Computation of Longitudinal Motions of a Viscoelastic Bar

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Abstract

A Gelerkin type method, based on trigonometric functions and Crank-Nicolson discretizations of the time variable, is applied to compute solutions of the initial boundary value problem associated with the equation

$$\rho_0 u_{tt} = a_1 u_{xx} + a_2 u_x^2 u_{xx} + a_3 u_{xtx} + f, \quad x \in [0, 1], \quad t > 0.$$

Error estimates are derived and a numerical example is presented.

1. Introduction

This paper is devoted to the problem of computation of the longitudinal motions of a bar with uniform cross-section and length L, presenting a viscoelastic behavior. Denoting by x the position of a cross-section (which is assumed to move as a vertical plane section) in the rest configuration of the bar, by $u(x, t)$ the displacement at time t of the section from its rest position, by $\tau(x, t)$ the stress on the section at time t, by $f(x, t)$ the given external field force at time t, and by $\rho_0 > 0$ the constant density, the equation of motion is

$$(1.1) \qquad \rho_0 u_{tt}(x, t) = \tau_x(x, t) + f(x, t), \quad x \in (0, L), \quad t > 0.$$

Assuming the ends of the bar to be clamped for all times and the initial state of motion specified by given functions $u_0(x)$ and $u_1(x)$, $x \in [0, L]$, we have the following boundary and initial conditions associated to (1.1):

$$(1.2) \qquad u(0, t) = u(L, t) \equiv 0, \quad t \in [0, \infty),$$

$$(1.3) \qquad u(x, 0) = u_0(x), \quad x \in [0, L],$$

$$(1.4) \qquad u_t(x, 0) = u_1(x), \quad x \in [0, L].$$

The medium is characterized by a stress-strain relation of the form

$$(1.5) \qquad \tau = a_1 u_x + \frac{a_2}{3}(u_x)^3 + a_3(u_x t_t,$$

with $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$, which describes, in the terminology of Duvaut-Lions[2], a material with "short memory".

We shall be concerned with approximations of the solution $u(x,t)$ of the mixed boundary-initial value problem represented by equations (1.1) – (1.5).

A theory of this type of equation was first developed by Greenberg, MacCamy and Mizel[3]. Under a smothness hypothesis on the initial data, precisely $u_0 \in C^4(0, L)$, and $u_1 \in C^2(0, L)$, and for $f \equiv 0$, they show the existence of a unique $u \in C^2((0, L) \times (0, \infty))$, such that $u_{txx} = u_{xtx} = u_{xxt}$ and u satisfies (1.1)-(1.5). Furthermore, for this "classical" solution, there exists a constant M which depends on

$$J = \sum_{i=0}^{2} \big(\max_{x\in[0,L]} |D^i u_0(x)| + \max_{x\in[0,L]} |D^i u_1(x)| \big),$$

and tends to zero as J goes to zero, such that

(1.6) $$|||u|||(t) = \sum_{i=0}^{2} \sum_{k=0}^{i} \max_{x\in[0,L]} \left| \frac{\partial^i u(x,t)}{\partial x^{i-k} \partial t^k} \right| \le M, \; t \in [0, \infty).$$

Moreover,

(1.7) $$\lim_{t\to\infty} |||u|||(t) = 0.$$

To prepare the ground for this work on approximations, in [4] the first author discussed "weak" solutions of (1.1)-(1.5) in $H^2((0,T) \times (0,L))$, for any $T > 0$. They were obtained as limit of semi-discretized Galerkin approximations.

Now, for the full numerical treatment of the problem, we use the trigonometric function space

$$\mathscr{H}_N = \left\{ \sum_{j=1}^{N} \alpha_j \sin \frac{j\pi x}{L} \,\middle|\, \alpha_j \in \mathbb{R} \right\},$$

and a discretization in the time variable defined by

$$t_n = n\Delta t, \;\; \Delta t = \frac{T}{M}, \;\; n = 0, 1, \ldots, M.$$

Here N and M are positive integers and T is a fixed time level. We shall consider approximations only on $[0, L] \times [0, T]$, specifically, at the levels $n\Delta t$.

