BOL. SOC. BRAS. MAT. 8(1977), 1–38

The Relation Between C^∞ and Topological Stability

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1. Introduction.

The basic problem we will discuss here is the stability of smooth (C^∞) proper mappings $f: N \to P$ between smooth manifolds (without boundaries). We denote $C^\infty_{pr}(N, P)$ the space of proper smooth mappings between N and P. Then, we first say that f is *C^∞-stable* if there is a neighborhood U ou f in $C^\infty_{pr}(N, P)$ so that if $g \in U$ then there are diffeomorphisms $\phi : N \to N$ and $\psi : P \to P$ such that $f = \psi \circ g \circ \phi$. If we weaken the condition to just require that ψ and ϕ be homeomorphisms, then we have instead the notion of *topological* (C^0) *stability*. When f and g are related by $f = \psi \circ g \circ \phi$ with ψ, ϕ diffeoomorphismis (homeomorphisms) then f and g are said to be C^∞-equivalent (C^0-equivalent).

The study of C^∞-stability is a natural attempt to generalize the earlier results of Morse and Whitney, which described dense sets of mappings between manifolds of specific dimensions. For instance for the Morse functions we recall:

1. *Definition by Local and Global Properties:* A Morse function of $f: M \to \mathbb{R}$ is a real valued function which satisfies the conditions a) at singular points x (where $D_x f = 0$), $D_x^2 f$ is a non-degenerate quadratic form, and b) images of the singular points via f give distinct values in $\mathbb{R}$.
2. *Local Classification and Local Normal Form:* For a singular point x of a Morse function f, coordinate systems can be chosen near x and $f(x)$ so that f is defined locally by

$$f(x) = x_1^2 + x_2^2 + \ldots + x_q^2 \quad x_{q+1}^2 - \ldots - x_m^2$$

(for some integer $1 \leq q \leq m$).
3. *Density:* The set of Morse functions is dense in $C^\infty_{pr}(M, \mathbb{R})$ (with the Whitney topology).
4. *Stability:* Morse functions are C^∞-stable.

Work partially supported by grants from the Research Foundation of the City University of New York and the National Science Foundation.

The author is indebted to John Mather and Andre Galligo for conversations regarding this work.

Whitney gives a similar description in terms of local and global properties and normal forms for the types of mappings which he considers from 2-manifolds to 2-manifolds and from n-manifolds to m-manifolds with $m \geq 2n - 1$. This suggests the following basic question for the theory of stability of mappings.

Question: Are these principal properties, namely, definition by local and global properties, local classification, local normal forms, and density, valid for stable mappings between arbitrary smooth manifolds N, P?

Mather's work [19] gives a complete answer to this question, describing exactly how these properties are related to the stability of mappings. Earlier in the Thom-Levine notes [16], it was shown that C^∞-stable mappings are not dense in all dimensions. Mather was able to determine exactly the range of dimensions, (n, p), with $n = \dim N$ and $p = \dim P$, where stable mappings are dense in $C^\infty_{pr}(N, P)$. This range of dimensions is called the *nice dimensions.*

Outside of the nice dimensions, this introduces the problem of ignoring entire open subsets of $C^\infty_{pr}(N, P)$ if we consider only C^∞-stability mappings. To avoid this difficulty, Thom and Mather [20], [21], [31], [33] considered the weaker notion of C^0-stability. Mather, following ideas proposed by Thom, proved that for all (n, p) C^0-stable mappings are dense. However, the proof gives no clue to the classification or local forms of singularities occurring for C^0-stable mappings.

One possibility for improving our understanding of these theories would be to study the relationship between them, with the hope of filling the gaps in each of them. Several particular questions which we should ask are:

1) In the nice dimensions does the topological classification of C^∞-stable mappings agree with the C^∞-classification?

2) Which properties of C^∞-stable mapping suggest potentially useful generalizations for C^0-stable mappings?

3) Do C^0-stable and C^∞-stable mappings differ in the nice dimensions where they are both dense?

What we will survey at this time is the present status of the answers to these questions. We begin with a review of the key results of Mather's theory of C^∞-stability as they pertain to the properties we have described. More thorough surveys can be found in [6], [34]. Next, we describe the key ideas used in proving that C^0-stable maps are dense. This will essentially be a description of Mather's paper [21]. Another description is contained in [6]. Then, we turn to comparing the theories by describing the results which have been obtained for the three questions just mentioned.

C^∞-stability: We examine Mather's theory of C^∞-stability, which includes both Morse functions and Whitney's examples as special cases. Interestingly, however, Mather's first characterization of C^∞-stable mappings differs strikingly from the geometric characterization of the examples. The only earlier hint of such an approach is given in the Thom-Levine notes for homotopic stability [16]. This characterization might be thought of as an algebraic characterization of stability.

Infinitesimal Stability: We begin with the reduction of stability to an infinitesimal condition. Let Diff(N) denote the group of C^∞-diffeomorphisms of N. Then, given an $f: N \to P$, there is a map $\Phi_f : Diff(N) \times Diff(P) \to C^\infty_{pr}(N, P)$ given by $(\phi, \psi) \to \psi \circ f \circ \phi$. Then, f being stable is equivalent to the image of Φ_f containing an open neighborhood of f in $C^\infty_{pr}(N, P)$. (The topology on $C^\infty(N, P)$ is the Whitney topology. When N is compact this corresponds to the uniform convergence of functions together with all partials.)

We may think of these as being infinite dimensional manifolds, and we will try to interpret $D\Phi_f$. Let f_t be a one-parameter family of functions $f_t : N \to P$. Then, for each x, $f_t(x)$ is a curve through $f(x)$ and $d/dt\, f_t(x)$ is a vector at $f(x)$. For all x this gives us a vector field over f, i.e., a mapping $\zeta : N \to TP$ such that

$$\begin{array}{ccc} & \overset{\zeta}{\nearrow} & TP \\ & & \downarrow \pi \\ N & \xrightarrow[f]{} & P \end{array}$$

commutes. We denote the set of vector fields over f by $\theta(f)$. For $1_N : N \to N$ we denote $\theta(1_N)$ by $\theta(N)$; this consists of vector fields on N. Then, let's consider families ψ_t, ϕ_t so that $\psi_0 = 1_P$ and $\phi_0 = 1_N$. We compute the derivative

$d/dt\,(\psi_t \circ f \circ \phi_t)|_{t=0} = d/dt\,(\psi_t)\,|_{t=0} \circ (f \circ \phi_0) + D\psi_0 \circ Df\, d/dt\, \phi_t|_{t=0} = \eta \circ f + Df\xi$

where $\eta = d/dt\,(\psi_t)\,|_{t=0} \in \theta(P)$ and $\xi = d/dt\,(\phi_{t|t=0}) \in \theta(N)$.

Thus, if we denote $\omega f : \theta(P) \to \theta(f)$ by $\omega f(\eta) = \eta \circ f$ and $tf : \theta(N) \to \theta(f)$ is defined by $tf(\xi) = Df(\xi)$, then

$$D\Phi_f = tf + \omega f : \theta(N) \oplus \theta(P) \to \theta(f).$$

If $Diff(N)$, $Diff(P)$ and $C^\infty(N, P)$ were finite dimensional or even Banach manifolds, then by the implicit function theorem, $D\Phi_f$ being subjective would be enough to conclude that Φ_f is locally onto; and f, stable. However, these are weaker forms of infinite-dimensional manifolds, Frechet manifolds; and the implicit function theorem is not valid in general for them [13, III, - 1]. Nonetheless, we define

Definition 1. *f is infinitesimally stable if $tf + \omega f : \theta(N) \oplus \theta(P) \to \theta(f)$ is surjective.*

Then, Mather has still proven.

Theorem 1. *For proper smooth mappings infinitesimal stability is equivalent to C^∞-stability.*

Local Classification. We examine the infinitesimal condition locally. Let $\theta_x(f)$ denote the germs of vector fields over f at x. Then, *if* $f^{-1}(y) = \{x_1, \ldots, x_k\}$, the condition in terms of germs becomes

$$\sum_{i=1}^{k} \theta_{x_i}(N) \oplus \theta_y(P) \to \sum_{i=1}^{k} \theta_{x_i}(f).$$

This says that given germs of vector fields $\tau_1, \ldots, \tau_k$ over f at $x_1 \ldots x_k$, there are germs of vector fields $\xi_1, \ldots, \xi_k$ at $x_1, \ldots, x_k$ respectively and η at y so that

$$\tau_i = tf(\xi_i) + \eta \circ f_{(x_i)} \quad 1 \le i \le k.$$

This can be seen to imply

i) $\theta_{x_i}(N) \oplus \theta_y(P) \to \theta_{x_i}(f)$ being onto, each i.

ii) $D_{x_i} f(T_{x_i} N)$ intersect transversally in $T_y P$, (i.e., each $D_{x_i} f(T_{x_i} N)$ is transverse to the intersection of the others).

Definition 2. f $f : N, x \to P, y$ is a germ of a mapping at x, then f is *infinitesimally stable* at x if $\theta_x(N) \oplus \theta_y(P) \to \theta_x(f)$ is onto. In this case, f is often simply called a *C^∞-stable map germ* (it is stable in a well defined sense).

We next describe the possibilities for these types.

Let $C_x(N)$ denote the germs of real-valued C^∞-functions at x. This is an algebra. Let $\mathscr{M}_x$ denote the ideal of germs vanishing at x. Then, $f : N, x \to P, y$ induces $f^* : C_y(P) \to C_x(N)$ by $f^*(g) = g \circ f$. This is an algebra homomorphism.

We define algebras for f:

$$Q(f) = C_x(N)/f^* \mathscr{M}_y \, . \, C_x(N)$$

$$Q_{p+1}(f) = C_x(N)/f^* \mathscr{M}_y \, . \, C_x(N) + \mathscr{M}_x^{p+2}.$$

Then, the classification for C^∞-stable germs is given by the next theorem of Mather [19, IV].

Theorem 2. *If f and $g : N, x \to P, y$ are C^∞-stable map germs then f and g are C^∞-equivalent as germs iff $Q_{p+1}(f) \simeq Q_{p+1}(g)$. Also, if $n \le p$ then $Q(f) \simeq Q_{p+1}(f)$; similarly for g.*

To evaluate this algebra, we use the fact that if $\mathbb{R}[[x_1, \ldots, x_k]]$ is the formal power series ring with maximal ideal $\mathscr{M}_n$ then

$$C_x(N)/\mathscr{M}_x^k \xrightarrow{\sim} \mathbb{R}[[x_1, \ldots, x_n]]/\mathscr{M}_n^k$$

with the isomorphism given by sending a germ to its Taylor expansion of deg $k-1$ (by choosing a coordinate system about x with x corresponding to 0).

Examples:

1) *Morse singularities.*

$$y = x_1^2 + \ldots + x_q^2 - x_{q+1}^2 - \ldots - x_n^2$$

$$Q_2(f) \simeq \mathbb{R}[[x_1, \ldots, x_n]]/(x_1^2 + \ldots + x_q^2 - x_{q+1}^2 - \ldots - x_n^2) + \mathscr{M}_n^3$$

2) *Whitney singularities* $\mathbb{R}^2 \to \mathbb{R}^2$ *fold*: $f(x_1, x_2) = (x_1^2, x_2)$

$$Q_3(f) \simeq \mathbb{R}[[x_1, x_2]]/(x_1^2, x_2) \simeq \mathbb{R}[[x_1]]/(x_1^2)$$

cusp: $f(x_1, x_2) = (x_1^3 + x_2 x_1, x_2)$

$$Q_3(f) \simeq \mathbb{R}[[x_1, x_2]]/(x_1^3 + x_2 x_1, x_2) \simeq \mathbb{R}[[x_1]]/(x_1^3).$$

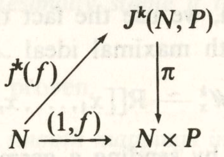
These algebras are the analogues of the local ring of a variety in algebraic geometry.

