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An Exceptional Decomposition of the Augmentation Ideal of PSL (2,7)

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In [2] (cf. also [1, pp 58/59]) K.W. Gruenberg and the author have given a criterium for the integral augmentation ideal of a finite group to decompose. In case of a solvable group, this criterium is necessary and sufficient. In this note we shall construct a decomposition of the integral augmentation ideal of PSL(2, 7), which can not be derived from our criterium.

If G is a finite group, we denote by $\mathfrak{g}$ its integral augmentation ideal; i.e. if $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation homomorphism, then $\mathfrak{g}$ is the kernel of ε (as additive group $\mathfrak{g}$ is free on the elements $\{g - 1\}_{g \in G \setminus \{1\}}$).

We recall from [2]:

Theorem. *Let U be a π-Hallgroup of G with $U^g \cap U = 1$ or U for every $g \in G$ and such that $N_G(U)$* is a Frobeniusgroup with kernel U unless $U = N_G(U)$.

Then
$$\mathfrak{g} \simeq A \oplus B,$$
where $\mathbb{Z}_\pi \oplus_{\mathbb{Z}} A$ is projective and $\mathbb{Z}_{\pi'} \oplus_{\mathbb{Z}} B$ is projective. (Here π' is the complementary set of divisors of $|G|$ to π and $\mathbb{Z}_\pi$ is the semi-localization at all the rational primes in π.)

We call such a decomposition a (π, π')-decomposition.

Remark 1. The 3-Sylow subgroup and the 7-Sylow subgroup of $PSL(2, 7)$ satisfy the hypotheses of the above theorem; and so there exists a $(\{3\}, \{2, 7\})$ – and a $(\{7\}, \{2, 3\})$-decomposition of the augmentation ideal of $PSL(2, 7)$. However the 2-Sylow group does not satisfy the conditions of the above theorem.

Proposition. *The augmentation ideal $\mathfrak{g}$ of $PSL(2, 7)$ has a $(\{2\}, \{3, 7\})$--decomposition.*

Proof. Let A be a cyclic subgroup of $G = PSL(2, 7)$ of order 4 and D a 2-Sylow subgroup of G:

$$A = \langle a : a^4 = 1 \rangle, \; D = \langle a, i : a^4 = i^2 = 1, \; a^i = a^{-1} \rangle.$$

Let $\mathfrak{a}$ be the integral augmentation ideal of A and denote by $\mathfrak{a}^{\cdot}$ the $\mathbb{Z}D$-module, $\mathfrak{a}$ on which i acts via conjugation. *Then*

$$M = (\mathfrak{a}^{\cdot})^G = \; : \mathbb{Z}G \oplus_{\mathbb{Z}_*} \mathfrak{a}^{\cdot}$$

is a left $\mathbb{Z}G$-module of $\mathbb{Z}$-rank 63. For a $\mathbb{Z}$-lattice X and a rational prime p we write X_p for the completion of X at p.

Claim 1. *M_3 is $\mathbb{Z}_3G$-projective and M_7 is $\mathbb{Z}_7G$-projective; moreover, $M_3 | \mathfrak{g}_3$ and $M_7 | \mathfrak{g}_7$ (where $X | Y$ indicates that X is isomorphic to a direct summand of Y).*

Proof $\mathfrak{a}^{\cdot}$ is a $\mathbb{Z}D$-lattice; i.e. a module over a 2-group, and so $(\mathfrak{a}^{\cdot})_3$ is $\mathbb{Z}_3D$--projective and $(\mathfrak{a}^{\cdot})_7$ is $\mathbb{Z}_7D$-projective. But then the corresponding induced modules are projective over $\mathbb{Z}_3G$ and $\mathbb{Z}_7G$ resp.

For a prime divisor p of $|G|$ we have a decomposition

$$\mathbb{Z}_pG = P_0(p) \oplus P_1(p),$$

where $P_0(p)$ is the uniquely determined projective $\mathbb{Z}_pG$-module such that $\mathbb{Q} | \mathbb{Q} \oplus_{\mathbb{Z}_p} P_0(p)$. Then $P_1(p) | \mathfrak{g}_p$. In fact, $P_0(p)$ is the projective cover of $\mathbb{Z}_p$, and so we the two exact sequences

$$0 \to \mathfrak{g}_p \quad \to \mathbb{Z}_pG \to \mathbb{Z}_p \to 0$$

$$0 \to X_p(p) \to P_0(p) \to \mathbb{Z}_p \to 0$$

and the uniqueness of the projective cover shows $P_1(p) | \mathfrak{g}_p$.

