{S \in \mathscr{S}} S = A$.

2) If $S \in \mathscr{S}$ then $\partial S = Cl(S) - S$ is a union of elements of $\mathscr{S}$ of lower dimension. Here $Cl(S)$ denotes the closure of S.

3) If $S \in \mathscr{S}$ and U is a submanifold of M transversal to S at $x \in S$ then U is transversal to all elements of $\mathscr{S}$ in a neighborhood of x.

The elements of $\mathscr{S}$ are called strata and A is called a stratified set. It is easy to see that if $f: M \to N$ is a C^∞ mapping transversal to all strata of a stratified set $Y \subset N$ then $f^{-1}(Y)$ is a stratified subset of M. The stratum through a point $x \in f^{-1}(Y)$ is the connected component of the preimage of the stratum through $f(x)$.

Consider the space $C^\infty(M, \mathbb{R}^c)$ endowed with the C^∞ topology. Here M is a compact manifold without boundary and dimension $m \geq c$.

Theorem. *There is an open and dense set $\mathscr{G} \subset C^\infty(M, \mathbb{R}^c)$ such that if $f \in \mathscr{G}$ then $\theta(f)$ is a stratified set of dimension $c-1$.*

This theorem will be proved in the next section. We now give an idea of the proof of a special case of it. Suppose dim $M = m \geq 2c-4$. It follows from Thom's transversality theorem [1] that there exists an open and dense set $G_0 \subset C^\infty(M, \mathbb{R}^c)$ such that if $f \in G_0$ then the singular set of f, $S(f)$, is a compact manifold of dimension $c-1$. Furthermore at each point $p \in S(f)$ df_p has rank $c-1$. For each $p \in S(f)$ denote by $N_f(p)$ the line through the origin of $\mathbb{R}^c$ orthogonal to the image of df_p. It follows that the mapping $N_f: S(f) \to \mathbb{R}P^{c-1}$ is C^∞. In $\mathbb{R}P^{c-1}$ we consider the stratification $\mathscr{S}$ defined by the coordinate subspaces of $\mathbb{R}^c$, namely, two lines belong to the same stratum of $\mathscr{S}$ iff the smallest coordinate subspace containing each line contains the other. Let $G_1 \subset G_0$ be the set of mappings $f \in G_0$ such that N^f is transversal to $\mathscr{S}$. If $f \in G$ then $f^{-1}(\mathscr{S})$ is a stratification of $S(f)$ and $\theta(f)$ is a stratified subspace. Thus it is enough to prove that G is open an and dense. For this we construct a stratification of the jet bundle $J^1(M, \mathbb{R}^c)$ such tha $f \in G$ iff $j^1 f: M \to J^1(M, \mathbb{R}^c)$ is transversal to this stratification. Therefore using Thom's transversality theorem we conclude that G is open and dense.

From example 4 it follows that the stratification constructed above is not fine enough to separate the local optima from the other critical points. In other words a stratum may contain critical Pareto points which are local optima and also critical Pareto points which are not local optima. To refine this stratification we have to deal with higher order jet bundles.

Conjecture. There is an open and dense subset $G \subset C^\infty(M, \mathbb{R}^c)$ such that if $f \in G$ then $\theta(f)$ is a stratified set and θ_{op} is a union of strata. This conjecture has been verified for $c = 2$ [5] and $c = 3$ [3]. Another interesting problem is to study the relations between this stratification and the topology of M. In [2] Wan described the Morse Inequalities for $c = 2$.

We now consider some local questions and state some open problems in this direction. Let $f: (\mathbb{R}^m, 0) \to (\mathbb{R}^c, 0)$ be a C^∞ map germ with $0 \in \theta(f)$. Let $j^k f(0)$ be the k-jet of f at 0 which can be identified with the Taylor polynomial of order k. We say that f is Pareto k-determined if for any map-germ $g: (\mathbb{R}^m, 0) \to (\mathbb{R}^c, 0)$ with $j^k g(0) = j^k f(0)$ we have: 0 is a local Pareto optimum of f iff 0 is a local Pareto optimum of g. Denote by $J^k(m, c)$ the space of k-jets at 0 of map-germs $f: (\mathbb{R}^m, 0) \to (\mathbb{R}^c, 0)$. We can identify $j^k(m, c)$ with the vector space of polynomial maps from $\mathbb{R}^m$ to $\mathbb{R}^c$ of degree less or equal k.

Conjecture. There exist semi-algebraic sets $A_1, A_2 \subset J^k(m, c)$ such that a map-germ $f: (\mathbb{R}^m, 0) \to (\mathbb{R}^c, 0)$ is Pareto *k-determined* if and only $j^k f(0) \in \in A_1 \cup A_2$. Furthermore if $j^k f(0) \in A_1$ then 0 is a local Pareto optimum of f and if $j^k f(0) \in A_2$ then 0 is not a local Pareto optimum.

For $k = 2$ the semi-algebraic set A_1 has been defined in [6].

2. Proof of the Theorem

Let $f: M \to \mathbb{R}^c$ be a C^∞ mapping. For each sequence of integers $I = (i_1, \ldots, i_s)$ with $1 \leq i_j \leq c$, consider the projection $\pi_I: \mathbb{R}^c \to \mathbb{R}^{c-s}$ which drops the coordinates with index in I. Given two such sequences $I = (i_1, \ldots, i_s)$ and $I' = (i_1, \ldots, i_{s'})$ we say that $|I| \geq |I'|$ if $s \geq s'$. We denote by $f_I: M \to \mathbb{R}^{c-s}$ the mapping $\pi_I \circ f$.

