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Essential selfadjointness of singular elliptic operators

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1. Introduction and statement of the theorems

Recently, some attention was given to the question of essential selfadjointness of the elliptic L defined by

$$Lu = -\Sigma_{ik}\partial_i(a_{ik}\partial_k u) + qu \qquad (i, k = 1, \ldots, n)$$

with $D(L) = C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $R(L) = L^2(\mathbb{R}^n)$

Under the assumptions

(1.1) $$q \in L^2_{loc}(\mathbb{R}^n), \text{ real},$$

(1.2) $$a_{ik} \in H^{1,\infty}_{loc}(\mathbb{R}^n), \text{ real}, \ i, k = 1, \ldots, n$$

(i.e. the a_{ik} are Lipschitz continuous), the operator L is defined and is elliptic if

(1.3) $$\Sigma_{ik} a_{ik}(x)\xi_i\xi_k > 0 \qquad (i, k = 1, \ldots, n).$$

L is called formally selfadjoint if

(1.4) $$a_{ik} = a_{ki} \qquad (i, k = 1, \ldots n)$$

In the case $n = 1$, if q is bounded below, it is well known and essentially due to Hermann Weyl [22] that L is essentially selfadjoint. In the case $n \geq 3$, however, examples of Ural'tseva [19] and Laptev [9], cf. also Maz'ya [10], show that this property of L may fail for higher dimensions. Both these examples rely on the fact that the a_{ik} are rapidly increasing since they take $q = 1$.

On the other hand, there are several results which guarantee the essential selfadjointness of L if the coefficients a_{ik} are well behaved. In the case $L = -\Delta$ there is already a large number of results. In particular, we mention the papers of B. Simon [15], T. Kato [7], and C. Simader [14] which treat the problem under the minimal regularity assumption $q \in L^2$ resp. $q \in L^2_{loc}$, as we do here.

In the case of variable coefficients a_{ik}, under suitable conditions on the potential q, the essential selfadjointness of L was established if the largest eigenvalue $\Lambda(x)$ of the matrix $(a_{ik}(x))$ does not grow substantially faster then

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$K|x|^2$ as $|x| \to \infty$. Some additional power of $ln|x|$ is also possible. This is due to Ikebe – Kato [4]. Results which allow a faster growth of $\Lambda(x)$ are, among others, due to K. Jörgens [5], H. Triebel [18], Stetkaer-Hansen [16], J. Walter [20], [21], and Kalf-Walter [6]; cf. in particular a recent paper of Devinatz [2] and the paper of Laptev [9] mentioned above.

In the case the largest eigenvalue $\Lambda(x)$ of $(a_{ik}(x))$ grows substantially faster than $K|x|^2$, it seems that the conditions given up to now for the essential selfadjointness of L imply that the lowest eigenvalue $\lambda(x)$ of $(a_{ik}(x))$ has the same growth behaviour as $\Lambda(x)$ in some sense. See e.g. Laptev's case $a_{ik}(x) = a(|x|)\delta_{ik}$ where L is essentially selfadjoint, and Devinatz' result [2] of which a variant is given in theorem 1.2 of this paper. Note, however, that Laptev's counterexample has the form

$$a_{ik}(x) = a(x)\delta_{ik}$$

where $\lambda(x) = \Lambda(x)$. So, the above comparison of the growth of $\lambda(x)$ and $\Lambda(x)$ has to be understood in a specific sense. For a precise formulation, see theorem 1.1 and 1.2.

In this paper, we give some contributions to this argument. The first result states that L is essentially selfadjoint if the largest eigenvalue $\Lambda(x)$ does not grow faster that $K|x|^2\lambda(x)$, in the sense that

$$\text{(1.5)} \qquad \sup\{\Lambda(x) \mid x \in B_{2R} - B_R\}/\inf\{\lambda(x) \mid x \in B_{2R} - B_R\} \leq KR^2 + K$$

for all $R > 0$ with some constant K. Here, B_R denotes the ball of radius R with the origin as center. – For simplicity of exposition we restrict ourselves to the case that q is bounded below, i.e. that there is a constant $c > 0$ such that

$$\text{(1.6)} \qquad q \geq -c$$

and we do not allow first order terms $b_i\partial_i u$. – Our first result reads

Theorem 1.1. *Under the assumptions* (1.1) – (1.6), *the operator L is essentially selfadjoint.*

Note that in (1.5) a slightly faster growth of the type $KR^2(lnR)^\mu$ instead of KR^2 is possible but we omit the proof (this can be done with a "hole-filling-method" from non-linear elliptic analysis). Theorem 1.1 is proved in 3. The proof employs an inequality which follows from the Moser-technique [11] in the theory of non-linear elliptic equations. With this method, we obtain also another version of Devinatz' result [2] assuming less regularity for the a_{ik}. For this purpose, the following condition is sufficient: *There exist constants K and $r > 0$ such that*

$$\text{(1.7)} \qquad \sup\{\Lambda(x) \mid x \in B_r(y)\}/\inf\{\lambda(x) \mid x \in B_r(y) \leq K, \; y \in \mathbb{R}^n$$

for all balls $B_r(y)$ with radius r and center y.

Theorem 1.2. *Under the assumptions* (1.1) – (1.4), (1.6), (1.7), *the operator L is essentially selfadjoint.*

The proof is given in 3. Note that (1.7) is almost a "pointwise" condition for the quotient $\Lambda(x)/\lambda(x)$ since r may be small. – We also intend to give a pointwise condition for a_{ik} which yield essential selfadjointness:

$$\text{(1.8)} \qquad \Lambda(x)/\lambda(x) \leq K, \quad x \in R^n$$

and

$$\text{(1.9)} \qquad |\nabla a_{ik}(x)|/\lambda(x) \leq K, \quad i, k = 1, \ldots, n, x \in R^n,$$

with some constant $K > 0$.

In this case we need more integrability for the potential q

$$\text{(1.10)} \qquad q \in L^n_{loc}(\mathbb{R}^n)$$

Theorem 1.3. *Under the assumption* (1.1) – (1.4), (1.6), (1.8) – (1.10), *the operator L is essentially selfadjoint.*

The proof is given in 3 and uses an interesting inequality about non-variational elliptic operators due to Pucci [12] and Alexandroff. Because this technique works only for $H^{2,n}_{loc}(\mathbb{R}^n)$ – solutions this explains why we need condition (1.10). Furthermore, let us mention that it is an open problem to prove essential selfadjointness of L if one assumes instead of $q \geq -c$ the condition, that L is bounded from below in L^2, i.e.

$$\text{(1.11)} \qquad (L\varphi, \varphi) \geq -\lambda \|\varphi\|_2^2, \qquad \varphi \in C_0^\infty(\mathbb{R}^n),$$

some constant λ. ($\|\cdot\|_2 = L^2$-norm; $(.,.)$ = scalar product in L^2). This is interesting even for the case $L = -\Delta + q$. For this problem, we present the following observation:

Theorem 1.4 *Let $q \in L^2_{loc}$ for $n = 2,3$ and let $q \in L^p_{loc}$, $p > n/2$ for $n \geq 4$. Furtheormore, let $|a_{ik}(x)| \leq K + K|x|^2$, $x \in \mathbb{R}^n$, with some constant K and suppose* (1.2) – (1.4), (1.11). *Then L is essentially selfadjoint.*

The proof is given in 3. For the case of regular coefficients, cf. Wienholtz [24]. The only difficulty is to obtain a certain regularity theorem (cf. theorem

2.2). – The proofs of the theorems are all based on the fact that L is essentially selfadjoint if there is a $\lambda \in R^n$ such that $Lu + \lambda u = 0$ has only the trivial L^2-distribution solution.

