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Geroch's conjecture for a hypersurface of $\mathbb{R}^4$

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Introduction

A special case of a conjecture by Geroch [1] can be stated in the following way.

"Let M be an oriented hypersurface of $\mathbb{R}^4$ which is euclidean at infinity (i.e., outside a compact set the hypersurface is isometric to 3-dimensional euclidean space minus some compact set) and has non-negative scalar curvature. Is M globally euclidean?".

In this paper we prove that the conjecture is true when the hypersurface is the graph of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$.

We show that a hypersurface which is euclidean at infinity is cylindrical outside a compact set. The arguments used are the same as of Hartman & Nirenberg [2], with slight modifications.

We use a divergence formula for the scalar curvature presented by Reilly in [3], to reduce the hypothesis of non-negative to that of zero scalar curvature.

Finally, we apply a theorem we proved in [4] to prove the conjecture in the graph case.

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Scalar curvature

Let $N : M \to S^3$ be the *Gauss normal map* defined by a differentiable choice of a unitary vector field orthogonal to M.

After identification of the parallel spaces T_pM and $T_{N(p)}S^3$, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the symmetric operator $dN_{(p)}: T_pM \to T_pM$ are called the *principal curvatures* of M at p.

The *scalar curvature* S of M is given by

$$S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.$$

We observe that

$$2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = (\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

thus obtaining the invariant expression

$$2S = [tr\,(dN)]^2 - tr\,[(dN)^2]$$

On the condition at infinity

By definition, M is euclidean at infinity if there exists a compact set K of M and an isometry $\phi : \mathbb{R}^3 - \bar{B}_\rho \to M - K$, for some closed disk B_ρ of radius ρ.

Our goal is to show that either $M - K$ is contained in a hyperplane of $\mathbb{R}^4$ or it is part of a cylinder spanned by a planar curve. To get there, we have to introduce some definitions and a lemma from [3].

Definition 1. *Let D be a connected open set of $\mathbb{R}^3$. A mapping $p : D \to \mathbb{R}^3$ is a gradient mapping if dp is symmetric; $r(x)$ denotes the rank of $dp(x)$ and $r^*(x)$ is the largest integer s with the property that any neighborhood of x constains a point x^* with $r(x^*) = s$.*

Definition 2. A 2-dimensional section of an open set D of $\mathbb{R}^n$ through x is the connected component, containing x, of the intersection of D with a 2-dimensional affine subspace of $\mathbb{R}^n$.

Lemma 1. *Let $p : D \to \mathbb{R}^3$ be a gradient mapping with rank not greater than 1. If $r^*(x_0) = 1$, then p is constant on a 2-dimensional section β of D through x_0. Also, $x \in \beta$ implies that $r^*(x) = 1$, and $r(x) = 1$ or 0 according to whether $r(x_0) = 1$ or 0. The section β on which p is constant is uniquely determined, even locally.*

Proof. See lemma 2 and corollaries 1 and 2 of [3].

Hartman and Nirenberg proved in [3] that an isometric imbedding of $\mathbb{R}^{n+1}$ is cylindrical. In what follows, we modify some of their arguments to give a stronger version of that theorem, valid only for dimensions greater than 2.

Theorem 1. *Let $\phi : \mathbb{R}^3 - \bar{B}_\rho \to \mathbb{R}^4$ be an isometric imbedding. Up to a rotation of $\mathbb{R}^3 - \bar{B}_\rho$,*

$$\phi(x_1, x_2, x_3) = x_1 A + x_2 B + C(x_3),$$

where A and B are constant vectors and the set $\{A, B, C(x_3)\}$ is orthonormal.

Proof. For each $x \in \mathbb{R}^3 - \bar{B}_\rho$, there exists a connected neighborhood V_x of x in $\mathbb{R}^3 - \bar{B}_\rho$ such that $\phi(V_x)$ can be parametrized as the graph of a smooth function F. That is, there exists a diffeomorphism $z : V_x \to z(V_x) \subset \mathbb{R}^3$ such that $\phi\,|_{V_x} = (z, F(z))$, up to change of axis in $\mathbb{R}^4$. So, $N\,|_{V_x} = \dfrac{(\text{-grad } F(z),\ 1)}{(1 + \|\text{grad } F(z)\|^2)^{1/2}}$, which shows that rank dN = rank d(grad F), enabling us to extend the definitions of r and r^* to the normal map N.