When we have ny function S, defined at the times $n\Delta t$, $n = 0, 1, \ldots, M$, including those previously defined for all times, we denote by S_n the function at $t = n\Delta t$, and define

$$S_{n+\frac{1}{2}} = \tfrac{1}{2}(S_{n+1} + S_n),$$

$$S_{n,\theta} = \theta S_{n+1} + (1 - 2\theta) S_n + \theta S_{n-1}, \quad \theta \in [0, 1],$$

$$\partial_t S_{n+\frac{1}{2}} = \frac{S_{n+1} - S_n}{\Delta t},$$

$$\partial_t^2 S_n = \frac{S_{n+1} - 2S_n + S_{n-1}}{(\Delta t)^2},$$

$$\delta_t S_n = \frac{S_{n+1} - S_{n-1}}{2\Delta t} = \frac{\partial t S_{n+\frac{1}{2}} + \partial_t S_{n-\frac{1}{2}}}{2}.$$

If we write $\phi_j(x) = \sin j\pi x/L$, the approximations we shall propose to the $u(x, t_n)$ is a sequence of functions $U_n(x)$ characterized by the following Galerkin-like conditions:

(1.8) (i) $U_n \in \mathscr{H}_N$, $n = 0, 1, \ldots, M$;

(1.8) (ii) $\langle U_0, \phi_j \rangle = \langle u_0, \phi_j \rangle$, $j = 1, \ldots, N$;

(1.8) (iii) $\langle U_1, \phi_j \rangle = \langle F(.,\Delta t), \phi_j \rangle$, $j = 1, \ldots, N$,

$$F(x, \Delta t) = u_0(x) + \Delta t\, u_1(x) + \frac{(\Delta t)^2}{2\rho_0} \{[a_1 + a_2(Du_0(x))^2]\, D^2 u_0(x) + a_3 D^2 u_1(x) + f(x, 0)\};$$

(1.8) (iv) $$\rho_0 \langle \partial_t^2 U_n, \phi_j \rangle + a_1 \langle (U_{n,\frac{1}{4}})_x, (\phi_j)_x \rangle + \frac{a_2}{3} \langle [(U_n)_x]^3, (\phi_j)_x \rangle + a_3 \langle (\delta_t U_n)_x, (\phi_j)_x \rangle = \langle f_n, \phi_j \rangle, \; j = 1, \ldots, N, \; n = 1, 2, \ldots, M-1.$$

We remark that

$$\langle f_1, f_2 \rangle = \textstyle\int_0^L f_1(x) f_2(x)\, dx,$$

that $F(x, \Delta t)$ is the Taylor approximation to $u(x, \Delta t)$ with $u_{tt}(x, 0)$ evaluated in the differential equation (1.1), and that (iv) is a second order correct in Δt scheme for the canonical weak form of (1.1).

Taking into account the orthogonality properties of the family $\{\sin j\pi x/L, \cos j\pi x/L \,|\, j = 0, 1, 2, \ldots\}$ and introducting the representation

(1.9) $$U_n(x) = \sum_{j=1}^{N} C_j^n \phi_j(x), \; n = 0, 1, \ldots, M,$$

we derive from (1.8), by a straightforward calculation, the following equivalent relations:

(1.10) $$C_j^0 = \frac{2}{L} \langle u_0, \phi_j \rangle,$$

$$C_j^1 = \frac{2}{L}\langle F(.,\Delta t), \phi_j\rangle, \tag{1.11}$$

$$C_j^{n+1} = \left[\frac{\rho_0 L}{2} + j^2\left(\frac{a_3\pi^2\,\Delta t}{4L} + \frac{a_1\pi^2\,(\Delta t)^2}{8L}\right)\right]^{-1} \tag{1.12}$$
$$\cdot\left\{\left[-\frac{\rho_0 L}{2} + j^2\left(\frac{a_3\pi^2\,\Delta t}{4L} - \frac{a_1\pi^2\,(\Delta t)^2}{8L}\right)\right]C_j^{n-1} + \right.$$
$$+\left[\rho_0 L - \frac{a_1\pi^2\,(\Delta t)^2}{4L}j^2\right]C_j^n - \frac{a_2\pi^4\,(\Delta t)^2}{24L^3}\cdot j\,\cdot$$
$$\cdot\sum_{i,k,l=1}^{N} ikl\,[\delta_{|i-k|,\,|l-j|} + \delta_{|i-k|,|l+j|} + \delta_{|i+k|,\,|l-j|} +$$
$$\left. + \delta_{|i+k|,\,|l+j|}]\,C_i^n C_k^n C_l^n + \langle f_n, \phi_j\rangle\right\},$$