We finally remark that some of the analytical tools of the "theory of essential-selfadjointness" can be used nicely to derive simple proofs of Liouville-type theorems (e.g. solutions of $\Delta u = 0$ which are bounded in all $\mathbb{R}^n$ must be constant). Some examples of this type are given in the appendix to this paper.

Throughout the paper, $L^p_{(loc)}$ and $H^{m,p}_{(loc)}$ denote the usual Lebesgue and Sobolev spaces which consist of all functions which are (locally) p-integrable (with $m-th$ derivatives in $L^p_{(loc)}$ for $u \in H^{m,p}_{(loc)}$).

The results of this paper were found by the author in 1973 during his stay in Berkeley and presented in talks in Göttingen and Linköping 1974. We did not publish these results yet because of the lack of physical applications. But the recent work, cf. references, gives now sufficient motivation to justify this. For a mathematical application or motivation see Triebel's paper [18] and also the work of H.O. Cordes on Banach algebras of pseudo differential operators (to appear), where he needs that powers of second order elliptic operators with variable coefficients are essentially selfadjoint, cf. [1].

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2. Regularity theorems.

A function $u \in L^2_{loc}(\mathbb{R}^n)$ is called a distributional solution of equ. $Lu = 0$ iff $(u, L^*\varphi) = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $L^*\varphi = -\Sigma_{ik}\partial_k(a_{ik}\partial_i\varphi) + q\varphi$, $(i, k = 1, \ldots, n)$. – For the proof of theorem 1.1 and 1.2 we need

Lemma 2.1. *Let $u \in L^2_{loc}(\mathbb{R}^n)$ be a distributional solution of $Lu = 0$. Assume* (1.1) – (1.3) *and that $q \geq 0$. Then*

$$u \in H^{1,2}_{loc}(\mathbb{R}^n)$$

Proof Let $\omega \in C_0^\infty(\mathbb{R}^n)$, $\omega \geq 0$, and $\int \omega\, dx = 1$. Set $\omega_h(x) = \omega(h^{-1}x)$ and let $\omega_h * v$ denote the convolution between the L^2_{loc}-function v and the mollifier ω_h. Furthermore, for $l > 0$ set $v_l = \min\{|v|, l\}$ sign v. Now, in equ. $(u, L\varphi) = 0$ set

$$\varphi = \omega_h * [\tau^2(\omega_h * u)_l]$$

where $\tau \in C_0^\infty(\mathbb{R}^n)$, $\tau \geq 0$, and $\tau = 1$ on a ball B_R of radius R. This yields – with $U = (\omega_h * u)_l$ –

$$O = (u, L\varphi) = A + B + C \tag{2.1}$$

where
$$A = (u, (\partial_k a_{ik})\, \partial_i\, [\omega_h * (\tau^2 U)]),$$
$$B = (u, a_{ik}\, [(\partial_k \omega_h) * \partial_i(\tau^2 U)])$$
$$C = (qu, \omega_h * (\tau^2 U)).$$

Here and in the following we use the summation convention $i, k = 1, \ldots, n$. Since $u \in L^2_{loc}$ and $a_{ik} \in H^{1,\infty}_{loc}$, we may estimate

$$|A| \leq K + K\, \|\tau \nabla U\|_2 \tag{2.2}$$

with a (generic) constant $K = K(\tau)$ not depending on h and l.

(In the following, we use the same letter K for different constants). We rewrite B in the form

$$B = (\partial_k(\omega_h * u), a_{ik}\partial_i(\tau^2 U)) + D \tag{2.3}$$

where

$$D = (u, a_{ik}[\partial_k\omega_h * (\partial_i(\tau^2 U)] - \partial_k\omega_h * [a_{ik}\partial_i(\tau^2 U)])$$

and estimate

$$|D| \leq K\, \|\nabla(\tau^2 U)\|_2\, \|\nabla\omega_h\|_\infty\, \sup\{\|a_{ik} - a_{ik}(\,.\, - t)\|_\infty \,|\, |t| \leq kh\} \tag{2.4}$$

where $\|\,.\,\|_{\infty;T}$ denotes the L^∞-norm taken over a neighbourhood $T$of supp τ. – Since a_{ik} is Lipschitz and $u \in L^2$, the last factor of the right hand side of (2.4) remains bounded for $h \to 0$, $l \to \infty$ and we obtain

$$|D| \leq K + K\, \|\tau \nabla U\|_2 \tag{2.5}$$

finally, we observe that for $h \to 0$

$$C \to (qu, \tau^2 u_l) \tag{2.6}$$

From (2.1) – (2.5) we conclude

$$(\partial_k(\omega_h * u),\ a_{ik}\partial_i\, [\tau^2(\omega_h * u)_l]) + C = E \tag{2.7}$$

where

$$|E| \leq K + K\, \|\tau\, \nabla(\omega_h * u)_l\|_2 \tag{2.8}$$

uniformly in h and l. Since $a_{ik} \in H^{1,\infty}_{loc}$, the ellipticity constant is uniform on compact domains of $\mathbb{R}^n$. Thus we conclude from (2.7)

$$(2.9)\qquad c_\tau \|\tau\nabla(\omega_h * u)_l\|_2^2 + C \leq E + F$$

where $c_\tau > 0$ and

$$F = -(\partial_k(\omega_h * u), a_{ik}(\partial_i\tau^2)(\omega_h * u)_l).$$

Performing partial integration we obtain

$$|F| \leq |(\omega_h * u, \partial_k[a_{ik}(\partial_i\tau^2)](\omega_h * u)_l + a_{ik}(\partial_i\tau^2)\partial_i(\omega_h * u)_l)|$$

and thus

$$(2.10)\qquad |F| \leq K + K\|\tau\nabla(\omega_h * u)_l\|_2$$

uniformly in h and l. From (2.8) – (2.10) we conclude

$$(2.11)\qquad \frac{1}{2}c_\tau\|\tau\nabla(\omega_h * u)_l\|_2^2 + C \leq K_\tau$$

uniformly in h and l. For fixed l, the number C remains bounded as $h \to 0$ on account of (2.6). Since this holds for all $\tau \in C_0^\infty(\mathbb{R}^n)$, $\tau \geq 0$, we obtain

$$u_l \in H_{loc}^{1,2}(\mathbb{R}^n)$$

and, by weak compactness and lower semi-continuity

$$\frac{1}{2}c_\tau\|\tau\nabla u_l\|_2^2 + (qu, \tau^2 u_l) \leq K_\tau$$

Since $q \geq 0$, we conclude from the last inequality

$$(2.12)\qquad \frac{1}{2}c_\tau\|\tau\nabla u_l\|_2^2 \leq K_\tau$$

uniformly in l. From this, it follows that also $u \in H_{loc}^{1,2}(\mathbb{R}^n)$, q.e.d.