Assuming that M-K does not lie on a hyperplane of $\mathbb{R}^4$, i.e., N is not constant, let $x_0 \in \mathbb{R}^3 - \bar{B}_\rho$ such that $r^*(x_0) = 1$. Applying lemma 1 to p = grad F, one sees that through $z(x_0)$ there passes a uniquely determined 2-dimensional section γ of $z(V_{x_0})$ on which grad F is constant and $r^* = 1$. Therefore, graph $F|_\gamma$ is contained in a uniquely determined 2-dimensional section $\tau \subset \phi(V_{x_0})$ on which the normal is constant. Since ϕ preserves geodesics, the pre-image of line segments of τ, being geodesics of $\mathbb{R}^3 - \bar{B}_\rho$, are line segments. Thus $\phi^{-1}(\tau)$ is part of a uniquely determined plane $\alpha(x_0)$ of $\mathbb{R}^3$ through x_0, on which N is constant and $r^* = 1$. Also $\phi|_{\phi^{-1}(\tau)}$ is linear, as an isometry which maps line segments into line segments.

Let $\beta(x_0) = (\mathbb{R}^3 - \bar{B}_\rho) \cap \alpha(x_0)$ and $A = \{x \in \beta(x_0) \mid N(x) = N(x_0)$ and $r^*(x) = 1\}$. Clearly, A is closed. Repeating the argument used for x_0, one sees that A is also open in $\beta(x_0)$. $\beta(x_0)$ being connected implies that N is constant and $r^* = $ on $\beta(x_0)$.

Furthermore, if $r^*(x_1) = 1$ and $x_1 \in \beta(x_0)$, then the sections $\beta(x_0)$ and $\beta(x_1)$ are parallel. Otherwise, $\beta(x_0) \cap \beta(x_1)$ would intersect $\mathbb{R}^3 - \bar{B}_\rho$ in a point x_2, contradicting the unique determination of $\beta(x_2)$. Thus, there exists a plane α through the origin in $\mathbb{R}^3$ such that $\beta(x) = (x + \alpha) \cap \mathbb{R}^3 - \bar{B}_\rho$.

Once the plane α is determined, repetition of previous arguments shows that for all points x with $r^*(x) = 0$, it still holds that N is constant, ϕ is linear and $r^* = 0$ on $\beta(x)$.

Assuming that α is orthogonal to the x_3-axis, we conclude that

$$\phi(x_1, x_2, x_3) = x_1 A(x_3) + x_2 B(x_3) + C(x_3).$$

Furthermore, the orthonormality of $\{\partial\phi/\partial x_1,\ \partial\phi/\partial x_2,\ \partial\phi/\partial x_3\}$ implies that A and B are constant.

Divergence formula for the scalar curvature

Definition 3. [3] The first Newton operator T on vector fields of a hypersurface M is defined by $T = [(tr\ dN) . I] - dN$, where I is the identity.

Lemma 2. [3] T satisfies the following properties:

a) $\operatorname{tr}(T \circ dN) = 2S$

b) $\operatorname{div} T = 0$, where div T is the 1-form whose value at a vector field X is the trace of the operator $(\nabla T)(X)$.

Proof. a) $tr(T_o dN) = tr[(tr\, dN) . dN] - tr[(dN)^2] =$
$= [tr(dN)]^2 - tr[(dN)^2] = 2S.$

b) $(\nabla T)(X) . Y = (\nabla_Y T)(X) = \nabla_Y(TX) - T(\nabla_Y X) =$
$= \nabla_Y[(tr\, dN) . X - (dN) . X] - (tr\, dN) . \nabla_Y X +$
$+ dN(\nabla_Y X) = [\nabla_Y(tr\, dN)] . X - [\nabla_Y(dN)] . X .$

Codazzi equation tells us that

$$[\nabla_Y(dN)] . X = [\nabla_X(dN)] . Y,$$

hence the trace of $[\nabla(dN)] . X$ is the trace of $\nabla_X(dN)$.