Transversality. To describe how the stability can be determined using transversality, we make use of the k-jet bundle. We recall that if $f, g : N, x \to P, y$ are germs of mappings at x, then they are said to be k-equivalent if using coordinate charts about x and y, the partial derivatives of order $\le k$ are equal. This does not depend on the coordinate charts chosen. Equivalence classes under k-equivalence are called k-jets. Then, the k-jet bundle $J^k(N, P)$ consists of all k-jets $f : N, x \to P, y, (x, y) \in N \times Y$. The projection $\pi : J^k(N, P) \to N \times P$ given by $f \to (x, y)$, the source and target of f, makes $J^k(N, P)$ a smooth fiber bundle with fiber

$$J^k(n, p) = \{k\text{-jets } f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0\}$$

(This can be thought of as the set of coefficients for k-th order Taylor expansions of functions.)

For a mapping $f : N \to P$, we have a lifting $j^k(f) : N \to J^k(N, P)$ the k-jet extension of f, defined by $j^k(f)(x) = k$-jet of f at x. It makes the diagram commute.



Using the k-jet *bundle we have* [16]

Theorem 3. (*Thom transversality*). *Let* $\Sigma \subset J^k(N,P)$ *be a submanifold and let* $W = \{f \in C^\infty(N,P), j^k(f)$ *is transverse to* $\Sigma\}$ *then* W *is a countable intersection of open dense subsets of* $C^\infty(N,P)$.

As $C^\infty(N,P)$ is a Baire space, this set is dense. To relate stability and transversality, we use two groups of germs of diffeomorphisms. First, we let

$\mathscr{R}$ = group of germs of diffeomorphisms $\mathbb{R}^n, 0 \to \mathbb{R}^n, 0$.

$\mathscr{L}$ = group of germs of diffeomorphisms $\mathbb{R}^p, 0 \to \mathbb{R}^p, 0$.

Then, $\mathscr{A} = \mathscr{L} \times \mathscr{R}$. For germs $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, $\mathscr{R}$ acts on f by composition on the right; $\mathscr{L}$ acts by composition on the left, and $\mathscr{A}$ acts by

$$(g,h).f = g \circ f \circ h^{-1}.$$

This in turn induces an action of $\mathscr{A}^k$, the k-jets of germs of diffeomorphisms, on $J^k(n,p)$.

As the orbits of $\mathscr{A}^k$ are invariant under the structural group of $J^k(N,P)$, they form a subbundle.

The other group we need plays a key role in Mather's theory of C^∞-stable germs.

Let $\mathscr{C}$ be the group of germs of diffeomorphisms of P parametrized by points of N. Specifically, $\mathscr{C}$ consists of germs of mappings

$$h: \mathbb{R}^n \times \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$$

such that for each $x \in \mathbb{R}^n$, $h(x,.)$ is a germ of a diffeomorphism of $\mathbb{R}^p$ fixing 0. Then $\mathscr{K} = \mathscr{R}.\mathscr{C}$ (semi-direct product); and $\mathscr{K}$ acts via $((g,h).f)(x) = h(x, f \circ g^{-1}(x))$. Geometrically $\mathscr{K}$ can be thought of as germs of diffeomorphisms of $\mathbb{R}^n \times \mathbb{R}^p$ which preserve the order of contact between the graphs of germs of functions, thought of as germs of submanifolds, with $\mathbb{R}^n \times \{0\}$ [13, VII, -3]. Algebraically, there is the result of Mather [19, III].

Proposition 4. *The* k*-jets* $j^k(f)$ *and* $j^k(g)$ *lie in the same* $\mathscr{K}^k$*-orbit iff* $Q_k(f) \simeq Q_k(g)$.

Again $\mathscr{K}^k$ acts on $J^k(n,p)$ and its orbits form subfiberbundles of $J^k(N,P)$. These orbits are called contact classes.

A standard argument from differential geometry shows that since $\mathscr{K}^k$ and $\mathscr{A}^k$ are Lie groups acting smoothly on $J^k(n,p)$, the orbits are immersed submanifolds. However, using an even stronger result from algebraic geometry, we can conclude that since $\mathscr{K}^k$ and $\mathscr{A}^k$ are algebraic groups acting algebraically on $J^k(n,p)$, the orbits are actually (semi-algebraic) submanifolds [19, V].

A simple argument using the Thom transversality theorem shows that for any stable mapping f and $k \geq 0$, $j^k(f)$ is transverse to all $\mathscr{A}^k$ and $\mathscr{K}^k$-orbits [34, -3], [19, V]. However, transversality is only a local condition and we saw earlier that infinitesimal stability also involves a relation between points in $f^{-1}(y)$. This extra condition ii) is contained in Mather's formulation of multi-transversality. If we let $N^k = N \times N \times \ldots \times N$ (k copies), then we define $N^{(k)} = \{(x_1, \ldots, x_k) \in N^k \mid x_i \neq x_j \text{ if } i \neq j\}$. The projection $\pi: J^l(N,P) \to$ $\to N \times P$ induces a product mapping $\pi^k: (J^l(N,P))^k \to N^k \times P^k$. We let ${}_kJ^l(N,P) = (\pi^k)^{-1}(N^{(k)} \times P^k)$. Then, ${}_kJ^l(N,P)$ is a fiberbundle over $N^{(k)} \times P^k$ with fiber $(J^l(n,p))^k$. There is also a lifting ${}_kj^l(f): N^{(k)} \to {}_kJ^l(N,P)$ given by ${}_kj^l(f)(x_1, \ldots, x_k) = (j^lf(x_1), \ldots j^lf(x_k))$. We obtain ${}_k\mathscr{A}^l$ orbits in ${}_kJ^l(N,P)$ by letting $Diff(N) \times Diff(P)$ act on each component of ${}_kj^lf$, and ${}_k\mathscr{K}^l$ orbits by letting $\mathscr{K}^l$ act independently on each component. These are again submanifolds of ${}_kJ^l(N,P)$. More importantly, the Thom transversality theorem extends to multitransversality [19, V].

Theorem 5. (*Multi-transversality*). *If* $\Sigma \subset {}_kJ^l(N,P)$ *is a submanifold, then* $\{f \in C^\infty(N,P) \mid {}_kj^lf$ *is transverse to* $\Sigma\}$ *is a countable intersection of open dense sets.*

Lastly, we can give Mather's characterization of stability using multi-transversality.

Theorem 6. *The following are equivalent for* $f: N \to P$ (*smooth, proper*). *Let* $k \geq p$, $r \geq p+1$.

i) f *is stable;*

ii) ${}_rj^kf$ *is transverse to all* ${}_r\mathscr{A}^k$*-orbits;*

iii) ${}_rj^kf$ *is transverse to all* ${}_r\mathscr{K}^k$*-orbits.*

To give a direct interpretation of multi-transversality, we suppose $f^{-1}(y) = \{x_1, \ldots x_k\}$. Suppose $j^lf(x_i) \in \Sigma_i$, a $\mathscr{K}^l$-orbit. Then ${}_kj^lf$ is multitransverse at $(x_1, \ldots x_k)$ iff i) j^lf is transverse to Σ_i at x_i and ii) the $D_{x_i}f(T_{x_i}\Sigma_i(f))$ are transverse $1 \leq i \leq k$. $(\Sigma_i(f) = \{x \in N \mid j^lf(x) \in \Sigma_i\})$.

This characterization of stability using multi-transversality proves useful in 1) determining (n, p) where stable mappings are not dense, and also in 2) determining whether C^0-stable mappings are C^∞-stable. Also, the use of multitransversality is a key idea in proving C^0-stable mappings are dense for all (n, p).

Normal Forms and Unfoldings. While theorem 2 classified stable map germs, it left open the question of for which algebras Q are there C^∞-stable germs $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ with $Q(f) \simeq Q$; and when there are, is there a normal form for such an f? We describe Mather's answer to this question.

If $Q \simeq Q(f)$ for some f, then Q is a local algebra with maximal ideal $\mathcal{M}$. Then, $Q/\mathcal{M}^{k+1} \simeq Q_k(f)$ which is a quotient algebra of $\mathbb{R}[[x_1, \ldots, x_n]]/\mathcal{M}_n^{k+1}$. We let $\hat{Q} = \varprojlim_k Q/\mathcal{M}^{k+1}$. This is a quotient algebra of $\varprojlim_k \mathbb{R}[[x_1, \ldots, x_n]]/\mathcal{M}^{k+1} \simeq \mathbb{R}[[x_1, \ldots, x_n]] = \mathbb{R}[[\mathbf{x}_n]]$.

We can, in fact, answer the question on the basis of Q. First, we write such a quotient algebra in the form

$$Q \simeq \mathbb{R}[[\mathbf{x}_n]]/I \quad \text{with} \quad I \subset \mathcal{M}_n^2.$$

We let $b =$ minimum number of generators of I, and denote $i(\hat{Q}) = a - b$. To attempt to construct an unfolding of $\hat{Q}$ we proceeed as follows: Suppose $n - p \leq i(\hat{Q})$, so we can choose d generators for I $(d = a + (p - n))$ $r_1, \ldots, r_d$. Let $\mathcal{M}_a^{(d)} = \mathcal{M}_a \times \mathcal{M}_a \times \ldots \times \mathcal{M}_a$ (d copies), similarly for $I^{(d)}$. Also, we let L be the submodule of $\mathcal{M}_a^{(d)}$ generated by

$$\partial_i = \left\langle \frac{\partial r_1}{\partial x_i}, \ldots, \frac{\partial r_d}{\partial x_i} \right\rangle \quad 1 \leq i \leq a.$$

Let V be the quotient module $V = \mathcal{M}_a^{(d)}/L + I^{(d)}$. Then, the condition we need is that $l = \dim_{\mathbb{R}} V \leq p - d$. If this is so, let $v_i \in \mathcal{M}_a^{(d)}$ $(1 \leq i \leq l)$ be elements whose projections form a basis for V. We write $v_i = (v_{i1}, \ldots, v_{id})$. Then, we define a map germ $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ by

$$\begin{cases} y_i = r_i + \sum_{j=1}^{l} t_j v_{ji} & 1 \leq i \leq d. \\ y_i = t_i & d < i \leq p. \end{cases}$$

Then, by the results of Mather [19, IV] this is a C^∞-stable map germ.

Let's first make some observations about this method. First, we compare the space $\mathcal{M}_a^{(d)}/L + I^{(d)}$ with the normal space to $\mathcal{K}(g)$ where g is defined by $g(\mathbf{x}_a) = (r_1(\mathbf{x}_a), \ldots, r_d(\mathbf{x}_a))$.

Because germs are only defined locally, $\theta_0(g)$ can not only be thought of as a $C_0(\mathbb{R}^a)$-module, it is, in fact, free of rank d generated by $\partial/\partial y_1, \ldots, \partial/\partial y_d$. If we combine the homomorphism $C_0(\mathbb{R}^a) \to \mathbb{R}[[\mathbf{x}_a]]$ obtained by sending a germ to its Taylor series. Then, the composition

$$\mathcal{M}_a\theta_0(g) \to \mathcal{M}_a C_0(R^a)^{(d)} \to \mathcal{M}_a^{(d)}$$

sends $tg(\mathcal{M}_a\theta_0(R^a)) + g^*\mathcal{M}_d\theta_0(g)$ to $\mathcal{M}_a L + I^{(d)}$ and induces an isomorphism

$$\mathcal{M}_a\theta_0(g)/(tg(\mathcal{M}_a\theta_0(R^a)) + g^*\mathcal{M}_d\theta_0(g)) \xrightarrow{\sim} \mathcal{M}_a^{(d)}/\mathcal{M}_a L + I^{(d)}.$$

The vector space on the left is exactly the normal space to $\mathcal{K} . g$, [19, III]. The germs for which this has finite dimension are $\mathcal{K}$-finitely determined, i.e., a finite jet the germ actually determines its C^∞-type up to $\mathcal{K}$-equivalence. Then, $\mathcal{M}_a^{(d)}/L + I^{(d)}$ is obtained from $\mathcal{M}_a^{(d)}/\mathcal{M}_a L + I^{(d)}$ by dividing by the vector space by $\partial_1, \ldots, \partial_a$. This subspace $< \partial_1, \ldots, \partial_a >$ is the tangent space to the image of the mapping $\mathbf{x}_a \to$ germ of g at $\mathbf{x}_a$. If we define

$$\tilde{g}(\mathbf{x}_a, \mathbf{t}) = (g_1(\mathbf{x}_a, \mathbf{t}), \ldots, g_d(\mathbf{x}_a, \mathbf{t}))$$

where

$$g_i(\mathbf{x}_a, \mathbf{t}) = r_i + \sum_{j=1}^{l} t_j v_{ji},$$

then the image of the mapping $\mathbf{t} \to \tilde{g}(\mathbf{x}, \mathbf{t})$ has a tangent space $= < v_1, \ldots, v_l >$. Thus, the mapping $(\mathbf{x}_a, \mathbf{t}) \to$ germ of g (as a function of only $\mathbf{x}_a$) at $(\mathbf{x}_a, \mathbf{t})$ gives a normal section to $\mathcal{K} . g$. This $\tilde{g}$ is called a *$\mathcal{K}$-versal deformation* of g (or I). See, for example, [24]. Thus, the stable map germ f is the "graph" of the versal deformation $\tilde{g}$: $f(\mathbf{x}, \mathbf{t}) = (\tilde{g}(\mathbf{x}, \mathbf{t}), \mathbf{t})$. Usually f is called a universal unfolding of g.

