On commuting nilpotent matrices

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The purpose of this note is to show that for a certain class of nilpotent matrices commutativity implies similarity.

Let F be a field, n a positive integer, A a nilpotent $n \times n$-matrix and V the vector space of n-tuples over F. A operates on V in a natural way.

Let $V = \sum_{i=1}^{k} V_i$ be a decomposition of V, where the V_i are A-invariant and A-indecomposable for $i = 1, \ldots, k$, and $\dim V_i \geq \dim V_{i+1}$ for $i = 1, \ldots, k-1$.

The k-tuple of positive integers $(a_1, \ldots, a_k) := (\dim V_1, \ldots, \dim V_k)$ is called the Segre-type of A (see [3], p. 48). We will also say that the A-space V is of type $(a_1, \ldots, a_k)$.

Let W be an A-invariant subspace of V. A induces nilpotent linear transformations on W and V/W. It is known that we can assign to these linear transformations Segre-types $(b_1, \ldots, b_k)$ and $(c_1, \ldots, c_k)$ with $b_i \geq b_{i+1}$ and $c_i \geq c_{i+1}$ for $i = 1, \ldots, k-1$; the b_i and c_i are non-negative integers with the property $b_i \leq a_i$ and $c_i \leq a_i$ for all $i = 1, \ldots, k$. We will also say that the A-spaces W and V/W are of type $(b_1, \ldots, b_k)$ and $(c_1, \ldots, c_k)$ respectively.

The following lemma gives a criterion for W to be $C(A)$-invariant, where $C(A)$ is the ring of all $n \times n$-matrices which commute with A.

Lemma 1. *Under the above hypotethis, W is $C(A)$-invariant if and only if $a_i = b_i + c_i$ for $i = 1, \ldots, k$.*

Proof. We consider V as $F[x]$-module via $(\Sigma\, \xi_i x^i)v := v\, \Sigma\, \xi_i A^i$. The submodules of V, which are invariant under all endomorphisms of the $F[x]$--module V are precisely the subspaces of V which are invariant under $C(A)$. Therefore the lemma is a consequence of a theorem of Chatelet ([2], p. 168). (Chatelet's theorem is formulated for finite abelian groups, but his result holds for finitely generated torsion modules over principal ideal domains; see [1], ex. 16, p. 144).

We recall without proof two basic facts on nilpotent matrices.

Lemma 2. a) *Let W_1, W_2 be subspaces of V such that $W_1 \geq W_2$ and $W_1 A \leq \leq W_2$. If dim $W_1/W_2 = r$, then dim $E(A) \geq r$, where $E(A)$ is the eigenspace of A (to the eigenvalue 0). b) Let A, B be nilpotent matrices such that dim $E(A^i) = $ = dim $E(B^i)$ for $i = 1, 2, \ldots$, where $E(A^i)$ and $E(B^i)$ are the eigenspaces of $E(A^i)$ and $E(B^i)$ respectively (to the eigenvalue 0). Then A and B are similar.*

Definition. *Let A* be a nilpotent matrix of Segre-type $(a_1, \ldots, a_k)$. We call A *gapped* if and only if $a_i - a_{i+1} \geq 2$ for $i = 1, \ldots, k-1$.

Theorem. *Let A, B be gapped nilpotent matrices. If A and B commute, then A and B are similar.*

Proof. Let $V = \sum_{i=1}^{k} V_i$ be a decomposition of V into A-invariant and A-indecomposable subspaces with dim $V_i \geq$ dim V_{i+1} for $i = 1, \ldots, k-1$. $(a_1, \ldots, a_k) := (\text{dim } V_1, \ldots, \text{dim } V_k)$ is the Segre-type of A.

We define a series of subspaces of V by

$$V^j = \sum_{i=1}^{k} V_i A^{j-(i-1)} \text{ for } j = 0, 1, \ldots, a_1;$$

if $j - (i-1) \leq 0$, put $A^{j-(i-1)} := E$, E the unit matrix.

I) We show first that B operates trivially on the series V^j, i.e. $V^j B \leq V^{j+1}$ for $j = 0, 1, \ldots, a_1 - 1$.

The type of V/V^j as A-space is $(c_1, \ldots, c_k) = (\max\{\{\min j, a_1\}, 0\}, \max\{\{\min j-1, a_2\}, 0\}, \ldots, \max\{\{\min j-k+1, a_k\}, 0\})$. On the other hand we can easily check that $a_i - c_i \geq a_{i+1} - c_{i+1}$ for $i = 1, \ldots, k-1$, and subsequently the type of V^j as A-space is $(a_1 - c_1, \ldots, a_k - c_k)$. Therefore we can apply Lemma 1 and find that V^j is $C(A)$-invariant, in particular B-invariant.

Now define $V^j(i) := V_1 A^j + \ldots + V_i A^{j-(i-1)+1} + \ldots + V_k A^{j-(k-1)}$ for $i = 1, \ldots, k$. Clearly it is either $V^j(i) = V^j$ or dim $V^j/V^j(i) = 1$.

If dim $V^j/V^j(i) = 1$, the type of $V/V^j(i)$ as *A-space* is $(c_1, \ldots, c_i + 1, \ldots, c_k)$; on the other hand $a_i - c_i - 1 \geq a_{i+1} - c_{i+1}$ since A is gapped. Therefore the type of $V^j(i)$ as A-space is $(a_1 - c_1, \ldots, a_i - c_i - 1, \ldots, a_k - c_k)$. Consequently, we can apply Lemma 1 and find that $V^j(i)$ is $C(A)$-invariant, in particular B-invariant. Since B is nilpotent, and $V^j(i) = V^j$ or dim $V^j/V^j(i) = 1$, we get $V^j B \leq V^j(i)$ for $i = 1, \ldots, k$. This implies $V^j B \leq V^{j+1}$, since $V^j = \bigcap_{i=1}^{k} V^j(i)$.

II) We show now that dim $E(A^i) \leq$ dim $E(B^i)$ for $i = 1, 2, \ldots$; it is sufficient to show dim $E(A^i) \leq$ dim (B^i) for $i = 1, \ldots, a_k$. If $(e_1, \ldots, e_{a_n}) := (k \; a_k$-times, $k-1 \; (a_{k-1} - a_k)$-times, $\ldots, 1 \; (a_1 - a_2)$-times), then dim $E(A^i) = \sum_{\nu=1}^{i} e_\nu$. Consider $V^{e_i - 1}/V^{e_i + i}$; we calculate: dim $V^{e_i - 1}/V^{e_i + i} = \sum_{\nu=1}^{i} e_\nu$. Applying I) we see that $V^{e_i - 1} B^i \leq V^{e_i + i}$ Lemma 2 a) theorefore shows that dim $E(A^i) \leq$ dim $E(B^i)$ for $i = 1, \ldots, a_1$.

III) The assumptions on A and B being symmetrical, we can interchange the roles of A and B and obtain dim $E(A^i) =$ dim (B^i) for all i. The theorem now is proved by apolying Lemma 2 *b*).

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Reference

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