where $j = 1, 2, \ldots N$, $n = 1, 2, \ldots, M-1$, and, for $p, q = 0, 1, 2, \ldots,$

$$\delta_{p,q} = \begin{cases} 2 & \text{if} \quad p = q = 0 \\ 1 & \text{if} \quad p = q \neq 0 \\ 0 & \text{if} \quad p \neq q \end{cases}$$

Equations (1.9)-(1.12) give us an explicit algorithm to compute step by step the approximations at the various time levels.

The object of this paper is the analysis of algorithm (1.9)-(1.12). We shall prove a convergence result which indicates the asymptotic behavior of the error, namely

$$\sup_{0\le n\le M} \|u(.,t_n) - U_n\|_1 = 0\left[(\Delta t)^2 + \frac{1}{N}\right],$$
$$\Delta t \to 0,\; N \to \infty,$$

where we have to assume some minimum natural smoothness for u and $\|.\|_1$ is the norm of the Sobolev space $H^1(0, L)$.

The proof of this result will be presented in section 3. In section 2 we discuss two stability lemmas and a result from the theory of approximation of functions by trigonometric polynomials. In section 4 we present results of numerical experiments performed with this algorithm.

2. Stability and Approximation Lemmas

The functions considered here are real valued and measurable, and C will denote a generic constant. We adopt the usual notation

$$\langle f, g\rangle = \textstyle\int_0^L f(x)\,g(x)\,dx,$$
$$\langle f, g\rangle_m = \sum_{i=1}^{m} \langle D^i f, D^i g\rangle + \langle f, g\rangle,\; m \ge 1,$$
$$\|f\| = \sqrt{\langle f, f\rangle},$$
$$\|f\|_m = \sqrt{\langle f, f\rangle_m},$$
$$|f|_\infty = \sup_{x\in[0,L]} |f(x)|,$$

for functions f and g defined on $[0, L]$.

A priori estimates for equations (1.10)-(1.12) will be derived. For the analysis we shall need some results from the theory of Sobolev spaces which we state now. Proofs are given in [5].

Let

$$L^2 = \{u : [0, L] \to R \mid \|u\| < \infty\},$$
$$H^m = \{u \in L^2 \mid D^i u \in L^2,\; i = 1, \ldots, m\},$$
$$H^0 = L^2,$$
$$L^\infty(H^m) = \{u : [0, T] \to H^m \mid \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_m < \infty\}.$$

The space H^m is a Hilbert space with scalar product $\langle u, v\rangle_m$, and $L^\infty(H^m)$ is a Banach space with norm $\operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_m$. The following two propositions are true:

(i) If $u \in H^1$, there exists a constant C, independent of u, such that

$$|u|_\infty \le C\,\|u\|^{\frac{1}{2}}\,\|u\|_1^{\frac{1}{2}}; \tag{2.1}$$

(ii) Let $f \in C^k(R)$, $k \ge 1$, with $f(0) = 0$. If $u \in L^\infty(H^k)$ then $f(u) \in L^\infty(H^k)$, and

$$\|f(u(t))\|_1 \le M\,\|u(t)\|_1, \tag{2.2}$$

or

$$\|f(u(t))\|_k \le C_k(1 + \|u(t)\|_{k-1}^{k-1})\,\|u(t)\|_k, \tag{2.3}$$

if $k \ge 2$, where M and C_k are constants.

We are ready now to go through our basic lemmas.