The third remark about the unfolding is the point of view taken by Mather [21, 13]. An unfolding f of g can be viewed as a diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^p \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i} \\ \mathbb{R}^a & \xrightarrow{g} & \mathbb{R}^d \end{array}$$

where by the definition of f, f is transverse to i, and the diagram is a fiber product. Thus, we have

Definition 3. An *unfolding* of a map germ $g: U, x \to V, y$ is a diagram of map germs

$$\begin{array}{ccc} U', x' & \xrightarrow{G} & V', y' \\ \uparrow j & & \uparrow i \\ U, x & \xrightarrow[g]{} & V, y \end{array}$$

where G is C^∞-stable, i and j are germs of immersions and the square is formed from the fiber product of i and G, which are transverse.

Remark. For the case $n \leq p$, we must have $\hat{Q} \simeq Q$ and $\dim_{\mathbb{R}} Q < \infty$.

First example. For the algebra $Q = \mathbb{R}[[x]]/(x^2)$
1) we can obtain a stable germ $\mathbb{R}^1 \to \mathbb{R}^1$ using 1-generator x^2 so $L = (x)$ and $V = (0)$. Thus $y = x^2$ is a *C^∞-stable germ.*
2) We can obtain a stable germ $\mathbb{R}^2 \to \mathbb{R}^2$ using 1-generator x^2 so again $V = (0)$ and we obtain $y_1 = x^2$, $y_2 = t_2$. This is the fold of Whitney.
3) We can obtain a germ $\mathbb{R}^2 \to \mathbb{R}^3$ by using 2-generators $(x^2, 0)$; $L = (x) \times (0)$ and $\dim_{\mathbb{R}} V = 1$ generated by $(0, x)$. Thus, the equations are $y_1 = x^2$, $y_2 = t_1 x$, $y_3 = t_1$. This is the Whitney umbrella.

Second example. $\mathbb{R}[[x]]/(x^3)$. We obtain a map $\mathbb{R}^2 \to \mathbb{R}^2$ using 1-generator x^3 so $L = (x^2)$, $\dim_{\mathbb{R}} V = 1$ generated by x so $y_1 = x^3 + t_1 x$, $y_2 = t_1$. This is a cusp.

Third example. $\mathbb{R}[[x_n]]/(f)$. If we let $\Delta = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, then we can compare the unfolding of f from the point of view of catastrophy theory. For catastrophy theory, a basis $\{v_1, \ldots, v_k\}$ is chosen for $\mathscr{M}_n/\Delta$. Then, the universal unfolding is given by $f + \sum_{i=1}^{k} t_i v_i$. For the theory of mappings we consider a basis for $\mathscr{M}_n/\Delta + (f)$, $\{w_1, \ldots, w^l\}$, and form $(x, t) \mapsto \to (f + \sum_{i=1}^{l} t_i w_i, t)$. In particular, when $f \in \Delta$, for example, when f is homogeneous as in the case of elementary catastrophies, the stable map germ is just the graph of the unfolding in the sense of catastrophy theory. Thus from $x^3 + y^3$ we obtain the hyperbolic umbilic

$$f(x, y, t) = x^3 + y^3 + t_1 xy + t_2 x + t_3 y,$$

and the stable germ

$$\tilde{f}(x, y, t) = (f(x, y, t), t_1, t_2, t_3).$$

2. Moduli and the Nice Dimensions.

As stable mappings must be transverse to all $\mathscr{K}^k$-orbits for all k, then by exhibiting an open set of mappings which are not transverse to all orbits we will have shown that stable mappings are not dense. This behavior is due to the presence of moduli.

Definition 4. We say that *moduli* occurs near-by a k-jet $f \in J^k(n, p)$ if any neighborhood of f in $J^k(n, p)$ intersects an uncontable number of $\mathscr{K}^k$-orbits.

To see how moduli can prevent transversality, consider the case where the lines $y =$ constant in the xy plane represent orbits and the parabola $y = x^2$ the image under jet-extension. Then, the parabola is not transverse at its minimum and any small perturbation of it will still have a minimum and not be transverse there. This example can be made precise whenever there is a manifold of codimension $\leq n$ in some $J^k(n, p)$ which consists of orbits which form moduli [19, V].

The basic fact used in determining the structure of moduli is:

Proposition 7. *Let G be an algebraic group acting algebraically on an algebraic set V(possibly affine). Then, there is a minimal closed algebraic subset Π invariant under the action of G, such that $V \backslash \Pi$ contains only finitely many orbits.*

This is applied to $\mathscr{K}^k$ acting on $J^k(n, p)$ to obtain $\Pi^k(n, p) \subset J^k(n, p)$. As the projection $\pi_{kl} : J^k(n, p) \to J^l(n, p)$, $l \leq k$, commutes with the action of $\mathscr{K}$, we have $\pi_{kl}^{-1}(\Pi^l(n, p)) \subset \Pi^k(n, p)$. In fact, there is a k such that for $l \geq k$ $\pi_{lk}^{-1}(\Pi^k(n, p)) = \Pi^l(n, p)$; we let $\Pi(n, p) = \Pi^k(n, p)$ and $\sigma(n, p) = \operatorname{codim} \Pi^k(n, p)$ ($\sigma(n, p) = \infty$ if $\Pi^k = \phi$). Then, Mather shows [19, V]

Theorem 8. *The set of stable mappings is dense in $C^\infty_{pr}(N, P)$ iff $n < \sigma(n, p)$.*

One part of this theorem is easy to see, for the set of functions which miss $\Pi(N, P)$ is the countable intersection of open, dense sets. Similarly, the set of functions which are multitransverse to the finite number of orbits in $J^p(N, P) \backslash \Pi^p(N, P)$ is also a countable intersection of open dense sets. Thus, so is the intersection of these two sets, and this intersection consists of C^∞-stable mappings by theorem 6.

The set of (n, p) for which $n < \sigma(n, p)$ yields the range of dimensions known as the *nice dimensions*. The computation of $\sigma(n, p)$ is carried out by decomposing $\Pi(n, p)$ using the Σ_r singularity types,

$$\Sigma_r = \{f \in J^1(n, p) \mid \dim \ker Df = r\}.$$

We let $\Sigma_r^k = \pi_{kl}^{-1}(\Sigma_r)$, and apply the proposition (slightly modified for Zariski open subsets) to obtain $\prod_r^k(n, p)$. As before there is a k' such that $\prod_{lk}^{-1}(\prod_r^{k'}(n, p)) = \Pi_r^l(n, p)$ $l \geq k'$. We denote it by $\Pi_r(n, p)$. Then, $\Pi(n, p) = \bigcup_{r=1}^{n} Cl(\Pi_r(n, p))$.

(Cl denotes closure in $J^k(n, p)$).

We now describe the case $n \leq p$, where by good fortune most $\Pi_r(n, p)$ can be described using the second intrinsic derivative. If $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is a map germ of type Σ_r with $K = \ker D_0 f$ and $C = \operatorname{coker} D_0 f$, then $D_0^2 f: S^2\mathbb{R}^n \to \mathbb{R}^p$ ($S^2\mathbb{R}^n$ denotes the symmetric product) induces the *second intrinsic derivative* $\tilde{D}_0^2 f: S^2 K \to C$ (this changes linearly with a change of coordinates). Then, f is of $\Sigma_{r,(j)}$-type if $\dim \ker \tilde{D}_0^2 f = j$ (where $\tilde{D}_0^2 f$ is considered as a linear map). The importance of $\tilde{D}_0^2 f$ for the $\mathscr{K}^2$-orbits is that

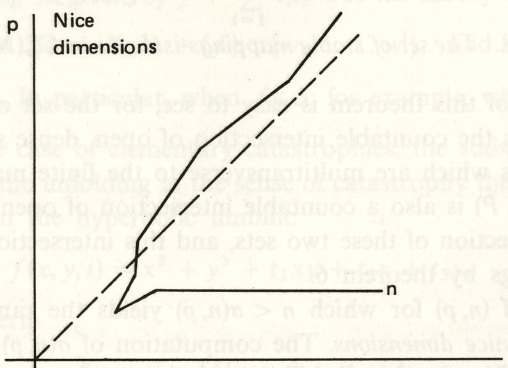
$$Q_2(f) \simeq \mathbb{R} \oplus K^* \oplus S^2 K^* / Im(\tilde{D}_0^2 f)^*$$

If we let $\bar{\Sigma}_{r,(i)} = \bigcup_{j>i} \Sigma_{r,(j)}$, then a calculation shows that moduli are dense in the relevant $\Sigma_{r,(i)}$ $r \geq 4$, $i \leq 2$ and $\Sigma_{3,(3)}$. In fact, $\Pi_r(n, p) = \Sigma_{r,(2)}$ $r \geq 4$, and $\Pi_3(n, p) \supset \Sigma_{3,(3)}$. Additional moduli must be computed to completely determine $\Pi_3(n, p)$ and to determine $\Pi_2(n, p)$. The computation yields

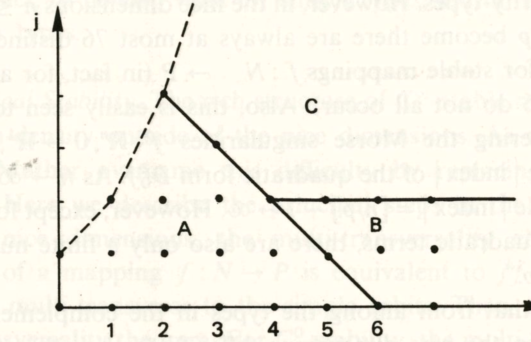
Theorem 9. *The dimensions (n, p) with $n \leq p$ belong to the nice dimensions iff*

$$n < \begin{cases} 6(p-n)+9 & 0 \leq p-n \leq 3 \\ 6(p-n)+8 & 4 \leq p-n \end{cases}$$

A similar type of computation works for $n > p$. When these two are combined we obtain the picture for the nice dimensions.



For a rough picture of the structure of $\Pi(n, p)$ for $n \leq p$, we look at the $\Sigma_{i,(j)}$-types.