Theorem 2.1. *Under the hypotheses of lemma 2.1*

$$u \in L_{loc}^\infty \cap H_{loc}^{2,2}(\mathbb{R}^n)$$

Proof: Since we know already $u \in H_{loc}^{1,2}$, we conclude from $(u, L\varphi) = 0$ the inequality

$$(2.13)\qquad (a_{ik}\partial_k u, \partial_i\varphi) + (qu, \varphi) = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. By approximation, (2.13) extends to all $\varphi \in L^\infty \cap H^{1,2}(\mathbb{R}^n)$ with compact support.

Thus we may set $\varphi = \tau^2 u_l|u_l|^{p-1}$ and obtain

$$(2.14)\qquad (a_{ik}\partial_k u, \partial_i(\tau^2 u_l|u_l|^{p-1})) \leq 0$$

where we could drop the term containing q since $q \geq 0$. – Inequality (2.14) is the starting point of the first part of the Moser-technique [11], and since all $\tau \in C_0^\infty(\mathbb{R}^n)$, $p \geq 1$, $l > 0$ are admissible, it is well known, cf. [8], [11], how to conclude the local boundedness of u from (2.14). Thus $u \in L_{loc}^\infty(\mathbb{R}^n)$ and the $H_{loc}^{2,2}$-property follows by the linear theory of elliptic operators since $-\partial_i(a_{ik}\partial_k u) = -qu \in L_{loc}^2$ in the sense of distributions. q.e.d.

Corollary to theorem 2.1. *If additionally $q \in L_{loc}^n$ then $u \in H_{loc}^{2,n}(\mathbb{R}^n)$.*

This follows from standard L^p-coerciveness results of the theory of elliptic equations.

Theorem 2.2. *Under the assumptions of theorem 1.4 – without assumption (1.11) – every distributional solution $u \in L_{loc}^2(\mathbb{R}^n)$ of $Lu = 0$ belongs to $L_{loc}^\infty \cap H_{loc}^{1,2}(\mathbb{R}^n)$.*

Proof: We omit the proof for $n = 2,3$ which is a simple consequence of Gardings inequality in L^s, $s < n/(n-1)$. – Thus, let $n \geq 4$. By hypothesis, $q \in L_{loc}^p$ with $p > n/2$ and we may write $p = (n + n\delta)/2$, $\delta > 0$. We first prove: If $(u \in L_{loc}^\beta(\mathbb{R}^n)$ with $\beta \geq 2$ then $u \in L^{\beta'}$ where

$$\beta' = \infty \quad \text{if} \quad n/\beta - 2\delta/(1+\delta) < 0,$$
$$\beta' \text{ is an arbitrary number} \quad \text{if} \quad n/\beta - 2\delta/(1+\delta) = 0,$$
$$\beta' = n/[n/\beta - 2\delta/(1+\delta)] \quad \text{if} \quad n/\beta - 2\delta/(1+\delta) > 0.$$

In fact, since $u \in L_{loc}^\beta$ and $q \in L_{loc}^p$, we have $uq \in L_{loc}^t$ and thus $Lu \in L_{loc}^t$ where t satisfies $1/t = 1/\beta + 1/p = 1/\beta + 2/(n + n\delta)$. By the L^p-regularity theorems of elliptic analysis, $\nabla^2 u \in L_{loc}^t$, and by Sobolev's inequality we obtain $u \in L_{loc}^{\beta'}$ where $\beta' = nt/(n-2t) = n/[n/t - 2] = n/[n/\beta - 2\delta/(1+\delta)]$ provide that $n/\beta - 2\delta/(1+\delta) > 0$. Otherwise, $\beta' = \infty$ or β' arbitrary. – By recursion we obtain $u \in L_{loc}^{\beta(i)}(\mathbb{R}^n)$ where $\beta(0) = 2$, $\beta(i+1) = (\beta(i))'$ and $(\,.\,)'$ is defined above. If $n/\beta(i) - 2\delta/(1+\delta) \leq 0$ for some i then $u \in L_{loc}^r(\mathbb{R}^n)$ for all $r \in [1, \infty]$. If $n/\beta(i) - 2\delta/(1+\delta) > 0$ for all $i = 0, 1, 2, \ldots$ we show that $\beta(i) \to \infty$ $(i \to \infty)$ and hence $u \in L_{loc}^r(\mathbb{R}^n)$ also in this case. – Suppose the sequence $(\beta(i))$ were bounded. Then it converges to some number β_∞ since $\beta(i+1) > \beta(i)$. However, this leads to the contradiction $\beta_\infty = n/[n/\beta_\infty - 2\delta/(1+\delta)] > \beta_\infty$. – From $u \in L_{loc}^r$, $r \in [1, \infty]$, and $q \in L_{loc}^p$, $p > n/2$, we conclude $qu \in L_{loc}^{p^*}$, with $p^* = p^*(p) > n/2$. Thus $Lu \in L_{loc}^{p^*}(\mathbb{R}^n)$ and $\nabla^2 u \in L_{loc}^{p^*}(\mathbb{R}^n)$. By Sobolev's theorem, $u \in L_{loc}^\infty \cap H_{loc}^{1,2}(\mathbb{R}^n)$ which proves the theorem.

3. Auxiliary lemmata and proof of the theorems.

As it is well known, cf. e.g. [3], the essential selfadjointness of L follows if one can prove the existence of a number $\lambda_0 \in \mathbb{R}$ such that $Lu + \lambda_0 u$ has no non-trivial distribution solutions $u \in L^2(\mathbb{R}^n)$. In the following, we prove this under the assumptions of theorem 1.1 – 1.4, respectively. The theorems then follow from the corresponding lemmata as a corollary.

During the calculations, the letter K denotes a generic constant, namely a constant not depending on the critical parameters which may be different in each use. Finally, since L is real, it suffices to consider the real $L^2(\mathbb{R}^n)$.

Lemma 3.1. *Under the hypotheses of theorem 3.1, there are no non-trivial distribution solutions $u \in L^2(\mathbb{R}^n)$ of the equation $Lu + \lambda_0 u = 0$ if $\lambda_0 \geq c$ where c is given by* (1.6).

Proof: We consider balls B_R of radius R and center 0. Let $\{b_{R/4}\}$ denote a set of k_n open balls of radius $R/4$ and center $y \in \partial B_{3R/2}$ which cover $\partial B_{3R/2}$ and let $b_{R/2}$ the corresponding concentric balls. The number k_n can be chosen independently on R. Now, let τ be any Lipschitz continuous function with support in $b_{R/2}$. Since $u \in L^\infty_{loc} \cap H^{1,2}_{loc}(\mathbb{R}^n)$ according to theorem 2.1 we have for all $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$(a_{ik}\partial_k u, \partial_i \varphi) + (q_0 u, \varphi) = 0, \qquad q_0 = q + \lambda_0 \geq 0,$$

and for any $p \geq 1$ by an approximation argument

$$(a_{ik}\partial_k u, \partial_i(\tau^2 u|u|^{p-1})) + (q_0 u, \tau^2 u|u|^{p-1}) = 0$$

Since $q_0 \geq 0$ we obtain

$$(a_{ik}\partial_k u, \partial_i(\tau^2 u|u|^{p-1})) \leq 0$$

The last inequality is the starting point of the first part of the Moser-technique [11] which is used in the theory of non-linear elliptic differential equations. With an appropriate choice of τ one obtains a recursion relation for the L^p-norms of u taken over certain balls $\subset b_{R/2}$, and, as it is well known, arrives at the inequality

$$\text{ess-max}\ \{|u(x)|^2 \mid x \in b_{R/2}\} \leq C \int_R |u|^2 dx$$

where $\int_R$ denotes integration over b_R and

$$C = KR^{-n}(\Lambda_+/\lambda_-)^{n/2}$$

where

$$\lambda_- = \inf\ \{\lambda(x) \mid x \in B_{2R} - B_R\}$$
$$\Lambda_+ = \sup\ \{\Lambda(x) \mid x \in B_{2R} - B_R\}$$