Because of the exact sequence

$$0 \to (\mathfrak{a}^{\cdot})^G \to (\mathbb{Z}A^{\cdot})^G \to \mathbb{Z}^G \to 0$$

($\mathbb{Z}A$ becomes a $\mathbb{Z}D$-module $\mathbb{Z}A^{\cdot}$ if i acts via conjugation on A) we see that $\mathbb{Q}$ is not a direct summand of $\mathbb{Q} \oplus_{\mathbb{Z}} (\mathfrak{a}^{\cdot})^G$ and with the first part of the claim and observing that the Krull-Schmidt-theorem holds of $\mathbb{Z}_pG$-lattices, we conclude $M_3 | P_1(3)$ and $M_7 | P_1(7)$; i.e. $M_3 | \mathfrak{g}_3$ and $M_7 | \mathfrak{g}_7$. This proves the claim.

Claim 2. *Let χ be the rational character of $\mathbb{Q} \oplus_{\mathbb{Z}} M$. Then*

$$\chi(g) = 0 \quad \textit{for } g \textit{ a 3-element}$$
$$\textit{or } g \textit{ a 7-element.}$$

For the 2-elements we have

$$\chi(a) = -1$$
$$\chi(a^2) = -1$$
$$\chi(a^3) = -1$$

All other 2-elements are conjugated to one of these.

Proof. Since M_3 and M_7 are projective $\mathbb{Z}_3G$-and $\mathbb{Z}_7G$-modules resp. we have

$$\chi(g) = 0 \quad \text{if } g \text{ is a 3-element and}$$
$$\chi(g) = 0 \quad \text{if } g \text{ is a 7-element.}$$

Moreover, every element in $PSL(2,7)$ is of prime-power order and so it is enough to compute $\chi(x)$ if x is a 2-element. To do so we first compute the character μ of $\mathbb{Q} \oplus_{\mathbb{Z}} \mathfrak{a}^{\cdot}$. $\mathbb{Q} \oplus_{\mathbb{Z}} \mathfrak{a}^{\cdot}$ is $\mathbb{Q}$-free on the elements $(a-1)$, (a^2-1) and (a^3-1). We have the following operations:

$$\begin{array}{llll}
a: & (a-1) \mapsto & (a^2-1)-(a-1) \\
a: & (a^2-1) \mapsto & -(a^3-1)-(a-1) \\
a: & (a^3-1) \mapsto & -(a-1) \\
 & & \\
a^2: & (a-1) \mapsto & -(a^3-1)-(a^2-1) \\
a^2: & (a^2-1) \mapsto & -(a^2-1) \\
a^2: & (a^3-1) \mapsto & (a-1)-(a^2-1) \\
 & & \\
a^3: & (a-1) \mapsto & -(a^3-1) \\
a^3: & (a^2-1) \mapsto & (a-1)-(a^3-1) \\
a^3: & (a^3-1) \mapsto & (a^2-1)-(a^3-1) \\
 & & \\
i: & (a-1) \mapsto & (a^3-1) \\
i: & (a^2-1) \mapsto & (a^2-1) \\
i: & (a^3-1) \mapsto & (a-1) \\
 & & \\
ai: & (a-1) \mapsto & -(a-1) \\
ai: & (a^2-1) \mapsto & (a^3-1)-(a-1) \\
ai: & (a^3-1) \mapsto & (a^2-1)-(a-1)
\end{array}$$

Moreover

$$ai\,a^{-1} = a^2 i \qquad \text{and}$$
$$a^3 i = i(a^i) = (ia)^i$$
$$ia = (ai)^i$$

Hence we have

$$\mu(1) = 3;\ \mu(a) = -1,\ \mu(a^2) = -1,\ \mu(a^3) = -1$$
$$\mu(i) = 1,\ \mu(ai) = -1,\ \mu(a^2 i) = \mu(i) = 1,\ \mu(a^3 i) = \mu(ia) = \mu(ai) = -1.$$

To compute the induce character, we put

$$\dot{\mu}(g) = \begin{cases} \mu(g) & \text{for} \quad g \in D \\ 0 & \text{for} \quad g \notin D. \end{cases}$$

