Differentiable Mappings Between Foliated Manifolds

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In [1] are announced some results on stability of equivariant maps between compact G-manifolds X and Y, where G is a compact Lie group. Let $C_G^\infty(X, Y)$, $Diff_G(X)$ and $Diff_G(Y)$ be the sets of C^∞-G-equivariant maps, C^∞-G-diffeos of X and C^∞-G-diffeos of Y respectively; there is a natural group action

$$Diff_G(X) \times Diff_G(Y) \times C_G^\infty(X, Y) \xrightarrow{\Phi} C_G^\infty(X, Y)$$

and for each $f \in C_G^\infty(X, Y)$, we define the corresponding orbit-map

$$Diff_G(X) \times Diff_G(Y) \xrightarrow{\Phi_f} C_G^\infty(X, Y)$$

Definition. f is stable if $I_m(\Phi_f)$ is a neighbourhood of $f \in C_G^\infty(X, Y)$.

Let $C^\infty(TX)^G$, $C^\infty(TY)^G$ and $C^\infty(f^*(TY))^G$ be the invariant C^∞-sections of the G-bundles TX, TY and f^*TY repectively. We have the usual linear mappings, see [2],

$$\begin{array}{ccc} C^\infty(TX)^G & & \\ & \searrow^{\beta_f} & \\ & & C^\infty(f^*TY)^G \\ & \nearrow_{\alpha_f} & \\ C^\infty(TY)^G & & \end{array}$$

Definition. f is infinitesimally stable if $\alpha_f + \beta_f$ is surjective.

The main theorem in [1] set: "if $f \in C_G^\infty(X, Y)$ is infinitesimally stable then f is stable".

In this paper we will deal with similar questions for mappings between foliated manifolds. The author wish to tank Professor Harold I. Levine of Brandeis University, U.S.A., for many helpful suggestions.

Let M and N be C^∞-manifolds of dimensions m and n respectively. Let J and L be C^∞-regular foliations on M and N of codimensions d and q respectively.

Definition 1. $f \in C^\infty(M, N)$ is stable (tangencial sense) if there is a neighbourhood $V_f \subset C^\infty(M, N)$ of f in the C^∞-fine-topology such that for each $g \in V_f$ satisfying, $g(x)$ and $f(x)$ belong to the same leaf of L for each $x \in M$, there are diffeomorphisms $h: M \to M$, taking each leaf of J onto itself, and $k: N \to N$, taking each leaf of L onto itself, such that $f = k \circ f \circ h$.

Let TJ be the subbundle of TM with fiber T_xJ the tangent space to the leaf of J at x.

Definition 2. $f \in C^\infty(M, N)$, is infinitesimally stable (tangential sense) if given $w \in C^\infty(f^*(TL))$ there are $u \in C^\infty(TJ)$ and $v \in C^\infty(TL)$ such that $w = df.u + v \circ f$.

Then we have:

Theorem 1. *If M is compact and $f: M \to N$ is infinitesimally stable (tangential sense), then f is stable (tangential sense).*

Proof We will use the same terminology of [7]. First we will prove that the infinitesimal stability (tangential sense) is equivalent to the condition: a. for each $y \in N$ and each finite subset S of $f^{-1}(y)$ with no more than $(n-q+1)$ points, it follows that

$$J^{n-q}(f^*TL)_S = (df)\ (J^{n-q}(TX)_S) + f^*(J^{n-q}(TY)_q).$$

If $f: M \to N$ is infinitesimal stable (tangential sense), is straightforward that. f satisfy condition a. We will prove the converse.

Let $\Sigma(J, L, f) = \{x \in M \mid \dim [df_x(T_xJ) \cap T_{f(x)}L] < n\text{-}q\}$. If $f: M \to N$ satisfacy a, then for each $y \in N$, $\overline{\Sigma(J, L, f)} \cap f^{-1}(y)$ has no more than n-q points. In fact, is easy to see that for each $y \in N$, $\Sigma(J, L, f) \cap f^{-1}(y)$ has no more than n-q points; now suppose that for some $y \in N$, there are $x_1, \ldots, x_{n-q+1}$ distincts in $\overline{\Sigma(J, L, f)}$ such that $f(x_i) = y$, $i = 1, \ldots, n-q+1$. Let $(x_{ij})_{j=1,2,\ldots}$ be sequences of points of $\Sigma(J, L, f)$ converging to x_i, $i = 1, \ldots, n-q+1$; we indicate by

$$H_i = \{ z \in T_yL \mid z = \lim_{j \to \infty} df_{x_{ij}} . u_{ij}, u_{ij} \ \varepsilon\ df_{x_{ij}}^{-1} (T_{f(x_{ij})} L) \cap T_{x_{ij}}J \}.$$