$\Lambda(x)$, $\lambda(x)$ being the largest resp. smallest eigenvalue of $(a_{ik}(x))$. Here we have used the fact that $b_{R/2} \subset B_{2R} - B_R$. The size of C above is well known in non-linear elliptic analysis (cf.e.g. [23]); it is a simple exercise to derive it via the Moser-technique. On can derive it also via homegeneity (even the factor $(\Lambda_+/\lambda_-)^{n/2}$ but this requires care). By assumption (1.5) it follows that C remains bounded as $R \to \infty$ and thus

$$\text{ess-max}\ \{|u(x)|^2 \mid x \in b_{R/4}(y)\} \leq K \int_* |u|^2 dx$$

where K doesn't depend on R and $\int_*$ denotes integration over $B_{2R} - B_R$. Since the $b_{R/4}(y)$ are open and cover $\partial B_{3R/2}$ we conclude

$$\text{ess-max}\ \{|u(x)|^2 \mid x \in U\} \leq K \int_* |u|^2 dx \tag{3.1}$$

where U is some neighbourhood of $\partial B_{3R/2}$. Now, we use the fact that weak solutions of $Lu + \lambda_0 u = 0$ satisfy a maximum principle in the sense that

$$\text{ess-max}\ \{|u(x)|^2 \mid x \in B_{3R/2}\} \leq \text{ess-max}\ \{|u(x)|^2\ x \in U\}. \tag{3.2}$$

This follows from the inequality $q + \lambda_0 \geq 0$ and a truncation argument, cf. e.g. [8], using testfunctions

$$\varphi = \text{ess-max}\ \{o, u - l\},\ l = \text{ess-max}\ \{u(x) \mid x \in U\}\ \text{etc.}$$ – From (3.1) and (3.2) we obtain the statement of the lemma if we pass to the limit $R \to 0$ since $\int_* |u|^2 dx \to 0$ $(R \to \infty)$.

Lemma 3.2. *Under the hypotheses of theorem 3.2 there are no non-trivial distribution solutions $u \in L^2(\mathbb{R}^n)$ of the equation $Lu + \lambda_0 u = 0$ if $\lambda_0 \geq c$ where c is given by* (1.6).

Proof: Let B_R the ball of radius R and center 0, and $y \in \partial B_R$. We consider balls $B_r(y)$ of radius r and center y. Let τ be any Lipschitz continuous function with support in $B_{2r}(y)$. As in the proof of lemma 3.1 we conclude for all $p \geq 1$ (recall $u \in L^\infty_{loc} \cap H^{1,2}_{loc}(\mathbb{R}^n)$ by (2.1)

$$(a_{ik}\partial_k u, \partial_i(\tau^2 u|u|^{p-1})) \leq 0$$

From this inequality, we obtain with the Moser-technique [11],

$$\|u\|^2_{\infty,r} \leq C \int_{2r} |u|^2 dx$$

where $\|\,.\,\|_{\infty,r}$ denotes the L^∞-norm taken over $B_r(y)$, $\int_{2r}$ the integration over $B_{2r}(y)$. The cônstant C depends on r and the eigenvalues of $(a_{ik}(x))$, in fact,

$$C = Kr^{-n}(\sup\{\Lambda(x) \mid x \in B_{2r}(y)\}/\inf\{\lambda(x) \mid x \in B_{2r}(y)\})^{n/2}.$$

By our assumption (1.7), C is bounded uniformly with respect to $y \in \partial B_R$ and $R \in \mathbb{R}$, if r is fixed. – Since $u \in L^2$, $\int_{2r} |u|^2 dx < \varepsilon$ for $R \geq R(\varepsilon)$ und thus

$$\| u \|^2_{\infty, r} \leq C\varepsilon, \quad R \geq R(\varepsilon).$$

From the maximum principle (cf. the proof of lemma 3.1) we conclude

$$\text{ess-max } \{|u(x)|^2 \mid x \in B_R\} \leq \text{ess-max } \{|u(x)|^2 \mid x \in \partial B_R\} \leq C\varepsilon.$$

Passing to the limit $R \to \infty$ we obtain the statement $u = 0$. q.e.d.

Lemma 3.3. *nder the hypotheses of theorem 3.3 there are no non-trivial distribution solutions $u \in L^2(\mathbb{R}^n)$ of the equation $Lu + \lambda_0 u = 0$ If $\lambda_0 \geq c$ where c is given by (1.6).*

Proof: Let B_R the ball of radius R with the origin as center, and for each $y \in \partial B_R$ let us consider balls $B_r(y)$ of radius r and y as center. Let $\tau \in H^{2,\infty}$ such that $\tau = 1$ on $B_r(y)$, $\tau = 0$ on $\mathbb{R}^n - B_{2r}(y)$, $\tau \geq 0$, and $|\nabla\tau| \leq Kr^{-1}$, $|\nabla^2\tau| \leq Kr^{-2}$. From the corollary to theorem 2.1 we know that $u \in H^{2;n}_{loc}(R^n)$ and thus, if $Lu + \lambda_0 u$ in the sense of distributions,

$$a_{ik}\partial_i\partial_k u = (\partial_i a_{ik})\partial_k u + q_0 u \quad \text{a.e. in } \mathbb{R}^n$$

where $q_0 = q + \lambda_0$. Thus

$$a_{ik}\partial_i\partial_k(u^2\tau^n) = 2a_{ik}\tau^n\partial_i u\partial_k u + 4na_{ik}u\tau^{n-1}\partial_i u\partial_k\tau + \\ + (n\tau\partial_i\partial_k\tau + n(n-1)\partial_i\tau\partial_k\tau)a_{ik}u^2\tau^{n-2} + q_0 u^2\tau^n.$$

We now use the fact that $q_0 \geq 0$ and that, by hypothesis, $\Lambda(x) \leq K\lambda(x)$ and $|\nabla a_{ik}(x)| \leq K\lambda(x)$ where $\Lambda(x)$ is the largest and $\lambda(x)$ the smallest eigenvalue of $(a_{ik}(x))$. Applying Young's inequality we easily obtain

$$a_{ik}\partial_i\partial_k(u^2\tau^n) \geq -K_r\lambda(x)u^2\tau^{n-2}$$

with a constant K_r which may alter it's value during the estimates. The last inequality yields

$$A(u^2\tau^n) \geq -K_r u^2\tau^{n-2}$$

where A is the *uniformly* elliptic operator

$$A = \lambda^{-1}(x)a_{ik}(x)\partial_i\partial_k$$

From a theorem of Pucci [12] – Alexandroff we conclude

(3.3) $$\| u^2\tau^n \|^n_\infty \leq K_r \int (u^2\tau^{n-2})^n dx$$

i.e. the inequality $\| v \|^n_\infty \leq K \int |f|^n dx$ if $Av \geq f$, $v \geq 0$, and v satisfies zero boundary conditions. – From (3.3) we obtain