We have that the only allowable region is below the dotted line as $j \leq \binom{i+1}{2}$. Then, the region C consists of moduli and $C \subset \Pi(n, p)$; for $i \geq 4$, $\prod_i(n, p)$ is the region of C lying over i. The complement of $\prod(n, p)$ lies in the interior of $A \cup B$ (including the i-axis). Next, to describe those algebra types within the region $A \cup B$ which occur in the nice dimensions we use the fact that if $\delta = \dim_{\mathbb{R}} Q < \infty$ and Σ_Q denotes the $\mathscr{K}$-orbit corresponding to Q then

$$\operatorname{codim} \Sigma_Q = (\delta - 1)(p - n) + \gamma$$

(where δ is as above and γ is a number computable from Q [19, VI]). Here codim Σ_Q refers to codimension of the $\mathscr{K}^k$-orbit for $k \geq p+1$. Then, for Q to occur as an algebra in the nice dimensions for some (n, p), $n \leq p$, we must have

$$n < 6(p-n) + \begin{cases} 9 & 0 \leq p-n \leq 3 \\ 8 & 4 \leq p-n \end{cases}$$

and

$$n \geq (\delta - 1)(p - n) + \gamma.$$

These two inequilities together determine whether Q can occur, and if so, for which (n, p).

The surprising result when actually determining these algebras is the following [10]

Theorem 10. *In the complement of $\Pi(n, p)$ $n \leq p$, there are a (countably) infinite number of distinct algebra types. However, in the nice dimensions $n \leq p$, there are only 76 algebra types.*

The number itself is not as surprising as the fact that it is finite because an analysis of the low dimensions indicates that as n and p increase so do the number of singularity-types. However, in the nice dimensions $n \leq p$, no matter how large n and p become there are always at most 76 distinct germ types which can occur for stable mappings $f: N \rightarrow P$ (in fact, for any particular (n, p) the entire 76 do not all occur). Also, this is easily seen to be false for $n > p$, by considering the Morse singularities $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^1, 0$. They are determined by the | index | of the quadratic form $D_0^2 f$. As $n \rightarrow \infty$, the number of different possible | index | $= [n/p] + 1 \rightarrow \infty$. However, except for this phenomena of adding quadratic terms, there are also only a finite number of possibilities for $n > p$.

We also see that from among the types in the complement of $\Pi(n, p)$, those that occur in the nice dimensions differ from the other types only in that their codimensions satisfy an inequality. We would expect any other general properties to be equally shared by all types in the complement of $\Pi(n, p)$.

Definition 5. We say that a C^∞-stable germ type $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^p, 0$ is of *simple type* if f belongs to the complement of $\Pi(n, p)$ (i. e., if $j^k f \in J^k(n, p) \backslash \Pi^k(n, p)$). A finite dimensional *algebra Q* is of *simple type* if $Q \simeq Q(f)$ for some f of simple type.

Furthermore, if we consider the moduli sets $\Pi_r(n, p)$ in Σ_r, then any $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^p, 0$ with $j^1 f \in \Sigma_r$ but f in the complement of $\prod_r(n, p)$ may now have moduli occurring nearby, but it will occur in a Σ_j-type with $j < r$. For these we define.

Definition 6. A C^∞-stable germ type $f: \mathbb{R}^n, 0 \rightarrow \mathbb{R}^p, 0$ is of *discrete algebra type* if f is of Σ_r-type and belongs to the complement of $\Pi_r(n, p)$. An *algebra Q* is of *discrete algebra type* if $Q \simeq Q(f)$ for f of discrete algebra type.

For C^∞-stable map germs f we can describe these two conditions in terms of the $\mathscr{K}^{p+1}$-orbits.

A *simple type* will have a neighborhood in $J^{p+1}(n, p)$ which intersects only finitely many $\mathscr{K}^{p+1}$-orbits (this is analogous to Arnold's definition for functions [1]). A *discrete algebra type* will have a neighborhood in $J^{p+1}(n, p)$ which intersects only finitely many $\mathscr{K}^{p+1}$-orbits of the same Σ_r-type. Algebras of discrete algebra type of Σ_r-type can be described as the largest class of algebras with generators, closed under deformations, which can be classified without using moduli.

The pleasant fact about discrete algebra types is that topological properties we will describe for C^∞-stable map germs in the nice dimensions turn out, in fact, to be also valid for simple types and in many cases for discrete algebra types.

Topological Stability. The rich structure of C^∞-stable mappings contrasts with the non-density outside of the nice dimensions. As mentioned earlier, Thom and Mather overcame this difficulty by replacing C^∞-stability by C^0-stability. Here we describe the principal steps in the proof.

In the nice dimensions, the multi-transversality characterization of C^∞-stability of a mapping $f: N \rightarrow P$ is equivalent to $j^k f$ missing $\Pi^k(N, P)$ and f being multi-transverse to the simple orbits. Then, density follows by the multitransversality theorem. For C^0-stability, the multi-transverse characterization is also used, except that $\Pi^k(N, P)$ is replaced by $\Sigma_{N,P} \subset J^l(N, P)$ and the simple orbits are replaced by a decomposition of $J^l(N, P) \backslash \Sigma_{N,P}$ into manifolds which form a Whitney stratification. Again codim $\Sigma_{N,P} >$ dim N, density follows by the multi-transversality theorem.

The construction of $\Sigma_{N,P}$, the decomposition of $J^l(N, P) \backslash \Sigma_{N,P}$, and the verification that mappings missing $\Sigma_{N,P}$ and multi-transverse to the stratification are C^0-stable depende upon a number of key ideas. The principal ones are:

1) stratifications satisfying Whitney's conditions a) and b) a generalization of condition a) for mappings due to Thom.

2) a result of Lojasiewicz which guarantees the existence of a Whitney stratification of semi-analytic sets;

3) Thom's isotopy theorems,

4) Mather's universal unfolding of germs and his generalization of this to a global unfolding of mappings.

We briefly describe these ideas and the part each of the plays in the C^0-stability theorem.

First of all, we state the whitney conditions for submanifolds $M, N \subset \mathbb{R}^k$ with $x \in M$:

Definition 7. (1) M, N satisfies *Whitney's condition a)* at x if for any sequence of points $x_i \in N$ such that $x_i \rightarrow x$ and $T_{y_i} N$ converges in the Grassmannian of n-planes in $\mathbb{R}^k$, say $\lim_{i \to \infty} T_{x_i} N = V$, then $V \supset T_x M$.

2) M, N satisfies *Whitney's condition b)* at x if for sequences $\{x_i\} \in N$, $\{y_i\} \in M$ with $x_i \rightarrow x$, $y_i \rightarrow x$. We consider the sequence of lines $\{\overline{x_i y_i}\}$ in

$\mathbb{R}P^{k-1}$ and suppose that it converges to a line l and as before $T_{x_i}N$ converges in the Grassmannian to V, then

$$l \subset V.$$

These conditions are invariant under diffeomorphisms so they can, in fact, be defined for submanifolds of a manifold. Also, condition *b)* implies condition *a)*.

These Whitney conditions are important for determining the local structure of stratifications.

Definition 8. A *Whitney stratification* of a subset $A \subset X$ (a C^1-manifold) is a decomposition of A into a union of disjoint submanifolds $\{N_\alpha\}$ (at least C^1) such that

1) each N_α is locally closed;
2) $\{N_\alpha\}$ is locally finite;
3) if $Cl(N_1) \cap N_2 \neq \phi$ then $N_2 \subset Cl(N_1)$;
4) if $N_2 \subset Cl(N_1)$ then N_2, N_1 satisfy conditions b) (and hence a)) at any point of N_2. (Mather prefers to call this a prestratification and to work with the associated Whitney stratification formed by taking at each point $x \in A$, the set germ at x of the stratum containing x.)

Because of the Whitney conditions, Whitney stratifications have two fundamental properties:

1) For any two sufficiently close points in the same strata, the stratification is topologically the same in neighborhoods of the points [21, - 8].

2) In a neighborhood of a point, the stratification is homeomorphic to a cone on a stratified set [21, - 8].

This controls quite strongly the structure of Whitney stratifications.

Next, Lojasiewicz proved a result which implies that there is a Whitney stratification of a strongly analytic submanifold X of a real analytic manifold M. By this we mean that X is a analytic submanifold and a semi-analytic set. Also, X is semi-analytic if at each point $x \in Cl(X)$, X is defined locally at x by a finite number of equations $f_i(x) = 0$, and inequalities $f_j(x) \geq 0$, $f_k(x) > 0$ with the $f_i(x)$ real analytic.

The condition for $Cl(X)$ instead of X is to avoid the situation where

$$X = \{(x, y) : y < \sin \frac{1}{x}, x > 0, \text{ and } y < 0,\ x \leq 0\}$$

As X is an open subset of $\mathbb{R}^2$, it is analytic; however, at 0 it is not locally semi-analytic. Then, Lojasiewicz shows [21, - 4]

Theorem 11. *If $X, Y \subset M$ are strongly analytic submanifolds with $Y \subset Cl(x)\setminus X$, then the set of points $x \in Y$ where Y, X fail to satisfy condition a) (respectively b)) is a semi-anlytic subset of M and nowhere dense in Y.*

This theorem can be used to construct a Whitney stratification of a semi--analytic set Y: By induction on k, construct a semi-analytic subset $N_k \subset Y$ of codimensions k consisting of points where N_{k-1} is not locally a submanifold of codimension k or (N_{k-2}, N_{k-1}) does not satisfy condition b). Then $Y = \bigcup_{i=0}^{n-1} (N_i \setminus N_{i+1})$ where $n = \dim Y$ with $N_{i-1}\setminus N_i$ analytic and the $\{N_{i-1}\setminus N_i\}$ satisfy condition *b)*.

Mather modifies this consctruction to obtain the following stratification of the jet-space [21- 9].

Theorem 12. *For each pair of positive integers (n, p) there exists an integer k, a closed semi-algebraic subset $\Sigma \subset J^k(n, p)$ invariant under $\mathscr{A}^k$ and a Whitney stratification $\mathscr{S}$ of $J^k(n, p)\setminus\Sigma$, also invariant under $\mathscr{A}^k$ such that*

1) codim $\Sigma > n$;

2) $\mathscr{S}$ is a stratification of $J^k(n, p)\setminus\Sigma$ with only finitely many strata, each of which is semi-algebraic;

3) Let $f : N \to P$ be a proper smooth mapping. Let $\Sigma_{N,P}$ denote the subfiber bundle of $J^k(N, P)$ formed from Σ, and $\mathscr{S}_{N,P}$ the Whitney stratification of $J^k(N, P)\setminus\Sigma_{N,P}$ formed from $\mathscr{S}$. Then, if f satisfies $j^k f(N) \cap \Sigma_{N,P} = \phi$ and $j^k f$ is multi-transverse to the stratification $\mathscr{S}_{N,P}$ then f is topologically stable.

Then the multi-transversality theorem implies that the set of topologically stable mappings is (open) dense in $C^\infty(N, P)$.

The proof of 3) of the theorem uses a global version of the local unfolding described earlier. Given an $f : N \to P$, which misses $\Sigma_{N,P}$, with N compact, there is a fiber diagram

$$\begin{array}{ccc} N' & \xrightarrow{F} & P' \\ \uparrow i & & \uparrow j \\ N & \xrightarrow{f} & P \end{array}$$

with F C^∞-stable and j and i embeddings. This is a global unfolding of f. If a mapping $g : N \to P$ is sufficiently near f, then a homotopy f_t from f to g factors through F.

$$\begin{array}{ccc} N' & \xrightarrow{F} & P' \\ \uparrow i_t & & \uparrow j_t \\ N \times I & \xrightarrow{f_t} & P \times I \end{array}$$

with the diagram a fiber diagram for each t. Furthermore, the f_t can be chosen to miss $\Sigma_{N,P}$ and be multi-transverse to $\mathscr{S}$ for all t. Then, this turns out to be equivalent to j_t being transverse to a stratification of P' defined from Theorem 12. Then, it is enough to use Thom's second isotopy theorem. This applies to mappings (at least C^2)

$$X_1 \xrightarrow{f} X_2 \xrightarrow{\pi} Y$$

with $A_1 \subset X_1$ and $A_2 \subset X_2$ closed subsets with Whitney stratifications $\mathscr{P}_1$ and $\mathscr{P}_2$. Suppose that rank $D(f \mid N_\alpha)$ is constant on each stratum $N_\alpha \in \mathscr{P}_1$. Then, we say that f satisfies Thom's condition (a_f) if for each pair of strata (N, M) with $x \in M$ and $x_i \in N$ so that $x_i \to x$ and ker $D_{x_i}(f \mid N)$ converge in an appropriate Grassmannian to V, then $D_x(f \mid M) \subset V$. Then, we can state

Theorem 13. (*Thom's second isotopy theorem*). *Suppose the mappings* $X_1 \xrightarrow{f} X_2 \xrightarrow{\pi} Y$ *are as above and f satisfies Thom's condition* (a_f). *Also, suppose f maps each stratum of $\mathscr{P}_1$ submersively to a stratum of $\mathscr{P}_2$, and each stratum of $\mathscr{P}_2$ is mapped submersively onto Y by π. Also, suppose $f \mid A_1 : A_1 \to A_2$ and $\pi \mid A_2 : A_2 \to Y$ are proper. Then f is locally trivial over Y, i.e., for $y \in Y$ there is a neighborhood $U \subset Y$ and a mapping $g : B_1 \to B_2$ and homeomorphisms $U \times B_1 \xrightarrow{\sim} (f \circ \pi)^{-1} U = A_1 \mid U$ and $U \times B_2 \xrightarrow{\sim} \pi^{-1}(U) = A_2 \mid U$ so that the following diagram commutes*

$$\begin{array}{ccccc} A_1 \mid U & \xrightarrow{f \mid U} & A_2 \mid U & \xrightarrow{\pi} & U \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq 1_U \\ B_1 \times U & \xrightarrow{g \times 1} & B_2 \times U & \xrightarrow{\pi} & U \end{array}$$

Then the theorem follows using the Thom isotopy theorem and the connectedness of the interval.