We have: H_i is a vector subspace of T_yL and $\dim H_i < n-q$, $i = 1, \ldots, n-q+1$, since $\dim [\, df_{x_{ij}} (T_{x_{ij}}J) \cap T_{f(x_{ij})} L \,] < n-q$.

Then $\operatorname{cod} H_i \geq 1$ in T_yL and $\operatorname{cod} H_1 + \ldots + \operatorname{cod} H_{n-q+1} \geq n-q+1$. Therefore $H_1, \ldots, H_{n-q+1}$ are not in general position in T_yL. This mean that, there exist $\overline{w}_1, \ldots, \overline{w}_{n-q+1}$ in T_yL such that the equations $\overline{w_i} = h_i + z$ has no solution for $h_i \in H_i$ and $z \in T_yL$. But this is in contradiction with the hypothesis, since we know that condition a, implies that f is simultaneously infinitesimally stable (tangential sense), for all subset $S = \{p_1, \ldots p_r\}$ with no more than $n-q+1$ points and $f(p_1) = \ldots = f(p_r)$.

Now if we use $\overline{\Sigma(J, L, f)}$ instead of $\Sigma(f)$ in the proof of theorem 4.1, see [3] pp. 313-317, we get that condition a implies infinitesimal stability (tangential sense). Note that we only need the condition $f: M \to N$ is a proper map.

Second we prove that, if f is infinitesimally stable (tangential sense) then there is a neighbourhood W_f of f in $C^\infty(M, N)$ with the C^∞-fine topology, such that, all $g \in W_f$ and satisfying $f(x)$ and $g(x)$ belong to the same leaf of L for each $x \in M$, is locally infinitesimally stable (tangential sense). This can be done if we choose coordinates $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ in the neighbourhoods V of x and U of $y = f(x)$, such that $f(\overline{V}) \subset U$, J is defined on V by $\varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_d)$ and L is defined on U by $\psi(y_1, \ldots, y_n) = (y_1, \ldots, y_q)$. We have $\tilde{f}: J^{n-q}(TJ)_x \oplus J^{n-q}(TL)_y \to J^{n-q}(f^*TN)_x$ given by $\tilde{f}(J^{n-q}u(x), J^{n-q}v(y)) = J^{n-q}(df\,u + v \circ f)(x)$, is a linear map, onto $J^{n-q}(f^*TL)$ and depends continuously on x and $J^{n-q+1}f$. Then there is a neighbourhood W_f of f such that if $g \in W_f$, and $g(x)$ and $f(x)$ belong to the same leaf of L for each $x \in M$, then $\tilde{g}$ is onto $J^{n-q}(f^*TL)$. This is true since in the above coordinates, we have $g(x) = (f^1(x), \ldots, f^q(x), g^{q+1}(x), \ldots, g^n(x))$, for each $x \in V$. Note that all the second step can be done even if M is not compact, using the Mather's caracterization of C^r-fine topology, see [2] p. 268.

Finally we can apply the same ideas of [7], pp. 117-131, and finish the proof when M is compact.

Remark. If we drop the hypothesis that M is compact, the above theorem remain true for the proper maps. The proof follows that of [3] theorem 4.1, pg. 313.

2. Transverse Stability

We know that the set of the maps $f: M \to N$ such that f is (J, L)-transverse, i.e $df_x(T_xJ) + T_{f(x)}L = T_{f(x)}N$, is not dense in $C^\infty(M, N)$ with the C^∞-fine topology. Also let $\varphi: V \subset M \to R^d$ and $\psi: U \subset N \to R^q$ be local representations of J and L respectively, where $V \subset M$ is open, $U \subset N$ is open, $x \in V$, $y \in U$ and $f(V) \subset U$; if $m \geq d + q$ then x is a (J, L)-transverse point of f, if and only if, x is a regular point of $(\varphi, \psi \circ f): V \to R^d \times R^q$. This suggest the following definition: $x \in M$ is a "nice (J, L) – point" if x is a regular point of $(\varphi, \psi \circ f)$, or x is a nice singular point of $(\varphi, \psi \circ f)$ in the Thom-Boardman sense, see [5].