(3.4) $$\| u^2\tau^n \|_\infty \leq K_r \textstyle\int_r u^2 dx$$

where $\int_r$ denotes integration over $B_{2r}(y)$. If we fix r, the right hand side of (3.4) is smaller than ε if $R \geq R(\varepsilon)$ since $u \in L^2(\mathbb{R}^n)$. Now it follows that $\max\{|u(x)| \mid x \in \partial B_R\} < \varepsilon$ for $R \geq R(\varepsilon)$. From the maximum principle – cf. the proof of lemma 3.2 – we conclude

$$|u(x)| < \varepsilon \quad \text{on} \quad B_R, \quad R \geq R(\varepsilon),$$

and thus $u = 0$. *q.e.d.*

Lemma 3.4. *Under the hypotheses of theorem 1.4 there are no non-trivial distribution solutions $u \in L^2(\mathbb{R}^n)$ of the equation $Lu + \lambda_0 u = 0$ with $\lambda_0 > \lambda$ where λ is given by (1.11).*

Proof: Let $u \in L^2(\mathbb{R}^n)$ and $Lu + \lambda_0 u = 0$ in the sense of distributions. Set $L_0 = L + \lambda_0 id$, $q_0 = q + \lambda_0$. Then

(3.5) $$(L_0\varphi, \varphi) \geq (\lambda_0 - \lambda) \| \varphi \|^2_2, \quad \varphi \in C^\infty_0(\mathbb{R}^n).$$

By theorem 2.2, $u \in L^\infty_{loc} \cap H^{1,2}_{loc}(\mathbb{R}^n)$, and hence

$$(a_{ik}\partial_k u, \partial_i\varphi) + (q_0 u, \varphi) = 0, \quad \varphi \in C^\infty_0(\mathbb{R}^n)$$

Since $u \in L^\infty_{loc} \cap H^{1,2}_{loc}(\mathbb{R}^n)$ we obtain with an approximation argument that

$$(a_{ik}\partial_k u, \partial_i(u\tau^2)) + (q_0 u, u\tau^2) = 0$$

where τ is any Lipschitz-continuous function with compact support. By a simple calculation, using the symmetry of (a_{ik}), we obtain

$$(a_{ik}\partial_k u, \partial_i(u\tau^2)) = (a_{ik}\partial_k(u\tau), \partial_i(u\tau)) - (a_{ik}u^2\partial_i\tau, \partial_k\tau)$$

and thus – using also (3.5) –

(3.6) $$(\lambda_0 - \lambda) \| u\tau \|^2_2 \leq (a_{ik}\partial_k(u\tau), \partial_i(u\tau)) + (q_0 u\tau, u\tau) \leq (a_{ik}u^2\partial_k\tau, \partial_i\tau)$$

We now choose τ such that $\tau = 1$ on B_R, $\tau = 0$ on $\mathbb{R}^n - B_R$, $|\nabla\tau| \leq R^{-1}$ on $B_{2R} - B_R$. – By hypothesis, $|a_{ik}| \leq K + KR^2$ on $B_{2R} - B_R$. So, the right hand side of (3.7) can be estimated by $K \| u \|^2_{2;*}$ where $\| \cdot \|_{2;*}$ denotes the L^2-norm taken over B_{2R}-B_R. Thus

$$(\lambda_0 - \lambda) \| u\tau \|^2_2 \leq K \| u \|^2_{2;*}$$

and the right hand side of the last inequality tends to zero as $R \to \infty$ since

$u \in L^2$. The left hand side tends to $(\lambda_0 - \lambda) \| u \|_2^2$ and we conclude $u = 0$ which proves the lemma.

Appendix. Liouville's theorem for elliptic equations.

The purpose of the following lines is to present simple proofs of Liouville type theorems, cf. e.g. [13] for classical (and new) results and references concerning this argument. We use some of the analytical tools and ideas of previous sections. We present Liouville type theorems for non-linear elliptic systems up to dimension $n = 4$ and for linear elliptic systems in any dimension (i.e. bounded solutions must be constant). – For large dimensions, a recent counterexample of Nečas showing non-regularity of solutions to non-linear elliptic systems is also a counterexample for the conjecture that a Liouville type theorem holds in this setting. – Nečas' example is a system of Euler equations $\partial_i F_i(\nabla u) = 0$ where u and F_i have n^2 components. In his example the weak solution u has the components $x_i x_k |x|^{-1}$, cf. Nečas, Proc. Soviet Conference on Partial Differential Equations, Moscow 1976. Passing to the differentiated equation $\partial_i \partial_j F_i(\nabla u) = 0, j = 1, \ldots, n$, one obtains a linear elliptic system in n^3 equations with measurable coefficients and the solution $v = \nabla u$ which is bounded in all $\mathbb{R}^n$.

For scalar non-linear elliptic equations, say

(A 1) $$-\Sigma_{i=1}^n \partial_i F_i(\nabla u) = 0$$

Liouville's theorem follows from de Giorgi's theorem concernin the oscillation reduction property of solutions of *(A 1)*

(A 2) $$\operatorname{osc}\{u(x) \mid x \in B_R\} \leq \theta \operatorname{osc}\{u(x) \mid x \in B_{2R}\}$$

where $0 < \theta < 1$. This is usually used in the small for proving C^α-regularity. However, it can also be applied in the large to prove a Liouville theorem. (Iterate A 2). Note that some care is required for proving de Giorgi's theorem for equ. (A 1) in the form (A 2) without the additional summand KR^α on the right hand side of (A 2). Apparently, there is a connection between Liouville's theorem and the regularity of solution to elliptic equations. It is not known whether there holds a Liouville type theorem for the uniformly elliptic non--variational equation

$$a_{ik}\partial_i\partial_k u = 0, \quad u \in H^{2,n}_{loc}$$

with $a_{ik} \in L^\infty$. This seems to be connected with the problem of obtaining zero order estimates for u (i.e. C^α-estimates for u where α depends only on the

quotient of the maximal and minimal eigenvalue of (a_{ik}), and of $\| u \|_\infty$ and n.) We anticipate that a Liouville type theorem will be easier to prove than a regularity theorem.

We do not claim originality concerning the following theorems. E.g., it was pointed out by *R*. Finn to the author, that theorem A 1 was proved by him with a similar techinique. We hope that the general remarks above are illustrative.

(i) *A two dimensional Liouville theorem for nonlinear elliptic systems.*

Let $F_i(x, \eta)$, $i = 1, 2$, be r-vector functions on $\mathbb{R}^2 \times \mathbb{R}^{2r}$ satisfying the following conditions:

(A 1) *Regularity.* $F_i(x, \eta)$ *is measurable in* $x \in \mathbb{R}^2$ *and continuous in* $\eta \in \mathbb{R}^{2r}$.

(A 2) *Coerciveness.* $\Sigma_{i=1}^2 F_i(x, \eta)\eta_i \geq C|\eta|^2$, $x \in \mathbb{R}^2$, $\eta_i \in \mathbb{R}^r$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^{2r}$.

(A 3) *Growth & zero condition*: $|F_i(x, \eta)| \leq K|\eta|$, $x \in \mathbb{R}^2$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^{2r}$ *with some constant K.*

Let $Lu = -\Sigma_{i=1}^n \partial_i F_i(x, \nabla u)$ in the sense of distributions.