Because of the existential nature of this proof almost nothing is known about the local structure of these topologically stable mappings and this provides the *big unanswered question for topological stability*. However, the

properties of the construction of the stratification do allow us to state several properties.

1) If $f, g\ \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ are germs of topologically stable mappings in the preceding sense and $j^k f$ and $j^k g$ are in the same connected component of a strata of $\mathscr{S}$, then f and g are topologically equivalent. Because there are only a finite number of components, there are only a finite number of germ types.

2) If $f : N \to P$ is multi-transverse to $\mathscr{S}$ (and $j^k f(N) \cap \Sigma = \phi$), then so is $f \times 1 : N \times \mathbb{R} \to P \times \mathbb{R}$ (i.e., the property is preserved under suspension).

3) The fact that $\mathscr{S}$ is invariant under $\mathscr{A}^k$ implies that C^∞-stable mappings are also multi-transverse to $\mathscr{S}$.

3. C^0-Classification of C^∞-Stable Mappings – Local Problem.

We begin to examine here the first of three questions asked earlier about the relation between the C^∞-stable and C^0-stable mappings. As might be expected from our description of stable mappings, this question has both a local form and a global one. We begin with the local problem which can be rephrased: Are there distinct C^∞-stable map germs in the nice dimensions which are topologically equivalent? If so, then, in fact, the C^∞-classification of stable germs would have repetitions from the topological point of view. We will restrict ourselves to the case of $n \le p$.

An optimistic approach to this problem would be to seek enough topological invariants for C^∞-stable map germs to distinguish between any two of them in the nice dimensions. Preferably these invariants should depend on the associated algebra. This is the approach we will take. We begin by looking at one quite natural topological invariant of a map, namely, its fibers.

Real Multiplicity. As an example of this we recall the results of Milnor [28, 5.3] for the fiber of a complex algebraic map $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with an isolated singularity at 0. He has shown that for z_0 sufficiently close $0 \in \mathbb{C}$, and a ball B_ε of radius ε sufficiently small about $0 \in \mathbb{C}^{n+1}$, that the fiber

$$F_{z_0} = f^{-1}(z_0) \cap B_\varepsilon$$

only has reduced homology in the middle dimension n and that $\tilde{H}_n(F_{z_0}) = \mathbb{Z}^\mu$ where μ, the Milnor number, can be computed as

$$\mu = \dim_{\mathbb{R}} C[z_1, z_2, \ldots, z_n] \Big/ \left(\frac{\partial f}{\partial z}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

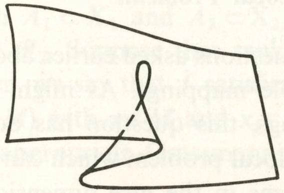
Question. Is there any analogous result for map germs that we are now considering?

Two principal differences are:

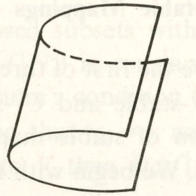
1) Our mappings are real mappings $f:\mathbb{R}^n,0\to\mathbb{R}^p,0$, which tend to behave badly.

2) $n\le p$ so stable map germs will have fibers consisting of a finite number of points.

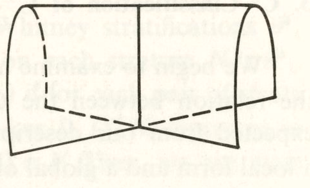
A few examples with their associated algebras indicate that the fibers can vary considerably. However, there is equality between the maximum fiber and $\delta(f)=\dim_{\mathbb{R}}Q(f)$. This leads us to define



cusp: $\mathbb{R}[[x]]/(x^3)$



fold: $\mathbb{R}[[x]]/(x^2)$



Whitney umbrella: $\mathbb{R}[[x]]/(x^2)$

Definition 9. *The real multiplicty of a map germ $f:\mathbb{R}^n,0\to\mathbb{R}^p,0$ is defined as $m(f)=\{max\ k\mid$ given a neighborhood $V\in 0$ in $\mathbb{R}^n$, there is a $y\in\mathbb{R}^p$ so that $|f^{-1}(y)\cap V|=k\}$. ($|S|$ = cardinality of a set S.)*

Then, a basic result used in the topological analysis of stable germs is a theorem proven with Andre Galligo [7].

Theorem 14. *If $f:\mathbb{R}^n,0\to\mathbb{R}^p,0$ is a stable map germ of Σ_2-type or of discrete algebra type, then*

$$m(f)=\delta(f).$$

Remark. In the case $n=p$, there is another approach of Levine and Eisenbud, which takes into account whether at points $f^{-1}(y)$, f preserves or reverses orientation. Their method does not even require that f be stable [12], [17].

We give a brief idea of the proof. There are three steps:

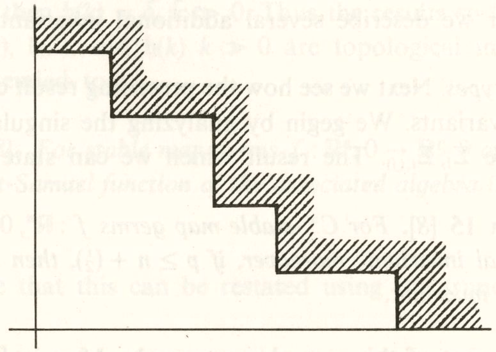
1) It is shown that for any two C^∞-stable germs $f_i:\mathbb{R}^{n_i},0\to\mathbb{R}^{p_i},0$ $i=1,2$ with $Q(f_1)\simeq Q(f_2)$ that $m(f_1)=m(f_2)$. Thus, $m(f)$ is independent of f and only depends on $Q(f)$. (This much is true without any restriction on $Q(f)$.)

2) Next, because for a germ f with $dim_{\mathbb{R}}Q(f)<\infty$, it is known $m(f)\le\delta(f)$ [12, VII, - 2). It is sufficient to find a not necessarily stable $g:\mathbb{R}^m,0\to\mathbb{R}^q,0$ with $Q(g)\simeq Q$ and $m(g)\ge\delta(g)$. From this g an appropriate f can be constructed by unfolding.

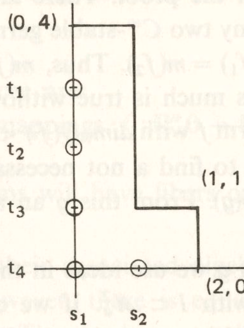
3) Lastly, to find such a g we use ideas in the theory of deformations of ideals. Let $Q\simeq\mathbb{R}[[x_a]]/I$ with $I\subset\mathcal{M}_a^2$. If we choose generators $r_1,\dots,r_b$ for I, then a deformation of I is a mapping $R:\mathbb{R}^{a+l}\to\mathbb{R}^b$ given by $R(x,t)=(R_1(x,t),\dots,R_b(x,t))$ such that $R_i(x,0)=r_i(x)$. This induces a mapping $\tilde{R}(x,t)=(R(x,t),t):\mathbb{R}^{a+l}\to\mathbb{R}^{b+l}$. This is an unfolding, although it may not be universal. Also, $Q(\tilde{R})\simeq Q$. We construct such an $\tilde{R}$ so that $m(\tilde{R})\ge\delta(\tilde{R})=\delta(Q)$.

It is in the construction of $\tilde{R}$ that we are limited by the specific types. Let us consider a simple example to illustrate how $\tilde{R}$ is constructed.

Let $Q\simeq\mathbb{R}[[x,y]]/(x^2+y^3,xy)$. We construct a diagram of (x^2+y^3,xy) in the first quadrant of the xy-plane as follows: *If* $f\in(x^2+y^3,xy)$, we choose a point (a,b) for f. This (a, b) is determined by the smallest non-zero term. cx^ay^b of f, with respect to the reverse lexicographical ordering of (a b) (so that $(a,b)<(a',b')$ if $b<b'$ or $b=b'$ and $a<a'$). If we form the set of all such points for all $f\in(x^2+y^3,xy)$ we obtain a set which is called the "stairs" of the ideal (x^2+y^3,xy). This was first used by Hironaka [14]. The stairs of a typical ideal has the form



In our case we have for x^2+y^3, $(2,0)$; for xy, $(1,1)$; and for y^4, $(0,4)$. These determine the stairs for $I=(x^2+y^3,xy)$.



We notice that the number of points in the complement of the stairs contains 5 points and $= \dim_{\mathbb{R}} Q$. We will construct an R which has one point for each such point in the complement. Then, $m(\tilde{R}) \geq 5$ as desired. Consider

$$R_1 = (x - s_1)(x - s_2) + (y - t_2)(y - t_3)(y - t_4)$$

$$R_2 = (y - t_1)(x - s_1)$$

$$R_3 = (y - t_1)(y - t_2)(y - t_3)(y - t_4).$$

(R_1, R_2, R_3) is a deformation of $(x^2 + y^3, xy, y^4) = I$. We count the solutions to $R_i = 0$:

If $y = t_1$ is sufficiently small, R_1 will still have two solutions, $x = s_1', s_2'$; If $x = s_1$ then there are solutions $y = t_2, t_3, t_4$. These give five points for distinct t_i.

For the case of $I \subset \mathbb{R}[[x_1, \ldots, x_n]]$, $n > 2$, only I with special stairs will have the approriate deformations [7]. We will return to discuss this situation after we describe several additional invariants.

$\Sigma_i, \Sigma_{i,(j)}$*-types.* Next we see how the preceding result can be used to obtain additional invariants. We gegin by analyzing the singularity structures near points of type $\Sigma_i, \Sigma_{i,(j)}$. The result which we can state is

Theorem 15 [8]. *For C^∞-stable map germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, the Σ_i-type is a topological invariant; moreover, if $p \geq n + \binom{i}{2}$, then the $\Sigma_{i,(j)}$-type is also a topological invariant.*

The first part of this was also proven by May in [25].

We describe the idea of the proof for Σ_i; the case for $\Sigma_{i,(j)}$ is similar. The basic fact we use is that for a stable map germ f, $\Sigma_i(f) \cup \Sigma_{i-1}(f)$ is not locally a topological manifold near points of $\Sigma_i(f)$ when $i > 1$ [11].

Then the idea is to inductively "peel away" the Σ_i-types by the topological procedure of removing at the i-th step all remaining points where the remaining singularity set is locally a topological manifold. This will work once we have removed $\Sigma_0(f)$. To do this topologically, we note that at a point $x \in \Sigma_0(f)$, *f is locally an immersion so* $m(f_{(y)}) = 1$. For points $x \in \Sigma_1(f)$, $m(f_{(y)}) = \delta(f_{(x)}) > 1$. Thus, we can remove Σ_0 by removing those x where $m(f_{(x)}) = 1$. The proof for $\Sigma_{i,(j)}$ is similar, working in Σ_i and using induction on j again beginning with $m(f)$ to remove $\Sigma_{i,(0)}$.