S-Transversality. Let S_r be a submanifold of $J^r(m, d+q)$, invariant under the action of $L^r(m) \times L^r(d+q)$. Take $S(V, R^d \times R^q) \subset J^r(V, R^d \times R^q)$ the subbundle with fiber S. If $\{V_i\}$ is a locally finite covering of M, with $\bar{V}_i$ compact and $\{U_i\}$ is a locally finite covering of N; suppose that J is represented on V_i by $\varphi_i : V_i \to R^d$, and L is represented on U_i by $\psi_i : U_i \to R^q$; finally we define:

$$Z = \left\{ z \in J^r g(x) \;\middle|\; \begin{array}{l} x \in V_i, g(\bar{V}_i) \subset U_i \\ J^r(\varphi_i, \psi_i \circ g)(x) \in S(V_i, R^d x R^q) \end{array} \right\}$$

Then we can prove

Theorem 2. *If $S \subset J^r(m, d+q)$ is a invariant submanifold under the action of $L^r(m) \times L^r(d+g)$ and Z is defined as above, then Z is void or submanifold of $J^r(M, N)$ of the same codimension as S in $J^r(m, d+q)$. Moreover $J^r f : M \to J^r(M, N)$ is transverse to Z at $x \in M$, iff $J^r(\varphi, \psi \circ f) : V \to J^r(V, R^d x R^q)$ is transverse to $S(V, R^d \times R^q)$ at x, where $V \subset M$, $U \subset N$, $\varphi : V \to R^d$ and ψ $U \to R^q$ are as above.*

For proof see [4].

Now using the Thom's transversality theorem and the theorem 2 we see that the set of the maps $f \in C^\infty(M, N)$ which have only "nice-(J, L)-points" is residual in $C^\infty(M, N)$.

The above construction suggest also a stratification for the set of the (J, L)-non-transverse points, similar to the Thom-Boardman' stratification of the singular points. In fact, let $I = (k_1, \ldots, k_r)$ be a admissible sequence of integers, see [5], and let $\Sigma_I(m, d+q)$ the corresponding manifold of singularities. We define

$$\Sigma_I(J, L) = \left\{ z \in J^r g(x) \;\middle|\; \begin{array}{l} x \in V_i, g(\bar{V}_i) \subset U_i \\ J^r(\varphi_i, \psi_i \circ g)(x) \in \Sigma_I(V_i, R^d \times R^q \end{array} \right\},$$

where $\{V_i\}$ and $\{U_i\}$ are open covering as above. If $\Sigma_I(J, L, f) = (J^r f)^{-1}(\Sigma_I(J, L))$, we have:

Proposition 1. *If $J^s f$ is transverse to $\Sigma_{I_s}(J, L))$, $I_s = (k_1, \ldots, k_s)$, $s < r$, then*

$$\sum_{k_1 \cdots k_r} (J, L, f) =$$

$$\{ x \in \Sigma(J, L, f)_{k_1 \ldots k_{r-1}} \mid \dim df_x(T_x J \cap T_x \Sigma(J, L, f)_{k_1 \ldots k_{r-1}}) + T_{f(x)} L \,] =$$

$$= \dim L + \dim [T_x J \cap T_x (\sum_{k_1 \cdots k_{r-1}} (J, L, f))] - k_r \}$$

Proof. We have

$$\sum_{k_1 \ldots k_r} (J, L, f) = (J^r f)^{-1} (\sum_{k_1 \ldots k_r} (J, L)). \text{ Then}$$

$$\sum_{k_1 \cdots k_r} (J, L, f) \cap V = (J^r F)^{-1} (\sum_{k_1 \cdots k_r} (V, R^d \times R^q)) = \sum_{k_1 \cdots k_r} (F),$$

where $F : V \to R^d \times R^q$, is defined by $F = (\varphi, \psi \circ f)$.