Let us recall Thom's stratification of the singular set of generic mappings. Consider the jet bundle $J^1(M, \mathbb{R}^c)$ which is a fiber bundle over $M \times \mathbb{R}^c$ whose fiber $J^1(m, \mathbb{R}^c)$ is the space of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^c$. The structural group of this bundle is $L(m)$, the space of isomorphisms of $\mathbb{R}^m$. The set $S_h(m, c)$ of linear mappings of rank $c-h$ is a submanifold of codimension $(m-h)(c-h)$ which is invariant by the group $L(m)$. Therefore it induces a subbundle $S_h(M, \mathbb{R}^c)$ of $J^1(M, \mathbb{R}^c)$. By Thom's transversality theorem the set G_0 of mappings $f: M \to \mathbb{R}^c$ such that the jet extension $j^1 f: M \to \mathbb{R}^c)$ is transversal to $S_h(M, \mathbb{R}^c)$ for all h is open and dense in $C^\infty(M, \mathbb{R}^c)$. Since $m \geq c$, the singular set of f, $S(f)$, is equal to $(j^1 f)^{-1}(S_1(M, \mathbb{R}^c))$. Thus the above decomposition of $S_1(M, \mathbb{R}^c)$ induces a stratification $\mathscr{S}_0(f)$ of $S(f)$. This is Thom's stratification. We observe that $A_0 = S_1(m, c)$ is an algebraic set. The projection $\pi_I: \mathbb{R}^c \to \mathbb{R}^{c-s}$ induces a projection $\pi_I: J^1(m, c) \to J^1(m, c-s)$. Let $A_I = \pi_I^{-1}(S_1(m, c-s))$. Then A_I is an algebraic set which is invariant by the group $L(m)$. It follows from a theorem of Lojasiewicz [2] that there is a stratification $\mathscr{S}(m, c)$ of $J^1(m, c)$ which is compatible with the stratification of A_I considered above. This means that if $z \in A_I$ then the stratum through z is containned in the stratum of A_I induced by π_I. Furthermore this stratification is $L_1(m)$-invariant and therefore gives rise to a stratification $\mathscr{S}(M, \mathbb{R}^c)$ of $J^1(M, \mathbb{R}^c)$ with the same properties.

Let $G \subset C^\infty(M, \mathbb{R}^c)$ be the set of mappings f such that $j^1 f$ is transversal to the stratification $\mathscr{S}(M, \mathbb{R}^c)$. It follows from Thom's transversality theorem that G is open and dense. For $f \in G$ we have a stratification $\mathscr{S}(f)$ of $S(f)$ induced by j^1f: the stratum through $x \in S(f)$ is $(j^1f)^{-1}(S_{j^1f(x)})$ where $S_{j^1f(x)}$ is the stratum of $\mathscr{S}(M, \mathbb{R}^c)$ through $j^1f(x)$. It is clear from the above construction that $\mathscr{S}(f)$ refines Thom's stratification of $S(f)$. Furthermore if I is a sequence of integers as above then $S(f_I)$ is a substratified set of $S(f)$ and the stratification induced in $S(f_I)$ refines Thom's stratification.

It remains to prove that $\mathscr{S}(f)$ induces a stratification in $\theta(f)$. Denote by $\theta_h(f)$ the set of points $p \in S_h(f)$ such that the image of df_p does not intersect the closure of the positive cone of $\mathbb{R}^c$ except at the origin. Then $\theta_h(f)$ is open in $S_h(f)$. Let $x \in Cl(\theta_h(f)) - \bigcup_{k>h} \theta_k(f)$, where $Cl(\theta_h(f))$ denotes the closure of $\theta_h(f)$. Then there exists a sequence of integers $I = (i_1, \ldots, i_s)$, $1 \leq i_1 < i_2 < \ldots < i_s \leq c$, such that $x \in \theta(f_I)$ and for any other sequence I' with $|I'| \geq |I|$, $x \notin S(f_{I'})$. Therefore, there exists an integer k such that $x \in \theta_k(f_I)$. Hence $\theta(f) = \bigcup_I \bigcup_{h \geq 1} \theta_h(f_I)$. From this fact it follows that $\mathscr{S}(f)$ defines a stratification on $\theta(f)$.

Remark. If the dimension of M is less then c we can prove, using the same arguments as above, that for a generic mapping f, $\theta(f)$ is a stratified set of the same dimension as M.

References

[1] H.J. Levine, *Singularities of Differentiable Mappings,* Proc. of Liverpool Sing. Symp. I, Springer Lecture Notes, 192, (1971).

[2] S. Lojasiewicz, *Ensembles Semi-Analytiques,* Lecture Notes, I.H.E.S., 1965.

[3] W. de Melo, *Stability and Optimization of Several Functions,* Topology, 15 (1976).

[4] S. Smale, *Global Analysis and Economics I, Pareto optimum and a generalization of Morse Theory,* Proc. Symp. Dynamical Systems at Salvador, Brazil, Ac. Press, New York (1973).

[5] Y.H. Wan, *Morse Theory for two Functions,* Topology 14, (1975).

[6] Y.H. Wan, *On local Pareto Optima,* J. Math. Economics 2, (1975).

Instituto de Matemática Pura e Aplicada
Rio de Janeiro
Brasil