Theorem A 1. *Assume the conditions (A3) – (A5), let* $n = 2$ *and let u be an r-vector function such that* $u \in H^1_{loc} \cap L^\infty(\mathbb{R}^n)$ *and* $Lu \leq 0$ *in the sense of distribuitions. Then u is a constant vector.*

Remark: If $r = 1$, one may replace the condition $u \in L^\infty$ by "u is bounded from above".

Proof of theorem A 1: By a translation, we may assume $u \geq 0$. Let $B_R = \{x \in \mathbb{R}^n \| x| \leq R\}$ and $\tau \in H^{1,\infty}(\mathbb{R}^n)$ such that $\tau = 1$ on B_R, $\tau = 0$ on $B_{2R} - B_R$, $\tau \geq 0$, and $|\nabla\tau| \leq R^{-1}$ on $B_{2r} - B_R$. Since $u\tau^2 \geq 0$, and $u\tau^2 \in H^1(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$, we have $\langle Lu, u\tau^2\rangle \leq 0$ and thus

$$\Sigma_{i=1}^n \int F_i(\,.\,, \nabla u)\partial_i(u\tau^2)\,dx \leq 0$$

Calculating $\partial_i(u\tau^2)$ by Leibniz's rule using A 4 and A 5 we obtain

(A 6) $$\int |\nabla u|^2 \tau^2 dx \leq K \int |\nabla u| |u| \tau |\nabla\tau| dx$$

with some constant K. Since u is uniformly bounded and $|\nabla\tau| \leq R^{-1}$, we may estimate the right hand side of (A 6) and obtain

(A 7) $$\int |\nabla u|^2 \tau^2 dx \leq K\sigma \int_* |\nabla u|^2\tau^2 dx + K\sigma^{-1}R^{-2} \int_* 1 dx$$

Here, $\int_*$ denotes integration over $B_{2R}-B_R$. We have used the fact that $\nabla\tau$ has its support in $B_{2R}-B_R$. Now, choosing σ small enough, we conclude from (A 7) that

$$\int|\nabla u|^2\tau^2dx \leq K\sigma^{-1}R^{-2}\int_* 1dx \leq C$$

uniformly for $R\to\infty$ and thus $\nabla u\in L^2(\mathbb{R}^n)$. Once knowing this we look at (A 7) again, choose σ large, and pass to the limit $R\to\infty$. This yields

$$\int|\nabla u|^2\,dx \leq K\sigma^{-1}R^{-2}\int_* 1dx \leq \sigma^{-1}C'$$

since $\int_*|\nabla u|^2dx\to 0$ for $R\to\infty$. The theorem follows by passing to the limit $\sigma\to\infty$.

In the scalar case $r=1$ – where one needs only the hypothesis that u is bounded from above – one has to take the function $\max\{u-l,0\}\tau^2$ as test function and concludes in the same manner as before. One will arrive at the statement $\max\{u-l,0\}=$ const., and Liouville's theorem follows too.

(ii) *The case $n=3$ and $n=4$.*

Let F_i, $i=1,\dots,n$, be r-vector functions on $\mathbb{R}^{nr}$ satisfying the following conditions

(A 8) F_i is uniformly Lipschitz continuous on $\mathbb{R}^{nr}$

(A 9) Zero condition: $F_i(o)=0$, $i=1,\dots,n$.

(A 10) Monotonicity/Ellipticity: There is a constant $c>0$ such that

$$\Sigma_i(F_i(\eta)-F_i(\zeta))(\eta_i-\zeta_i)\geq c|\eta-\zeta|^2, \quad i=1,\dots,n,$$

for $\eta=(\eta_1,\dots,\eta_n)\in\mathbb{R}^{nr}$, $\eta_i\in\mathbb{R}^r$, $\zeta=(\zeta_1,\dots\zeta_n)\in\mathbb{R}^{nr}$

We shall consider the equation

(A 11) $\quad\partial_iF_i(\nabla u)=0,\quad u\in H^{1,2}_{loc}(\mathbb{R}^n),\ u=(u_1,\dots,u_r),$

in the sense of distributions (summation convention!)

Theorem A 2. *Let $u\in H^{1,2}_{loc}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ be a distributional solution of equation (A 11) and assume (A 8) – (A 10).*

Then u is a constant vector.

Proof. We consider only the case $n=4$. The proof for the case $n=3$ is almost the same. – From (*A* 8) and (*A* 9) we conclude a growth condition $|F_i(\eta)|\leq K|\eta|$ and from (A 9) and (A 10) a coerciveness condition, namely

$$\Sigma_iF_i(\eta)\eta_i\geq c|\eta|^2,\quad (i=1,\dots,n).$$

As in the preceding proof we derive the estimate $\int_R|\nabla u|^2\,dx\leq KR^{-2}\int_{2R}dx$ where $\int_R$ etc. denotes integration over $B_R=\{x\in\mathbb{R}^n\,|\,|x|\leq R\}$. Since $n=4$, we conclude that

(A 12) $$R^{-2}\int_R|\nabla u|^2\,dx\leq K \quad\text{as}\quad R\to\infty$$

Now, we set $\varphi=D_j^{-h}(\tau^2D_j^hu)$ where τ is defined as in the proof of theorem A 1 and $D_j^{\pm h}w(x)=\pm h^{-1}\,(w(x\pm he_j)-w(x))$, e_j being the *j-th* unit vector. By (A 11)

$$(F_i(\nabla u),\,\partial_i\varphi)=0$$

and thus

$$(D_j^hF_i(\nabla u),\partial_i(\tau^2D_j^hu))=0$$

Using the Lipschitz continuity fo F_i we obtain

$$(D_j^hF_i(\nabla u),\tau^2\partial_iD_j^hu)\leq K\int|\nabla D_j^hu\,\|\nabla\tau|\,\tau\,|D_j^hu|\,dx$$

and from the monotonicity condition

$$\int|\nabla D_j^hu|^2\tau^2dx\leq K\int|\nabla D_j^hu|\,|\nabla\tau|\,\tau\,|D_j^hu|dx.$$

By a standard argument, it follows that $u\in H^{2,2}_{loc}$ and

$$\int|\nabla\partial_ju|^2\,\tau^2dx\leq K\int|\nabla\partial_ju|\,|\nabla\tau|\,\tau\,|\partial_ju|\,dx,\ j=1,\dots,n,$$

and – cf. the proof of th. A1 –

(A 13) $$\int|\nabla\partial_ju|^2\,\tau^2dx\leq\sigma\int_*|\nabla\partial_ju|^2\tau^2dx+\sigma^{-1}kR^{-2}\int_{2R}|\partial_ju|^2dx$$

where $\int_*$ denotes integration over $B_{2R}-B_R$. By (A 12) the quantity $R^{-2}\int_{2R}|\partial_ju|^2dx$ is uniformly bounded as $R\to\infty$. Choosing $\sigma=1/2$ we obtain that $\int_R|\nabla\partial_ju|^2dx$ is uniformly bounded and thus $\nabla\partial_ju\in L^2(\mathbb{R}^n)$, $j=1,\dots,n$. Once knowing this, we go back to (A 13) with another σ. Passing to the limit $R\to\infty$ we have $\int_*|\nabla\partial_ju|^2\tau^2dx\to 0$ since $\nabla\partial_ju\in L^2(\mathbb{R}^n)$, and we obtain

$$\int|\nabla\partial_ju|^2dx\leq\limsup_{R\to\infty}\ KR^{-2}\int_{2R}|\partial_ju|^2dx\leq\sigma^{-1}K$$

(recall A 12). Passing to the limit $\sigma\to\infty$ we arrive at the equation

$$\int|\nabla\partial_ju|^2dx=0,\quad j=1,\dots,n$$

and thus *u must be linear*. Since u is bounded, we obtain $u=$ const.