Therefore, from [5],

$$\sum_{k_1 \cdots k_r} (J, L, f) \cap V = \{ x \in \sum_{k_1 \cdots k_{r-1}} (F) \,|\, \ker dF_y \cap T_y \sum_{k_1 \cdots k_{r-1}} (F) = k_r \}$$

Now we have

$$\sum_{k_1 \cdots k_r} (J, L, f) \cap V = \{ x \in \sum_{k_1 \cdots k_{r-1}} (J, L, f) \cap V$$

$$\ker d\varphi_x \cap \ker d(\psi \circ f)_x \cap T_x \sum_{k_1 \cdots k_{r-1}} (J, L, f) = k_r \}$$

$$= \{ x \in \sum_{k_1 \cdots k_{r-1}} (J, L, f) \cap V \mid \dim [df_x(T_x J \cap T_x \sum_{k_1 \cdots k_{r-1}} (J, L, f) + T_{f(x)} L]$$

$$= \dim L + \dim [T_x J \cap T_x \sum_{k_1 \cdots k_{r-1}} (J, L, f)] - k_r \},$$

and the proof is complete.

If we take $\Sigma_I(m, d)$ and $\Sigma_I(m, q)$ we can define $\Sigma_I(J)$ and $\Sigma_I(L)$ respectively. Then from the multitransversality's theorem and theorem 2, we have:

Proposition 2. *If I_1, I_2, I_3 and I_4 are admissibles sequences, then the set of the maps $f : M \to N$ satisfying the normal-crossings conditions relatively to the manifolds $\Sigma_{I_1}(f)$, $\Sigma_{I_2}(J, f)$, $\Sigma_{I_3}(L, f)$ and $\Sigma_{I_4}(J, L, f)$ is residual $C^\infty(M, N)$.*

The definition of stability associate to the above stratification is the following

Definition 3. We say that $f \in C^\infty(M, N)$ is (J, L)-stable if there is neighbourhood V_f of f in $C^\infty(M, N)$ such that for each $g \in V_f$ there is a diffeomorphism $h : M \to M$ which takes each leaf of J on itself, and a diffeomorphism $k : N \to N$ taking leaves of L to leaves, such that $g = kf \circ h$.

Let $\pi : TN \to \dfrac{TN}{TL}$ the canonical projection.

Definition 4. We say that $f \in C^\infty(M, N)$ is infinitesimally (J, L)-stable if given $w \in C^\infty(f^*TN)$ there are $u \in C^\infty(TJ)$ and $v \in C^\infty(TN)$, such that $\pi(v)$ is locally constant along the leaves and $w = df \cdot u + v \circ f$.

Question. Are the above definition equivalents for proper maps$^\Omega$ We will give a parcial answer for this questions in the local case.

If only the manifolds N is foliated by L, the above definitions give raise to the L-stability and infinitesimal L-stability respectively.

The local L-stability is given by the following commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{f} & U & \xrightarrow{\psi} & R^q \\ {\scriptstyle h}\downarrow & \circlearrowright & {\scriptstyle k}\downarrow & \circlearrowright & \downarrow{\scriptstyle t} \\ V & \xrightarrow[g]{} & U & \xrightarrow[\psi]{} & R^q \end{array}$$

Then the L-stability is equivalent to the stability of the composite map $\psi \circ f$ with the second map ψ fixed.

We recall the definitions of stability of compositions as given by [6].

Let $f_i : M_i \to M_{i+1}$ be C^∞-map, $i = 1, 2$. We say that (f_1, f_2) is stable if there are neighbourhoods V_{f_1} and V_{f_2} of f_1 and f_2 respectively such that for each $(g_1, g_2) \in V_{f_1} \times V_{f_2}$ there are diffeomorphisms $h_i : M_i \to M_i$ such that

$$\begin{array}{ccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 \\ {\scriptstyle h_1}\downarrow & \circlearrowright & {\scriptstyle h_2}\downarrow & \circlearrowright & \downarrow{\scriptstyle h_3} \\ M_1 & \xrightarrow[g_1]{} & M_2 & \xrightarrow[g_2]{} & M_3 \end{array} \quad \text{commutes.}$$

We say that (f_1, f_2) is infinitesimally stable if given $w_i \in C^\infty(f_i^* TM_{i+1})$ there are $u_i \in C^\infty(TM_i)$ such that $w_i = df_i \cdot u_i + u_{i+1} \circ f_i$.