The trace of the map $Y \to [\nabla_Y(tr\, dN)] . X$ is given by its value on X, $\nabla_X[tr(dN)]$, since the image is spanned by X.

Therefore,

$$(\operatorname{div} T) . X = \nabla_X[tr(dN)] - tr[\nabla_X(dN)] = 0.$$

Remark The definition of div X as the trace of ∇X is equivalent to the usual definition of div $X = *(d * \tilde{X})$, where $\tilde{X}$ is the 1-form dual to X.

In what follows we extend to hypersurfaces a lemma and a theorem proved in [3] for graphs of functions.

Lemma 3. *Let A_0 be a constant vector of $\mathbb{R}^4$ and let A be the vector field on M defined as the orthogonal projection of A_0 onto M. Then the covariant derivative of A is a multiple of dN, i.e., $\nabla_X A = - < A_0, N > . dN(X)$.*

Proof. $A = A_0 - \langle A_0, N \rangle N$. For every vector field Y,

$$\langle \overline{\nabla}_X A, Y \rangle = \langle \overline{\nabla}_X (A_0 - \langle A_0, N \rangle N), Y \rangle =$$
$$= \langle \overline{\nabla}_X A_0, Y \rangle - \langle A_0, N \rangle \langle \overline{\nabla}_X N, Y \rangle = - \langle A_0, N \rangle \langle dN(X), Y \rangle,$$

showing that the tangential component of $\nabla_X A$ is $- \langle A_0, N \rangle\, dN(X)$. Consequently, $\nabla_X A = - \langle A_0, N \rangle\, dN(X)$.

Theorem 2. *The divergence formula for the scalar curvature of a hypersurface is given by*

$$div\,(T . A) = - 2 \langle A_0, N \rangle\, S.$$

Proof. By lemma 2-b,

$$0 = (div\, T) . A = tr(Y \to \nabla_Y(TA) - T(\nabla_Y A)).$$

Therefore, $div(TA) = tr(Y \to T(\nabla_Y A))$. By lemma 3,

$$\nabla_Y A = - \langle A_0, N \rangle . dN(Y),$$

hence $\operatorname{div}(TA) = - tr(To \langle A_0, N \rangle\, dN) = - \langle A_0, N \rangle\, tr(TodN) = - 2 \langle A_0, N \rangle\, S.$

The Conjecture in the Graph Case

Theorem 3. *Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and $M = graph\, f$. If M is euclidean at infinity and has non-negative scalar curvature, then M is euclidean everywhere.*

Proof. By theorem 1, there exists a compact set K of M such that $M - K$ is a subset of an orthogonal cylinder M_1. Let a subindex 1 refer to M_1, e.g., S_1 denotes the scalar curvature of M_1, etc.

Setting $A_0 = (0, 0, 0, 1) \in \mathbb{R}^4$, the divergence formula for the scalar curvature and Stokes' theorem imply that

$$\iint_K - 2S \langle A_0, N \rangle = \int_{\partial K} \langle TA, n \rangle = \int_{\partial K} \langle T_1 A_1, n_1 \rangle = \iint_K - 2S_1 \langle A_0, N_1 \rangle = 0,$$

where n stands for the outer normal to ∂K; the second equality holds because $M - K = M - K_1$, hence their boundaries coincide as well as $A = A_1$ and $T = T_1$ on ∂K; the fourth equality holds because the scalar curvature of a cylinder is zero.

Recalling that $N = (\text{-grad } f, 1)/(1 + \| \text{grad } f \|^2)^{1/2}$, we have that $\langle A_0, N \rangle = (1 + \| \text{grad} f \|^2)^{-1/2} > 0$. Therefore, $\iint_K - 2S \langle A_0, N \rangle = 0$ holds if and only if $S = 0$ everywhere.

Now we apply a theorem, proved in [4], which states that. "If a hypersurface M of $\mathbb{R}^4$, flat at infinity, has zero scalar curvature then its normal map is surjective, unless M is flat everywhere".

The normal vector at any point of graph f lies on the northern hemisphere of S^3; thus, N is never surjective.

We conclude that graph f is globally flat, hence isometric to 3-dimensional euclidean space.

References

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