(iii) *A Liouville-theorem for elliptic systems with constant coefficients, n arbitrary.*

Let $a_{ik}^{\nu\mu}$, $i, k = 1, \ldots, n$, and $\nu, \mu = 1, \ldots, r$ be real numbers and

$$a(u, v) = \int a_{ik}^{\nu\mu} \partial_k u_\nu \partial_i v_\mu dx$$

where u and v are r-vector-functions with components u_ν and v_μ in $H^{1,2}(\mathbb{R}^n)$, and a summation convention is used, $\nu, \mu = 1, \ldots, r$, $i, k = 1, \ldots, n$.

We assume that the bilinear form a is strongly elliptic, namely

(A 14) $$a_{ik}^{\nu\mu} \lambda_i \lambda_k \xi_\nu \xi_\mu > 0$$

for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_r) \in \mathbb{R}_r$, $\lambda \neq 0$, $\xi \neq 0$.

Theorem A 3 *Let $u \in L^\infty(\mathbb{R}^n)$ be an r-vector-function which satisfies the strongly elliptic equation*

(A 15) $$a(u, v) = 0$$

for all r-vector-functions v with components $\varepsilon\, C_0^\infty(\mathbb{R}^n)$ in the distribution sense. Then u is a constant vector.

Proof: By well known regularity results, $u \in C^\infty(\mathbb{R}^n)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be multi-indices with $\alpha_i \in \{0, 1, 2, \ldots\}$ and ∂^α be the corresponding higher order derivative. Then $a(\partial^\alpha u, v) = 0$ for all Lipschitz-continuous functions v, and we may set $v = \tau^2 \partial^\alpha u$ where τ is a Lipschitz continuous function such that $\tau = 1$ on $B_R = \{x \in \mathbb{R}^n \mid |x| \leq R\}$, $\tau = 0$ on $\mathbb{R}^n - B_{2R}$, $|\nabla\tau| \leq R^{-1}$.

By a simple calculation

$$0 = a(\partial^\alpha u, \tau^2 \partial^\alpha u) = a(\tau\partial^\alpha u, \tau\partial^\alpha u) + B$$

where

$$|B| \leq \varepsilon \, \| \nabla(\tau\partial^\alpha u) \|_2^2 + K(\varepsilon) \, \| \, |\nabla\tau| \partial^\alpha u \|_2^2$$

Since a is a strongly elliptic bilinear form with constant coefficients, we may estimate

$$a(\tau\partial^\alpha u, \tau\partial^\alpha u) \geq C \, \| \nabla(\tau\partial^\alpha u) \|_2^2$$

where c is a positive constant. Thus

$$\| \nabla(\tau\partial^\alpha u) \|_2^2 \leq K \, \| \, |\nabla\tau| \partial^\alpha u \|_2^2$$

and from the properties of τ we conclude

$$\| \nabla\partial^\alpha u \|_{2;R}^2 \leq KR^{-2} \, \| \partial^\alpha u \|_{2;2R}^2$$

where $\| \, . \, \|_{2;R}$ denotes the L^2-norm taken over B_R. Since this holds for all α of order m, we obtain

$$\| \nabla^{m+1} u \|_{2;R}^2 \leq KR^{-2} \, \| \nabla^m u \|_{2;2R}$$

and by recursion

(A 16) $$\| \nabla^n u \|_{2;R}^2 \leq KR^{-2n} \, \| u \|_{2;ZR}$$

where $Z = 2^n$.

Since $u \in L^\infty$

$$\| u_{2;ZR}^2 \leq KR^n$$

and the right hand side of (A 16) tends to zero as $R \to \infty$. This yields $\| \nabla^n u \|_2 = 0$ and u must have components which are polynomials. Since $u \in L^\infty$ by hypothesis it follows that u must be a constant vector. q.e.d.

We remark that it should be possible to extend theorem *A* 3 to strongly elliptic systems with variable coefficients which approach rapidly to constants as the argument tends to infinity.

(iv) *Zero-order Bernstein theorems.*

The famous theorem of Bernstein states that any solution to the minimal surface equation defined on the whole plane must be linear. It is known today that this holds up to dimension 7. Until now, it has not been possible to prove this in a simple manner as in (i) – (iii) of this appendix. We observe, however, that there is a class of equations for which a "zero-order" Bernstein-theorem can be proved in a simple way. By this we mean that any solution of the equation (not only those which are $\in L^\infty(\mathbb{R}^n)$) must be constant. Although the theorem below is not a deep one, we included it here because it leads to the analogue question whether a zero order Bernstein-theorem holds in higher dimensions. For simplicity we consider scalar elliptic equations in two variables of the type

$$Lu = \partial_i(a_{ik}(x, u)\partial_k u = 0, \; x \in \mathbb{R}^2$$

(summation convention $i, k = 1, \ldots, n$), and assume the following conditions for the a_{ik}, $i, k = 1, 2$,

(A 17) *Regularity*: $a_{ik}(x, \eta)$ is measurable in $x \in \mathbb{R}^2$ and continuous in $\eta \in \mathbb{R}^1$.
(A 18) *Ellipticity*: $a_{ik}(x, \eta)\xi_i\xi_k > 0$ for $\xi = (\xi_1, \xi_2) \neq 0$, $x \in \mathbb{R}^2, \eta \in \mathbb{R}^1$.
(A 19) *Symmetry*: $a_{ik} = a_{ki}$, $i, k = 1, 2$
(A 20) *Growth condition*: There are constants $K > 0$ and $\delta > 0$ such that

$$a_{ik}(x, \eta) \leq K(1 + |\eta|^{-1-\delta}), \; x \in \mathbb{R}^2, \; \eta \in \mathbb{R}^1.$$

Theorem A 4. *Let $u \in H^{1,2}_{loc}(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$ be a weak solution of the equation $Lu = 0$ in all $\mathbb{R}^2$ and assume (A 17) – (A 20). Then u is constant.*

Proof: For $v \in H^{1,2}(\mathbb{R}^n)$ with compact support we have

$$a_{ik}(\,.\,, u)\partial_k u \partial_i v dx = 0$$

and we set $v = \tau^2 f_\varepsilon(u)$ where τ is defined as in (iii) and $f_\varepsilon(\xi) = \xi[\varepsilon + |\xi|^2]^{(\delta-1)/2}$. Note that $f_\varepsilon(u) \in H^{1,2}_{loc}$. With this notations we obtain

$$a_{ik}(\,.\,, u)\tau^2 f_\varepsilon'(u)\partial_k u \partial_i u dx = -2\int a_{ik}(\,.\,, u)\tau f_\varepsilon(u)\partial_k u \partial_i \tau dx.$$

We observe that the right hand side of the above equation remains bounded as $\varepsilon \to 0$. Since (a_{ik}) is positive definite and

$$f_\varepsilon'(\xi) = \varepsilon + \delta|\xi|^2/[\varepsilon + |\xi|^2]^{(1-\delta)/2+1} \geq 0$$
$$f_\varepsilon'(\xi) \to \delta|\xi|^{\delta-1} \quad (\varepsilon \to 0),\ \xi \neq 0,$$

we obtain via Fatou's lemma that

$$\delta \int a_{ik}(\,.\,, u)\tau^2 |u|^{\delta-1}\partial_k u \partial_i u dx \leq -2\int a_{ik}(\,.\,, u)\tau u^\delta \partial_k u \partial_i \tau dx$$

where the integral on the left hand side has to be extended only over the set where $u \neq 0$. Using Young's inequality for positive definite forms we obtain