From [6] we know that for proper maps f_i then (f_1, f_2) is stable iff is infinitesimally stable. This theorem and the proposition 3 and 4 bellow, will answer our question in the local case, when M is not foliated.

Let $\psi : N \to Q$ be a fixed C^∞- stable mapping. We say that $f : M \to N$ is ψ-stable if there is a neighbourhood V_f of f in $C^\infty(M, N)$ such that for all $g \in V_f$, there are diffeomorphisms $h : M \to M$, $k : N \to N$ and $t : Q \to Q$ such that

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{\psi} & Q \\ {\scriptstyle h}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle t} \\ M & \xrightarrow[g]{} & N & \xrightarrow[\psi]{} & Q \end{array} \quad \text{commutes.}$$

If ψ is infinitesimally stable, we say that $f : M \to N$ is infinitesimally ψ-stable if given $w \in C^\infty(f^*TN)$ there are $u \in C^\infty(TM)$, $v \in C^\infty(TN)$ and $z \in C^\infty(TQ)$ such that $w = df.u + v \circ f$ and $0 = d\psi.v + z \circ \psi$.

Then we have:

Proposition 3. *If ψ is stable, then (f, ψ) is stable iff f is ψ-stable.*

Proof. It is obvious that if (f, ψ) is stable then f is ψ-stable.

Suppose that f is ψ-stable. Since ψ is stable there is a neighbourhood W_ψ of ψ in $C^\infty(N, Q)$ such that, for each $\varphi \in W_\psi$ there are diffeomorphisms $\varepsilon : N \to N$ and $\lambda : Q \to Q$ such that $\psi = \lambda\varphi\varepsilon^{-1}$. We can choose W_ψ such that ε is near to the identity $I_N : N \to N$.

Since f is ψ-stable, there is a neighbourhood V_f of f in $C^\infty(M, N)$ such that, for each $\bar{f} \in V_f$ there are diffeomorphisms $a : M \to M$, $b : N \to N$ and $c : Q \to Q$ such that

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{\psi} & Q \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ M & \xrightarrow[\bar{f}]{} & N & \xrightarrow[\psi]{} & Q \end{array} \quad \text{commutes}$$

Let U_f be a neighbourhood of f such that $\varepsilon \circ g \in V_f$ for all $g \in U_f$. Then we have that

$$\begin{array}{ccccccccccc} M & \xrightarrow{g} & N & \xrightarrow{\varepsilon} & N & \xrightarrow{\varepsilon^{-1}} & N & \xrightarrow{\varphi} & Q & \xrightarrow{\lambda} & Q \\ {\scriptstyle a}\downarrow & & & & {\scriptstyle b}\downarrow & & & & & & \downarrow{\scriptstyle c} \\ M & & \xrightarrow[f]{} & & N & & & \xrightarrow[\psi]{} & & & Q \end{array}$$

commutes. Therefore (f, ψ) is stable.

Proposition 4. *If ψ is infinitesimally stable, then (f, ψ) is infinitesimally stable iff f is infinitesimally ψ-stable.*

Proof. If (f, ψ) is infinitesimally stable, it follows that f is infinitesimally ψ-stable.

Suppose that f is infinitesimally ψ-stable. Let $w_1 \in C^\infty(f^*TN)$ and $w_2 \in C^\infty(\psi^* R^q)$ (be given sections. Since ψ is infinitesimally stable, there are $v_2 \in C^\infty(TN)$ and $z_2 \in C^\infty(TQ)$ such that $w_2 = d\psi . v_2 + z_2 . \psi$. Since f is infinitesimally ψ-stable, there are $u_1 \in C^\infty(TM)$, $v_1 \in C^\infty(TN)$ and $z_1 \in C^\infty(TQ)$, such that $df . u_1 + v_1 . f = w_1 - v_2 . f$ and $d\psi . v_1 + z_1 . \psi = 0$.

Finally if we take $u = u_1$, $v = v_1 + v_2$ and $z = z_1 + z_2$, we have $w_1 = df . u + v . f$ and $w_2 = d\psi . v + z . \psi$; then the proof is complete.

The above propositions provide an easier way to get the theory of stability of compositions developed by N. A. Baas, see [6], from the Mather's theory of stability; i.e, it is enough to study sequences of the type $(f_1, f_2, \ldots, f_n)$, where $f_2, \ldots, f_n$ are fixed stable mappings.

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