Now we seem to have two distinct types of topological invariants, δ and $\Sigma_i, \Sigma_{i,(j)}$. The $\Sigma_{i,(j)}$-types are defined by properties of Df, D^2f, while δ has a direct geometric interpretation. The next step is to see that, in fact, the invariants fit into a common framework.

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be of $\Sigma_{i,(j)}$-type with $Q(f) \simeq Q$, and maximal ideal $\mathcal{M}$. Then we have

$$\dim_{\mathbb{R}} Q/\mathcal{M}^2 = 1 + i$$

and

$$\dim_{\mathbb{R}} Q/\mathcal{M}^3 = 1 + i + j.$$

This suggests using the Hilbert-Samuel function of the algebra.

Hilbert-Samuel Function. We recall that if Q is a local algebra with maximal ideal $\mathcal{M}$ then the Hilbert-Samuel function of Q is defined by

$$\mathfrak{h}(n) = \dim_{\mathbb{R}} Q/\mathcal{M}^{n+1} \quad n \geq 0.$$

If $\dim_{\mathbb{R}} Q = \delta < \infty$ then $\mathfrak{h}(k) = \delta$, $k \gg 0$. Thus the results so far can be restated as saying $\mathfrak{h}(1)$, $\mathfrak{h}(2)$, and $\mathfrak{h}(k)$ $k \gg 0$ are topological invariants. This can, in fact, be extended to

Theorem 16 [8]. *For stable map germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ of discrete algebra type, the Hilbert-Samuel function of the associated algebra is a topological invariant.*

Let us first see that this can be restated using the truncated algebras $Q_k(f)$. In fact,

$$Q_k(f) \simeq Q(f)/\mathcal{M}^{k+1}$$

Thus, if $\delta_k(f) = \dim_{\mathbb{R}} Q_k(f)$, then $\mathfrak{h}(k) = \delta_k$, $k \geq 0$. An equivalent version of the theorem is.

Theorem 17. *For stable map germs f of discrete algebra type, the $\delta_k(f)$, $k \geq 0$ are topological invariants.*

First, let us make several remarks about these theorems. Certainly one of the key questions concerns the limitations imposed on the algebra types.

Question. To what extent should we hope to extend these results in the regions where moduli occur?

The first step would be to extend theorem 14 relating $m(f) = \delta(f)$. Unfortunately, we already have here a serious problem due to a result of Iarrobino [15]. This result implies: *that for $k > 2$, there are ideals $I \subset \mathbb{C}[[x_k]]$ such that any (complex) deformation of I, R must satisfy*

$$m(\tilde{R}) < \delta(I) = \dim_{\mathbb{R}} \mathbb{C}[[x_k]]/I.$$

Furthermore, I can be chosen to have homogeneous polynomials as generators.

For $k = 3$, there are such I with $\delta = 102$; for $k = 4$, I with $\delta = 24$; $k = 5$, I with $\delta = 26$. Moreover, for $k \geq 6$, "most" ideals of type $\Sigma_{k,(4)}$ and for $k \geq 8$ "most" ideals of type $\Sigma_{k,(3)}$ satisfy Iarrobino's result. Thus this comes very close to describing the complement of the discrete algebra for $k \geq 6$; and without doubt Iarrobino's result is not the strongest possible!

This leads to three fundamental questions about Iarrobino's result.

1) As $m(f)$ only depends on $Q(f)$ for stable f, how can it be computed from Q when $m(f) < \delta(f)$?

2) The difficulty already occurs for complex ideals. Is it then true that if for a stable polynomial germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we define the complexification $f_{\mathbb{C}} : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, then

$$m(f) = m(f_{\mathbb{C}})?$$

(It would be interesting to even know this for $n = p$.)

3) Iarrobino's result is a purely existential result. Give a specific example of such an I.

Because of Iarrobino's result, the best place to begin extending the results for the Hilbert-Samuel function is for Σ_2-types. Here work has only just begun, but preliminary work with Andre Galligo using the stratification of Briançon [2] suggests that at least the first $k+1$ terms of $\mathfrak{h}(\ \ )$ will be topological invariants when $p - n = k$.

In fact, it would seem that discrete invariants like the Hilbert-Samuel function are exactly the type which would be useful in analyzing C^0-stable mappings because they are not sensitive to moduli.

To see how the theorem is proven, we look at a stable germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ as a deformation of either algebra $Q(f)$, or ideal $I(f)$ by $x \to Q(f_{(y)})$, or $I(f_{(y)})$, for $x \in \mathbb{R}$. Then, the idea is to inductively use information about the Q or I occurring in the deformation to derive information about $Q(f)$.

There are several facts needed to allow this inductive process to work. First, we have that for any fixed k, $\mathfrak{h}(k)$ is upper semi-continuous as a function of $Q(f_{(x)})$. Secondly, we look at what the theorem says for particular $\Sigma_{i,(j)}$-types. For $\Sigma_{i,(0)}$ and $\Sigma_{i,(1)}$, the H-S function has the values 1, $i+1$, $i+2, \ldots, \delta, \delta, \ldots$. Thus, it determined by the $\Sigma_{i,(j)}$-type and δ. Thus, this leaves the $\Sigma_{2,(2)}$, $\Sigma_{2,(3)}$ and $\Sigma_{3,(2)}$ types.

These are analyzed by using the inductive process on the $\Sigma_{i,(j)}$ types. In fact, it is not necessary to analyze all information about the $\mathfrak{h}(\ \ )$ for $Q_{(x)}$, for instance, information about $\delta(Q_{(x)})$, $\mathfrak{h}(3)$, and several other specific $\mathfrak{h}(\ \ )$ is enough to determine $\mathfrak{h}(\ \ )$ for these three cases.

Complex Algebra Type. Returning to the problem of the topological classification, we see that the Hilbert-Samuel function fails to distinguish between many algebra types. However, it is possible to improve upon the topological classification with the next result.

Theorem 18 [9]. *For $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ which are C^∞-stable map germs of simple type, $Q(f) \otimes_{\mathbb{R}} \mathbb{C}$ is a topological invariant.*

The restriction to complex algebra type is necessary because the invariants do no distinguish between real algebra types which differ only by a sign in a generator; for example, $\mathbb{R}[[x, y]]/(x^4 + y^4, xy)$ and $\mathbb{R}[[x, y]]/x^4 - y^4, xy)$. Very likely a more thorough analysis of the singularity sets or a modified version of the invariant of Levine and Eisenbud will allow us to distinguish the few remaining cases.

Similarly, the restriction to simple types is again due more to the lack of analysis of certain singularity types, than the expectation that the theorem is false for some discrete algebra type. In fact, a most natural conjecture would be

Conjecture. *$Q(f)$ is a topological invariant for C^∞-stable map germs of discrete algebra type. This would then say that in the region where moduli are not needed to classify algebra types, our most optimistic expectation of C^∞-type being equivalent to C^0-type would be correct.*

For the proof of theorem, we have to look more closely at the deformation $x \to Q(f_{(x)})$ in terms of complex algebra types. For this we use

Definition 10. If $f: \mathbb{R}^n, 0$ is a C^∞ stable germ, then the algebra type Q is a *near-by algebra type* if for any neighborhood of 0, $U \subset \mathbb{R}^n$, there is an $x \in U$ with $Q(f_{(x)}) \simeq Q$.

This is analogous to the idea of adjacency used for functions [29].

It is easy to see that Q_1 being near-by $Q(f)$ corresponds to $\mathscr{K}^{p+1}$. f being in the closure of $\mathscr{K}^{p+1} \cdot Q_1$ ($\mathscr{K}^{p+1}$ orbit with $Q_{p+1} \simeq Q_1$). Then, one way to verify the theorem would be to inductively analyze the structure of the sets where given algebra types occur in the deformation. In fact, there is a very simple way to obtain a good deal of information about near-by algebra types Q_1 when $\delta(Q_1) = \delta(Q(f))$.

If we define an invariant $\lambda = \gamma - \delta$ (δ and γ defined earlier), then a codimension argument gives:

Proposition 19: *If Q_1 is near-by Q_2 with $\delta(Q_1) = \delta(Q_2)$ then $\lambda(Q_1) < \lambda(Q_2)$.*

This invariant λ can be used to destinguish between types in a series of deformations.

Type			λ
$\Sigma_{2,1}$	$(x^2 \pm y^k, xy^2)$		1
$\Sigma_{3,(1)}$	$(x^2 \pm z^k, y^2 \pm z^k, xy, xz, yz)$		2
$1\gamma 2$		$(x^2 \pm z^k, y^2 \pm z^k, yz, xz)$	1
		$(x^2 \pm z^k, y^2, xz, yz)$	2
		$(x^2, y^2, z^{k+1}, xz, yz)$	3
$1\varepsilon 1$	$(x^2 + yz, xy, xz, z^2 \pm y^k)$		2
	$(x^2 + yz, xy, xz + y^k, z^2)$		3
	$(x^2 + yz, xy, xz, z^2, y^{k+1})$		4
$1\varepsilon 2$		$(xy, xz, yz, z^2 \pm y^k, x^3)$	3
		$(xy, xz + y^k, yz, z^2, x^3)$	4
		$(xy, xz, yz, z^2, y^{k+1}, x^3)$	5

We illustrate this with an example. In the diagram the types $1\gamma 2$, $1\varepsilon 1$, and $1\varepsilon 1$, and $1\varepsilon 2$ are all $\Sigma_{3,(2)}$ types with the same Hilbert-Samuel function. We suppose that preliminary analysis has revealed

1) that the four groups are topologically distinct;
2) that the $\Sigma_{2,1}$ and $\Sigma_{3,(1)}$ types indicated are topological invariants; and
3) a line indicates there is a deformation from the lower to the upper type.

With this information and the λ invariant we can conclude that all the types are topologically distinct. We illustrate this for those of type $1\gamma 2$.

For example, for type $1\gamma 2$, the type with $\lambda = 3$ is topologically distinct from the other two because it has the $\Sigma_{3,(1)}$-type near-by. However, by the λ-invariant neither of the other two can have this one near-by. Similarly the one with the $\lambda = 2$ has the $\Sigma_{2,1}$-type near-by while the one with $\lambda = 1$ cannot. As the $\Sigma_{2,1}$ and $\Sigma_{3,(1)}$-types are topological invariants, the three $1\gamma 2$ types are topologically distinct. Then we use the $1\gamma 2$ to distinguish between the $1\varepsilon 1$, and the $1\varepsilon 1$, for the $1\varepsilon 2$; and conclude that they are all topologically distinct.

This type of argument is exactly what is used to refine the results for Hilbert-Samuel functions, to obtain the theorem.

The preceding discussion might be summarized by saying that there are enough easily assessible invariants, δ, b( ), λ, near-by algebray types, etc., to distinguish topologically between complex algebra types (and probably real algebra types). However, these do not provide a systematic theory to explain the singularity structures but rather only parts of the structure. A general question then is:

Question: Is there a natural model do describe the bifurcation theory of singularity or algebra types?

For instance, can we find models of the bifurcation sets (or their complements) similar to those obtained for the bifurcation of functions by Arnold, Brieskorn, and Looijenga [1], [4], [18]? Perhaps this question will be best answered for complex stable map germs $f: \mathbb{C}^n \to \mathbb{C}^p$, while in the real case, we must settle for the usual partial answers.