Then

$$\chi(g) = \sum_{n \in N_1} \dot{\mu}(ngn^{-1}),$$

where N_1 is a fixed 21-Hallsubgroup of G (e.g. N_1 is the normalizer of a 7-Sylowsubgroup). If $nan^{-1} \in D$, then $nan^{-1} \in A$, since $|nan^{-1}| = 4$; hence $na^2n^{-1} \in A$ and so $na^2n^{-1} = a^2$. But G has exactly 21 involutions, which are conjugate by the elements in N_1. Hence $n = 1$ and so

$$\chi(a) = \mu(a) = -1$$

similarly

$$\chi(a^{-1}) = \mu(a^{-1}) = -1.$$

The remaining elements in D are all involutions, which are conjugate in G, and so it suffices to compute $\chi(a^2)$. Hence there are well determined elements n_j, $0 \le j \le 3$ such that $(a^2)^{n_j} = a^j i$. Thus

$$\begin{aligned} \chi(a^2) &= \mu(a^2) + \mu(i) + \mu(ai) + \mu(a^2 i) + \mu(a^3 i) \\ &= -1 \quad +1 \quad -1 \quad +1 \quad -1 \\ &= -1. \end{aligned}$$

This proves the claim.

Remark 2. Let $\mathbb{Z}(G//N_1)$ be the permutation representation on the left coset space of G modulo N_1, where N_1 as above in the normalizer of a 7-Sylow subgroup of G. Then we have an epimorphism

$$\varepsilon_{N_1} : \mathbb{Z}(G//N_1) \to \mathbb{Z}, \quad gN_1 \to 1$$

and we put $\mathfrak{g}/N_1$ the kernel of ε_{N_1} ($\mathfrak{g}/N_1$ is $\mathbb{Z}$-free on the elements $gN_1 - N_1$). Then we have the exact sequence

$$0 \to \pi_1^G \to \mathfrak{g} \to \mathfrak{g}/N_1 \to 0.$$

If we complete at 2, then $(\pi_1{}^G)_2$ is $\mathbb{Z}_2G$-projective and so the above sequence is split exact; i.e.

$$\mathfrak{g}_2 \simeq (\pi_1{}^G)_2 \oplus (\mathfrak{g}/N_1)_2;$$

moreover,

$$(\pi_1{}^G)_2 = P_1(2) \quad \text{(cf. above).}$$

Let S_3 be a 3-Sylow subgroup of G, $|S| = 3$, and put $N_2 = N_G(S_3)$; then $|N_2| = G$. We can view σ_3, the augmentation ideal of S_3 as a $\mathbb{Z}N_2$-module $\sigma_{3\circ}$, by letting the involution in N_2 act by conjugation. We put

$$M_1 = (\sigma_{3\circ})^G \oplus \mathfrak{g}/N_1;$$

then M_1 has $\mathbb{Z}$-rank 63.

Claim 3. *Let ψ be the character of $\mathbb{Q} \oplus_{\mathbb{Z}} M_1$. Then $\psi = \chi$.*

Proof With similar arguments as above one concludes that

$$((\sigma_{3\circ})^G)_7 \ \text{ is } \ \mathbb{Z}_7G\text{-projective} \quad \text{and}$$
$$((\sigma_{3\circ})^G)_2 \ \text{ is } \ \mathbb{Z}_2G\text{-projective.}$$

Moreover as above one shows

$$((\sigma_{3\circ})^G)_7 \mid \mathfrak{g}_7 \quad \text{and}$$
$$((\sigma_{3\circ})^G)_2 \mid \mathfrak{g}_2.$$