(A 21) $$\int a_{ik}(\,.\,, u)\tau^2 |u|^{\delta-1}\partial_k u \partial_i u dx \leq$$
$$\leq \sigma \int_* a_{ik}(\,.\,, u)\tau^2 |u|^{\delta-1}\partial_k u \partial_i u dx + \delta^{-2}\sigma^{-1}\int_* a_{ik}(\,.\,, u)|u|^{\delta+1}\partial_i \tau \partial_k \tau dx$$

The integrals do not run over the sets where $u = 0$, and $\int_*$ denotes integration over $B_{2R} - B_R$. On account of the growth condition (A 20) we obtain that $|a_{ik}(\,.\,, u)||u|^{\delta+1}$ is uniformly bounded and also that the second summand in the right hand side of (A 21) is bounded by $K\sigma^{-1}$. We then proceed as in (i): Choosing first σ small, say $\sigma = 1/2$, we obtain that

(A 22) $$a_{ik}(\,.\,, u)|u|^{\delta-1}\partial_k u \partial_i u \in L^1(\mathbb{R}^2 - S)$$

where $S = \{x \in \mathbb{R}^2 \mid u(x) = 0\}$. Next we fix σ, perform the limit as $R \to \infty$, and the first summand in (A 21) tends to zero because of the integration $\int_*$ and (A 22). Passing to the limit as $\sigma \to \infty$ we obtain $a_{ik}|u|^{\delta-1}\partial_k u \partial_i u = 0$ for $u \neq 0$ and thus $\nabla u = 0$ a.e. The theorem follows.

(v) *Liouville's theorem in the discrete case.*

We define the set of grid points

$$\mathbb{R}^n_h = \{x = (m_1 h, \ldots, m_n h) \mid m_1 \text{ integers}\} \text{ where } h > 0.$$

The functions $u : \mathbb{R}^n_h \to \mathbb{R}$ are called "grid-functions", and the space of uniformly bounded grid-functions is denoted by $L^\infty(\mathbb{R}^n_h)$. The translation operators $E_i^{\pm h}$ on grid-functions w are defined by

$$E_i^{\pm h} w(x) = w(x \pm h e_i)$$

where e_i is the $i-th$ unit vector. The difference operators are defined by

$$D_i^{\pm h} = \pm h^{-1}(E_i^{\pm h} - I), \quad I \text{ identity},$$

and finally the discrete Laplacean by

$$\Delta_h = \Sigma_i D_i^h D_i^{-h},\ (i = 1, \ldots, n).$$

A grid-function u is called *discrete harmonic* if

(A 23) $$-\Delta_h u = 0 \quad \text{on} \quad \mathbb{R}^n_h$$

(For the purpose of our discussion, we could have chosen $h = 1$, but this notation seems to us more suggestive).

Theorem A 5. *Every discrete harmonic function $u \in L^\infty(\mathbb{R}^n_h)$ is constant.*

Proof: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi – index, $\alpha_i \in \{0, 1, 2, \ldots\}$, and D^h_α be the product of the difference operators $(D_i^h)^{\alpha_i}$. Let $(v, w)_h = h^n \Sigma_x v(x) w(x)$, $x \in \mathbb{R}^n_h$, where one of the grid-functions v, w must have finite support in $\mathbb{R}^n_h$. Let R be a multiple of h and let Q_R be the set of grid-point– $x \in \mathbb{R}^n_h$ such that $|x|_\infty \leq R$. Let τ be the grid-function with the properties $\tau \geq 0$, $\tau = 1$ on Q_R, $\tau = 0$ on $\mathbb{R}^n_h - Q_{2R}$, $|\nabla_h \tau| \leq KR^{-1}$, K being independent on R. Note that $\nabla_h \tau = 0$ on $\mathbb{R}^n_h - Q_{2R+h}$. From (A 23) we conclude via partial summation that

$$(\nabla_h D^h_\alpha u, \nabla_h(\tau D^h_\alpha u))_h = 0,\ \nabla_h = (D^h_1, \ldots, D^h_n),$$

and using the discrete Leibniz Leibniz formula, namely,

$$D^h_i(vw) = vD^h_i w + D^h_i v E^h_i w$$

we obtain

$$(\nabla_h D^h_\alpha u, \tau \nabla_h D^h_\alpha u)_h \leq \sum_{i=1}^n (|\nabla_h D^h_\alpha u|, |\nabla_h \tau||E^h_i D^h_\alpha u|)_h.$$

We estimate

$$|\nabla_h D^h_\alpha u| \leq h^{-1}|D^h_\alpha u| + h^{-1}\sum_{i=1}^n |E^h_i D^h_\alpha u|,$$

use Young's inequality to split the products $|D_\alpha^h u|\,|E_i^h D_\alpha^h u|$ and obtain (recall $\tau = 1$ on Q_R) $h^n \Sigma_x |\nabla_h D_\alpha{}^h u|^2 \le KR^{-1}h^{-1}h^n \Sigma_y |D_\alpha^h u|^2$, $x \in Q_R$, $y \in Q_{2R+h}$.

The inequality above holds for any multi-index α of order m, and consequently

$$h^n \Sigma_x |\nabla_h^{m+1} u|^2 \le KR^{-1}h^{-1}h^n \Sigma_y |\nabla_h^m u|^2, \quad x \in Q_R, \; y \in Q_{2R+h}$$

By recursion, we derive

(A 24) $$h^n \Sigma_x |\nabla_h^{n+1} u| 2 \le KR^{-n+1}h^{-n+1}h^n \Sigma_y |u|^2, \quad x \in Q_R, \; y \in Q_{MR},$$

with some constant M which does not depende on R.

Since $u \in L^\infty(\mathbb{R}_h^n)$ we have

$$h^n \Sigma_y |u|^2 \le KR^n, \quad y \in Q_{MR}$$

and the right hand side of (A 24) tends to zero as $R \to 0$. Therefore, $\nabla_h^{n+1} u = 0$, and we claim that, also in this case $u = \text{const.}$ In fact, if $\nabla_h^{n+1} u = 0$ then $\nabla_h^n u = 0$ then $\nabla_h^n u$ is constant and $\nabla_h^{n-1} u$ linear. Since $u \in L^\infty(\mathbb{R}_h^n)$, the linear grid-function $\nabla_h^{n-1} u$ must also be bounded and thus constant. Repeating this argument, we obtain successively $\nabla_h^{n-2} u = \text{const.}$, $\nabla_h^{n-3} u = \text{const.}$, until we arrive at the statement that $u = \text{const.}$ q.e.d.

With the above method one can easily derive discrete analogues of the theorems A 1 – A 4 and of theorem 1.4.

To our best knowledge it is an open problem to derive a discrete analogue of Bernstein's theorem for the difference approximation of the minimal surface equation:

$$\Sigma_i D_i^{-h} [D_i^h (1 + |\nabla_h u|^2)^{-1/2}] = 0, \quad i = 1, \ldots, n, \; n \le 7?$$

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