4. Global Topological-Classification of C^∞-stable Mappings.

We examine what conclusions we can make for global topological properties in the nice dimensions ($n \leq p$) from the local topological classification. Suppose that $f: N \to P$ is a stable mapping ($n \leq p$). Then we have a stratification of N by complex algebra-types. Let $\Sigma_{\tilde{Q}}(f) = \{x \in N \mid Q(f_{(x)}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \tilde{Q}\}$ for some complex algebra $\tilde{Q}$. Then, the local classification can be rephrased by saying that if f and g are C^0-equivalent (f, g C^∞*-stable mappings*), where $f = \psi \circ g \circ \phi$, then ϕ preserves the stratification of N given by $\{\Sigma_{\tilde{Q}}(f)\}$, $\{\Sigma_Q(g)\}$ and ψ preserves the images in P of these stratifications.

This, in turn, yields a simple consequence for the induced map on cohomology ϕ^*. Since ϕ preserves $\Sigma_{\tilde{Q}}$, and hence their closures $\overline{\Sigma}_{\tilde{Q}}$, ϕ^* preserves the dual cohomology classes to these singular submanifolds. These are the Thom polynomials for the singularity sets as described in Levine's lectures [17]. If $P_Q(f)$ denotes the Thom polynomial for $\Sigma_Q(f)$, then we can summarize this with

Corollary 20. *If* $f, g : N \to P$ *are* C^∞*-stable mappings in the nice dimensions* $n \le p$*; and they are* C^∞*-equivalent by* (ψ, ϕ) *as above, then* $\phi^* P_{\tilde{Q}}(g) = P_{\tilde{Q}}(f)$ *for all simple complex algebra types* $\tilde{Q}$.

Certainly we cannot expect the condition for $\{\Sigma_{\tilde{Q}}\}$ to be sufficient, so at this point we suppose that the conjecture of the topological invariance of $Q(f)$ is established for simple algebra types. Then, if $f : N \to P$ is stable, we have the multi-stratification of N given by $_r j^{p+1}(f)^{-1}(\Sigma_\alpha)$, $r \ge p+1$, where the Σ_α are multi-$\mathscr{K}^{p+1}$ orbits in $_r J^{p+1}(N, P)$. Then, as $Q(f_{(x)}) = Q_{p+1}(f_{(x)})$ are topological invariants, so are the multi-orbits. Thus any C^0-equivalence

$$\begin{array}{ccc} N & \xrightarrow{\ f\ } & P \\ \simeq \downarrow \phi & & \uparrow \psi \simeq \\ N & \xrightarrow{\ g\ } & P \end{array}$$

for f, g stable must in fact preserve the multi-stratification in N and its image in P. The basic global problem then is

Problem: Given a topological equivalence as above, can it be replaced by a C^∞-equivalence?

Unfortunately the general answer to this is no. We describe a general procedure for constructing counterexamples to it. For this we quote a theorem of Mazur [27], [30] from geometric topology.

Theorem 21. (*Mazur*) *Let* M_1 *and* M_2 *be smooth manifolds of dimension* n *such that there is a homotopy equivalence* $f : M_1 \to M_2$. *If* f^*TM_2 *and* TM_1 *are stably equivalent as vector bundles (i.e., there is a trivial bundle* θ^l *over* M_1 *such that* $TM_1 \oplus \theta^l \to f^*TM_2 \oplus \theta^l$*) then there is a diffeomorphism*

$F : M_1 \times \mathbb{R}^N \to M_2 \times \mathbb{R}^N$ *(sufficiently large* N*) such that* F *is homotopic to* $f \times 1^N$.

We choose a homeomorphism $f : M_1 \to M_2$ between smooth manifolds M_1, M_2, which are not diffeomorphic, such that f^*TM_2 is stably equivalent to TM_1. For example, we can choose M_1, and M_2 to be a Brieskorn sphere and a regular sphere, both of which are stably parallelizable. Let N be the smallest integer such that there is a diffeomorphism

$$\phi : M_1 \times \mathbb{R}^N \to M_2 \times \mathbb{R}^N.$$

We define two proper C^∞-stable mappings $g_1, g_2 : M_1 \times \mathbb{R}^N \to M_1 \times \mathbb{R}^N$ which are C^0-equivalent but C^∞-distinct. Let $k : \mathbb{R}^N \to \mathbb{R}^N$ be the suspension of the mapping $y = x^2$ given by $k(x_1, \ldots, x_N) = (x_1, \ldots, x_{N-1}, x_N^2)$. As the suspensions of C^∞-stable mappings are C^∞-stable, we have that

$$g_1 = 1 \times k : M_1 \times \mathbb{R}^N \to M_1 \times \mathbb{R}^N$$

and

$$g_2 = \phi^{-1} \circ (1 \times k) \circ \phi : M_1 \times \mathbb{R}^N \to M_1 \times \mathbb{R}^N$$

are C^∞-stable. They are clearly topologically equivalent as $1 \times k : M_1 \times \mathbb{R}^N \to M_1 \times \mathbb{R}^N$ and $1 \times k : M_2 \times \mathbb{R}^N \to M_2 \times \mathbb{R}^N$ are. However,

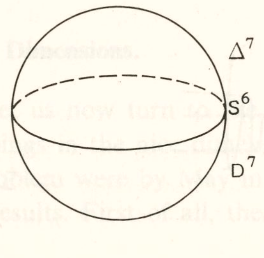
$$\Sigma_1(g_1) = M_1 \times \mathbb{R}^{N-1} \times \{0\},$$

and

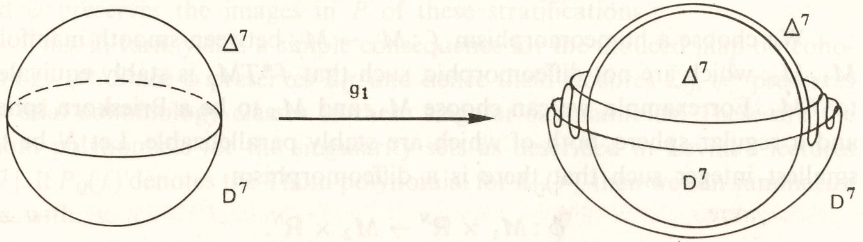
$$\Sigma_1(g_2) = \phi^{-1}(M_2 \times \mathbb{R}^{N-1} \times \{0\}).$$

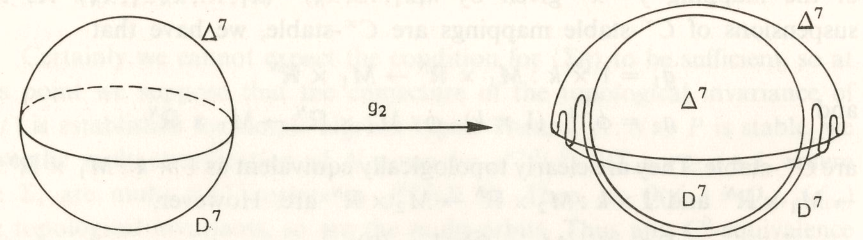
By assumption, $M_1 \times \mathbb{R}^{N-1}$ and $M_2 \times \mathbb{R}^{N-1}$ are not diffeomorphic; hence neither are $\Sigma_1(g_1)$, $\Sigma_1(g_2)$. Thus, g_1 and g_2 are C^∞-distinct. Also, the singularity k could be replaced by any other stable singularity type, with just a slight modification of the construction. In fact, it can be arranged that g_1 and g_2 are in the nice dimensions.

It might be thought that this phenomena can only occur for open manifolds such as $M_1 \times \mathbb{R}^N$. However, there is also a simple counter example for C^∞-stable mappings $M \to M$ where M is an exotic 7-sphere. Such an exotic sphere can be decomposed $M = \Delta^7 \cup_{S^6} D^7$ where (Δ^7, S^6) is a 7-disc with boundary S^6 homeomorphic to (D^7, S^6), but not diffeomorphic to it mod I_{s^6}.

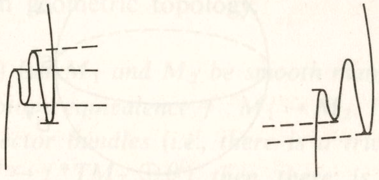


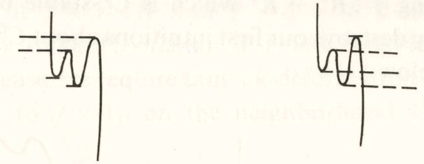
We consider mappings with just fold singularities indicated as follows:





The four fold-singularity sets decompose M into a Δ^7, D^7 and 3 cylinders $S^6 \times (0, 1)$. For g_1, we see that the Δ^7 has the preimages of three fold "curves", while D has the preimage coross section of g_1 of two. For g_2, we have that Δ^7 has two preimages and D^7 has three. Thus, the g_i are topologically equivalent by interchanging Δ^7 and D^7. However, a C^∞-equivalence (φ, ψ) would preserve preimage curves, and hence interchange Δ^7 and D^7 cross section of g_2 and the cylinders $S^6 \times$ I. Then, the restrictions to S^6 of $\varphi : D^7 \to \Delta^7$ and $\varphi : \Delta^7 \to D^7$ would be pseudo-isotopic, and hence isotopic. Then, this implies that M is of order 2 in the group of oxotic 7-spheres. Thus, for any M not of order 2, there cannot be a C^∞-equivalence (Mike Shub and Dave Tischler pointed out the possibility of order 2 occurring).





A first conjecture for dimensions where we can avoid these problems would be

Conjecture. *If dim $N < 7$, then the C^0 and C^∞ classification of C^∞-stable mappings $N \to P$ agree.*

To date, the only work in this direction follows from the result of Leslie Wilson described by Harold Levine, [17], [37]. The result of Wilson implies that if $f, g : M \to N$ are C^∞-stable mappings between 2-manifolds which are C^0-right equivalent (i.e., there is a homeaomorphism $h : M \to M$ such that $f \circ h = g$) then f and g are C^∞-equivalent. Beyond this nothing known.

A general conjecture for the dimensions $(n \le p)$ might be

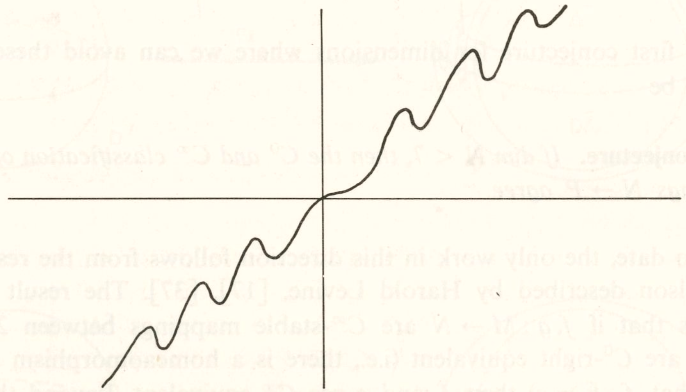
Conjecture. *Suppose that f and $g : N \to P$ are C^∞-stable mappings and C^0-equivalent. Suppose further that for the multi-stratification $\{\Sigma_\alpha\}$ and its image $\{\Gamma_\alpha\}$ we have $(\overline{\Sigma}_\alpha, \overline{\Sigma}_\alpha \backslash \Sigma_\alpha)\,(f)$ diffeomorphic to $(\overline{\Sigma}_\alpha, \overline{\Sigma}_\alpha \backslash \Sigma_\alpha)\,(g)$ for all α and and also for Γ_α in place of Σ_α, then f and g are C^∞-equivalent.*

This avoids the difficulties we have described; but can the pieces of diffeomorphisms be put together in a smooth way?

Certainly the success that we had with the local classification fades quickly when try to apply the results to the global C^0 classification. We next turn to the question of C^0-stability in the nice dimensions where the local C^0-classification proves to be invaluable.

5. C^0-stability in the Nice Dimensions.

Types of C^0-stability Let us now turn to the third problem mentioned earlier. Are C^0-stable mappings in the nice dimensions C^∞-stable? The first results obtained for this problem were by May in his thesis, 1973 [25]. We begin by describing these results. First of all, there is the example of May of a C^∞-proper mapping $f : R^1 \to R^1$ which is C^0-stable but not C^∞-stable. This example very nicely destroys our first intuitions about C^0-stable mappings. Its graph is shown below.