In addition we have that $(\mathfrak{g}/N_1)_7$ is $\mathbb{Z}_7G$-projective; in fact it suffices to show this when $(\mathfrak{g}/N_1)_7$ is viewed as a $\mathbb{Z}_7S_7$-module, where S_7 is the 7-Sylow subgroup of which N_1 is the normalizer. Let $x \in S_7$, then $x(gN_1 - N_1) = xgN_1 - N_1$ and if x has a fixed point, then $gN_1 = xgN_1$, i.e. $g^{-1}xg \in N_1$ i.e. $g^{-1}xg \in S_7$. but since $g \notin N_1$ and since S_7 is a trivial intersection group, $S_7 \cap S_7{}^g = 1$. Thus S_7 acts fixed point free on $gN_1 - N_1$ and so $(\mathfrak{g}/N_1)_7$ is $\mathbb{Z}_7G$-projective. Therefore $\psi(g) = 0$ if g is a 7-element. Let now g be a 3-element. We may assume without loss of generality that $N_1 = \langle S_7, i\rangle$ where i is the involution in D (cf. above). If ν is the character of $\mathbb{Q} \oplus_{\mathbb{Z}} \sigma_{3\circ}$, then $\nu(g) = -1$ and for $x \notin N_2$, $g^x \notin N_2$ and so the character of g in $\mathbb{Q} \oplus_{\mathbb{Z}} (\sigma_{3\circ})^G$ is -1. We have to compute the character of x in $\mathbb{Q}_{\mathbb{Z}}\, \mathfrak{g}/N_1$. This is done by observing

$$g(iN_1 - N_1) = iN_1 - N_1, \text{ since } i \in N_G(\langle g\rangle) \text{ and } g \in N_1.$$

If

$$g(a^jN_1 - N_1) = a^jN_1 - N_1,\ 0 \le j \le 3, \text{ then } a^{-j}ga^j \in N_1.$$

If

$$a^{-j}ga^j = g^{-1}, \text{ then } a^{-j} = i, \text{ a contradiction.}$$

Hence
$$a^{-j}ga^{-1} \in S_3^\tau \text{ for some } 1 \neq \tau \in N_1 \text{ with } |\tau| = 7.$$

Case 1 $a^{-j}ga^j = (g^{-1})^\tau$, *then* $a^{-j}ga^j = g^{i\tau}$ and so $a^j i\tau = ia^{-j}\tau$ stabilizes g; i.e., $a^{-j}\tau = i$.

But since a^{-j} and i lie in D, so must τ, a contradiction to $|\tau| = 7$.

Case 2 $a^{-j}ga^j = g^\tau$, then $a^j\tau$ stabilizes g and so $a^j\tau \in S_3$ but then $a^j \in N_1$, a contradiction.

Therefore $g(a^jN_1 - N_1) \neq a^jN_1 - N_1$. Similarly one shows
$$g(a^jiN_1 - N_1) \neq a^jiN_1 - N_1.$$

Hence $\psi(g) = 0$ if g is a 3-element. It remains to compute $\psi(g)$ if g is a 2-element. For the character ψ_1 of $\mathbb{Q} \oplus_{\mathbb{Z}} (\sigma_{3\,\circ})^G$ we have for a 2-element g, $\psi_1(g) = 0$ since $(\sigma_{3\,\circ})^G$ is $\mathbb{Z}_2G$-projective. Since
$$\mathfrak{g}_2 \simeq (\mathfrak{g}/N_1)_2 \oplus P_1(2)$$
we conclude that the character ψ_2 of $\mathbb{Q} \oplus_{\mathbb{Z}} \mathfrak{g}/N_1$ has value -1 at g. This proves the claim.

To proceed with the proof of the proposition, we observe that by Claim 3
$$\mathbb{Q} \oplus_{\mathbb{Z}} (\mathfrak{a}\cdot)^G \simeq \mathbb{Q} \oplus_{\mathbb{Z}} [(\sigma_{3\,\circ})^G \oplus \mathfrak{g}/N_1].$$

We now choose a $\mathbb{Z}G$-lattice M_0 such that
$$M_{0_2} \simeq ((\sigma_{3\,\circ})^G)_2 \oplus (\mathfrak{g}/N_1)_2$$
$$M_{0_3} \simeq (\mathfrak{a}\cdot)^G_3$$
$$M_{0_7} \simeq (\mathfrak{a}\cdot)^G_7$$

Then M_0 is a local direct summand of $\mathfrak{g}$, and so $\mathfrak{g}$ decomposes. This proves the proposition.

Remark. In the proof we have used heavily the structure of $PSL(2, 7)$. I have tried hard to extract the essentials or to derive a more general theorem; however I have not been able to do so.

References

[1] K. W. Gruenberg, *Relation Modules of Finite Groups*, Regional Conference Series in Mathematics 25 (1974).

[3] K. W. Gruenberg – K. W. Roggenkamp, *Decomposition of the Augmentation Ideal and of the Relation Modules of a Finite Group*, Proc. London Math. Soc. 31 (1975), 149-166.