Lemma 2.1. *Any possible solution of* (1.8) (*iv*) *satisfies*

$$\rho_0\,\|\vartheta_t U_{n+\frac{1}{2}}\|^2 + a_1\,\|(U_{n+\frac{1}{2}})_x\|^2 + a_3 \sum_{j=1}^{n} \Delta t\,\|(\delta_t U_j)_x\|^2 \le C, \tag{2.4}$$

for $n = 1, 2, \ldots, M-1$, *where* C *depends only on the data.*

Proof. We multiply equation (1.8) (iv) by $(C_j^{n+1} - C_j^{n-1})/2\,\Delta t$ and sum in j from 1 to N to get

$$\rho_0 \langle \partial_t^2 U_n, \delta_t U_n\rangle + a_i \langle (U_{n,\frac12})_x, (\delta_t U_n)_x\rangle + \frac{a_2}{3}\langle (U_n)_x^3, (\delta_t U_n)_x\rangle + a_3\|(\delta_t U_n)_x\|^2 = \langle f_n, \delta_t U_n\rangle.$$

Since

$$\delta_t U_n = \frac{\partial_t U_{n+\frac12} + \partial_t U_{n-\frac12}}{2},$$

$$\partial_t^2 U_n = \frac{\partial_t U_{n+\frac12} - \partial_t U_{n-\frac12}}{\Delta t},$$

we have from the above equation

(2.5) $$\frac{1}{2\Delta t}\{[\rho_0\|\partial_t U_{n+\frac12}\|^2 + \alpha_1\|(U_{n+\frac12})_x\|^2] - [\rho_0\|\partial_t U_{n-\frac12}\|^2 + \alpha_1\|(U_{n-\frac12})_x\|^2]\} + a_3\|(\delta_t U_n)_x\|^2 + \frac{a_2}{3}\langle (U_n)_x^3, (\delta_t U_n)_x\rangle = \langle f_n, \delta_t U_n\rangle.$$

By Cauchy-Schwarz and the arithmetic-geometric inequalities,

$$\sum_{j=1}^n \Delta t\,\langle f_j, \delta_t U_j\rangle \le \frac12\sum_{j=1}^n \Delta t\,\|f_j\|^2 + \frac12 \sum_{j=1}^n \Delta t\,\|\delta_t U_j\|^2 \le \frac12\sum_{j=1}^n\Delta t\,\|f_j\|^2 + \frac12\sum_{j=0}^n \Delta t\,\|\partial_t U_{j+\frac12}\|^2.$$

Hence, multiplying (2.5) by $2\Delta t$ and summing from 1 to n:

(2.6) $$\rho_0\|\partial_t U_{n+\frac12}\|^2 + a_1\|(U_{n+\frac12})_x\|^2 - \rho_0\|\partial_t U_{\frac12}\|^2 - a_1\|(U_{\frac12})_x\|^2 + 2a_3\sum_{j=1}^n\Delta t\,\|(\delta_t U_j)_x\|^2 + \frac23 a_2\sum_{j=1}^n \Delta t\int_0^1 (U_j)_x^3(\delta_t U_j)_x\,dx \le \sum_{j=1}^n\Delta t\,\|f_j\|^2 + \sum_{j=0}^n \Delta t\,\|\partial_t U_{j+\frac12}\|^2.$$

Defining the new variables

$$\lambda = \lambda(j) = (U_j)_x,$$

$$\Delta\lambda = \Delta\lambda(j) = (U_{j+1})_x - (U_{j-1})_x,$$

the sixth term of the left hand side of (2.6) can be written as

$$\frac{a_2}{3}\int_0^1\left(\sum_{\lambda=U_{1x}(x)}^{U_{nx}(x)}\lambda^3\,\Delta\lambda\right)dx = \frac{a_2}{3}\int_0^1\left(\sum_{\lambda=0}^{U_{nx}(x)}\lambda^3\,\Delta\lambda\right)dx - \frac{a_2}{3}\int_0^1\left(\sum_{\lambda=0}^{U_{1x}(x)}\lambda^3\,\Delta\lambda\right)dx,$$

with $\sum_{\lambda=0}^{(U_n)_x}\lambda^3\,\Delta\lambda\ge 0, \quad \forall x\in[0,1].$

Because of that and Gronwall's lemma, (2.6) becomes

$$\rho_0\|\partial_t U_{n+\frac12}\|^2 + a_1\|(U_{n+\frac12})_x\|^2 + 2a_3\sum_{j=1}^n\Delta t\,\|(\delta_t U_j)_x\|^2 \le C\left\{\rho_0\|\partial_t U_{\frac12}\|^2 + a_1\|(U_{\frac12})_x\|^2 + \sum_{j=1}^n\Delta t\,\|f_j\|^2 + \frac{a_2}{3}\int_0^1\left(\sum_{\lambda=0}^{(U_1)_x}\lambda^3\,\Delta\lambda\right)dx\right\},$$

which implies the estimation (2.4).