At the origin the Taylor series begins with x^3 and all other singularities are Morse singularities. It is C^∞-stable at all points except 0. There x^3 can be locally perturbed to $x^3 + ax$ but in either case we still have a function C^0-equivalent to f.

The difficulty occur because the domain manifold is not compact (alternately this situation could be avoided by requiring that the homeomorphisms used for the C^0-stability be within sufficiently small neighborhoods of the identity; this would give "ε-stability").

Thus, we will consider the problem for C^0-stable maps $f : N \to P$ with N compact. Also, before we describe the results we will give several definitions.

We recall that C^∞-stable mappings satisfy a number of properties:

1) *multi-transversality* to the $\mathscr{A}^k$ and $\mathscr{K}^k$-orbits;

2) *invariance under suspension*; if $f : N \to P$ is stable, so is $f \times 1 : N \times Y \to P \times Y$ for a manifold Y:

3) *stability under k-deformations*; if $f : N \to P$ is stable and $F : N \times \mathbb{R}^k \to P \times \mathbb{R}^k$ is a mapping such that $F(x, t) = (y, t)$ and $F(x, 0) = f(x)$ $x \in N$, then there is a neighborhood $0 \in U \subset \mathbb{R}^k$ such that $F \mid N \times U$ is C^∞-equivalent to $f \times 1_U$.

On the other hand C^0-stability in general does not have such properties. Hence, we consider various types of C^0-stability.

Definition 11. We say that a C^0-stable mapping $f : N \to P$ is

1) *mt-stable*: if f is multi-transverse to the stratification defined by Mather;

2) *S-stable*: if $f \times 1^k : N \times (S^1)^k \to P \times (S^1)^k$ is C^0-stable all $k \geq 0$;

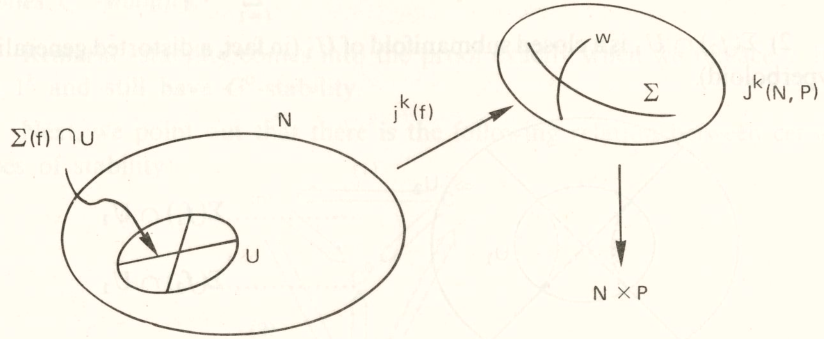
3) *Uniformly stable (U-stable)*: if f is C^0-stable under k-deformations, i.e., as in the C^∞-case, we require tant a k-deformation of f, $F : N \times \mathbb{R}^k \to P \times \mathbb{R}^k$ is C^0-equivalent to $f \times 1_U$ on the neighborhood $U \in 0$.

Relation of C^0 and C^∞-stability. In addition to proving that C^0-stability implies C^∞-stability for the Whitney embedding range $p \geq 2n$, and for $p = 1$ (the Morse functions), May's principal result may be stated.

Theorem 22. (*May*). *If $f : N \to P$ is in the interior of the set of uniformly stable mappings with N compact and (n, p) satisfying $n > p$, $p < 7$, and $n < 2\,(n - p + 2)$ then f is C^∞-stable.*

First we describe the meaning of the inequilities which (n, p) satisfies. In the nice dimensions $n > p$, two types of Σ_i-types occur generically, Σ_{n-p+1} and Σ_{n-p+2}. These inqualities are guaranteeing that only the Σ_{n-p+1} occur generically. The Σ_{n-p+1} singularities are singularities which occur as unfoldings of the simple singularities determined by Arnold of low codimension [38], [1].

The idea of the proof is to verify that such a mapping is multi-transverse to the $\mathscr{K}^k$-orbits. Suppose that $j^k(f)$ is not transverse at a point x to a $\mathscr{K}^k$-orbit Σ of codim $< n$. May shows that f may be perturbed in a neighborhood U of x so that for the perturbed mapping $\tilde{f}$, which is C^0-equivalent to f, $j^k(\tilde{f})$ is transverse to Σ except in one normal coordinate w. In the coordinate w, the coordinate function of $j^k(\tilde{f})$ has only a Morse singularity x in U.



Then $\Sigma(\tilde{f})$ $(= j^k(\tilde{f})^{-1}(\Sigma))$ near x is homeomorphic to an appropriate cone and hence is not a submanifold. If dim $N <$ codim Σ then a separate argument is needed (this is where uniform stability comes in). On the other hand, May finds a topological description of Σ which is valid for transverse maps. By comparing these he is able to show that they are topologically distinct.

Part of the difficulty which prevents May from completing the case of $n > p$ is that the topological information for Σ is obtained by looking at the relative structure of near-by orbits using only minimal transversality assumptions. We are going to describe how to modify May's construction to make use of the local C^0-classification of C^∞-stable germs for $n \leq p$.

We make three modifications to May's constructions:

1) we work with some $\overline{f} = f \times 1^k : N \times (S^1)^k \to P \times (S^1)^k$ which we will continue to denote by $f : N \to P$;

2) we make two perturbations of f, f_1 and f_2, with the following properties:

a) the f_i are perturbations only on a neighborhood U_1 of x, and are both C^0-equivalent to f;

b) for a smaller neighborhood $x \in U_2 \subset U_1$, $f_1 = f_2$ outside of U_2;

c) $j^k(f_i)$ are transverse to Σ except in one normal coordinate w.

d) the w-coordinate function of $j^k(f_1)$ has a Morse singularity at x;

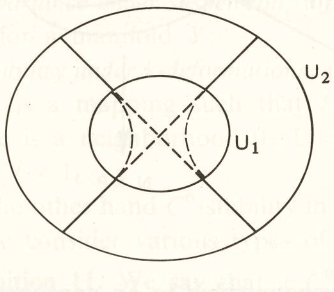
e) the w-coordinate function of $j^k(f_2)$ is a Morse function translated slightly on U_2.

3) f_1 and f_2 consist of C^∞-stable germs on $U_2 \backslash \{x\}$.

With this we are able to describe precisely the singularity sets $\Sigma(f_i) \cap U_1$. Then, assuming that Σ-type is a topological invariant for C^∞-stable germs, we will be able to describe the topological analogue for Σ. We have for these sets:

1) $\Sigma(f_1) \cap U_1$ is homeomorphic to a cone $\sum_{i=1}^{r} \lambda_i x_i^2 = 0$.

2) $\Sigma(f_2) \cap U_1$ is a closed submanifold of U_1 (in fact, a distorted generalized hyperboloid).



..............$\Sigma(f_1) \cap U_1$

..............$\Sigma(f_2) \cap U_1$

Next, we want to compute

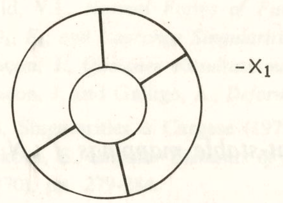
$\Sigma^0(g) = \{x \mid g_{(x)}$ is C^0-equivalent to a C^∞-stable germ of Σ-type$\}$.

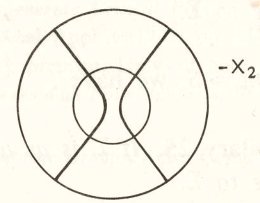
For a C^∞-stable germ g of Σ-type we have $\Sigma^0(g) = \Sigma(g)$ is the germ of a submanifold of dim $r-1$. Now, except at $\tilde{x}$ in U_2, we can compute Σ^0 from Σ because at all other points the f_i consist of C^∞-stable germs. For $f_1, \tilde{x} \notin \Sigma^0(f_1)$ because we would have $\Sigma^0(f_1) \simeq C$ near $\tilde{x}$, which is not locally a submanifold of the correct dimension. Similarly $x \notin \Sigma^0(f_2)$ because it would be an isolated point near $\tilde{x}$. Thus

$\Sigma^0(f_1) \cap U_2 = C - \{\tilde{x}\}$ which has as a deformation retract $\Sigma(f_1) \cap (U_2 \backslash U_1)$. Thus, if $X_0 = \Sigma(f_2) \cap U_1$ $X_1 = \Sigma^0(f_1) \cap (N - U_1)$, and $X_2 = \Sigma^0(f_2)$, then $X_2 = X_1 \cup \partial X_1$ X_0. Thus, a Mayer-Vietoris argument shows that $H_*(X_2) \neq$ $\neq H_*(X_1)$. Hence, they are topologically distinct. This contradicts the construction that they are topologically equivalent to f. Thus, f is not C^0-stable.

This can be used to prove both transversality to all $\mathcal{K}^k$-orbits corresponding to simple algebra types and also $j^k(f)$ $(N) \cap \Pi(n, p) = \phi$.

Lastly, multi-transversality follows by a perturbation argument for the image of the singularity sets in P; this gives

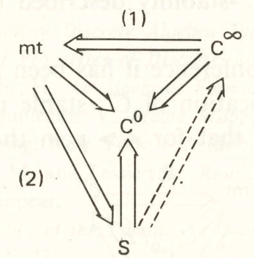




Theorem 23. [9] *In the nice dimensions $n \leq p$, with N compact, S-stability implies C^∞-stability.*

Remark. -stability comes into the proof exactly when we replace f by $f \times 1^k$ and still have C^0-stability.

Next, we point out that there is the following relation between certain types of stability:



1) follows because the Whitney stratification of theorem 12 is invariant under $\mathscr{A}^k$.

2) follows from the construction of the Whitney stratification.

Then, the theorem states that in the nice dimensions ($n \leq p$), we can complete the diagram with the dotted implication to obtain that C^∞, mt, and S are equivalent in the nice dimensions $n \leq p$.

There is another consequence of the method of proof. We define

Definition 12. A singularity subset $\Sigma \subset J^k(n, p)$ is a *local topological invariant* if there is a neighborhood U of Σ, invariant under $\mathscr{A}^k$ such that C^∞-stable germs $j^k(f) \in \Sigma$ are topologically distinct from those with $j^k(f) \in U \backslash \Sigma$.

Then, the method of proof really only relies on Σ being a local topological invariant. In fact, the same method shows

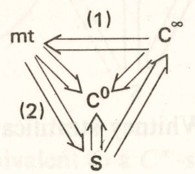
Proposition 24. *If $\Sigma \subset J^k(n, p)$, all $p - n = c$, is a local topological invariant which is invariant under $\mathscr{K}^k$ and satisfies a technical condition called the immersion condition (see* [9] *or* [25]*) then S-stable mappings $f: N \to P$ are transverse to Σ.*

As $mt \Rightarrow S$, we have

Corollary 25. *If Σ is as above, then mt-stable mappings $f: N \to P$ are transverse to Σ.*

This suggests one way of better understanding the stratification defined by Mather for C^0-stability; namely, construct a stratification of local topological invariants Σ as above. Then mt-stable maps must be transverse to this stratification. As we refine this stratification we obtain a better "approximation" to Mather's stratification. Unfortunately, almost nothing is known about the specific stratification in the region where moduli occur. This central problem will certainly prove to be much more difficult to solve than all of the problems relating to C^∞ and C^0-stability described up until now.

Addendum. While at the conference it has been possible to give a relatively simple topological classification of C^∞-stable map germs in the nice dimensions $n > p$. This implies that for $n > p$ in the nice dimensions



where we can reverse the arrow $C^\infty \Rightarrow U$ at least for $n > p$, $p < 7$, $n < 2(n - p + 2)$ by May's theorem. A natural conjecture is

Conjecture. *In the nice dimensions the four types of stability C^∞, mt, S, U are equivalent.*

A. DuPlessis has indicated in a private conversation that he has some results for reversing the arrow $C^\infty \Rightarrow U$. This leaves open the question of where C^0-stability is not equivalent to all of these. This would mean finding a C^0-stable map which is not S-stable. Even outside of the nice dimensions such a map would have to be different from the *mt*-stable ones.

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