Lemma 2.2. *Any possible solution of* (1.8) (*iv*) *satisfies*

(2.7) $$\|(U_n)_{xx}\| \le C,$$

for $n = 0, 1, 2, \dots, M$, *where* C *depends only on the data.*

Proof: If we multiply equation (18) (iv) by $-\frac12\pi^2 j^3/L^2\,(C_j^{n+1} + C_j^{n-1})$ and sum in j from 1 to N, we get, in view of (1.9),

$$\rho_0\left\langle\partial_t^2 U_n, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle + a_1\left\langle (U_{n,\frac12})_x, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xxx}\right\rangle + \frac{a_2}{3}\left\langle [(U_n)_x]^3, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xxx}\right\rangle + a_3\left\langle(\delta_t U_n)_x, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xxx}\right\rangle = \left\langle f_n, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle.$$

Integrating by parts in x where appropriated

(2.8) $$\frac{a_3}{2\Delta t}\left\langle (U_{n+1}-U_{n-1})_{xx}, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle = \frac{\rho_0}{\Delta t}\left\langle \partial_t U_{n+\frac12} - \partial_t U_{n-\frac12}, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle - \left\langle f_n, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle - a_1\left\langle (U_{n,\frac12})_{xx}, \left(\frac{U_{n+1}+U_{n-1}}{2}\right)_{xx}\right\rangle$$

$$- a_2 \left\langle \frac{\partial}{\partial x} G(U_{nx}), \left(\frac{U_{n+1} + U_{n-1}}{2}\right)_{xx} \right\rangle,$$

where $G(S) = S^3/3$.

The last three terms of the right hand side of (2.8) can be estimated in the following way:

$$\left|\left\langle f_n, \left(\frac{U_{n+1} + U_{n-1}}{2}\right)\right\rangle\right| \le C\{\|f_n\|^2 + \|(U_{n+1})_{xx}\|^2 + \|(U_{n-1})_{xx}\|^2\};$$

$$\left|a_1\left\langle (U_{n,\frac{1}{2}})_{xx}, \left(\frac{U_{n+1} + U_{n-1}}{2}\right)_{xx}\right\rangle\right| \le C\{\|(U_n)_{xx}\|^2 + \|(U_{n+1})_{xx}\|^2 + \\ + \|(U_{n-1})_{xx}\|^2\};$$

$$\left|a_2\left\langle \frac{\partial}{\partial x} G(U_{nx}), \left(\frac{U_{n+1} + U_{n-1}}{2}\right)_{xx}\right\rangle\right| \le C\{\|(U_{n+1})_{xx}\|^2 + \\ + \|(U_{n-1})_{xx}\|^2 + \|\frac{\partial}{\partial x} G(U_{nx})\|^2\} \le C\{\|(U_n)_{xx}\|^2 + \|(U_{n+1})_{xx}\|^2 + \|(U_{n-1})_{xx}\|^2\},$$

by (2.1)-(2.2). Hence, collecting the above inequalities into (2.8), multiplying the resulting expression by $4\Delta t/a_3$ and summing from 1 to $m-1$, we get

$$(2.9) \qquad \|(U_m)_{xx}\|^2 \le \|(U_0)_{xx}\|^2 + \|(U_1)_{xx}\|^2 + \\ + \frac{4\rho_0}{a_3}\sum_{j=1}^{m-1}\left\langle \partial_t U_{j+\frac{1}{2}} - \partial_t U_{j-\frac{1}{2}}, \left(\frac{U_{j+1} + U_{j-1}}{2}\right)_{xx}\right\rangle \\ + C_1 \sum_{j=1}^{M} \Delta t \,\|f_j\|^2 + C_2 \sum_{j=0}^{m} \Delta t \,\|(U_j)_{xx}\|^2.$$

Now the third term in the right hand side of (2.9) can be handled by summation by parts in j, Cauchy-Schwarz estimation for the resulting boundary terms and integration by parts in x for the sum. The following inequality comes out

$$(2.10) \qquad \left|\frac{4\rho_0}{a_3}\sum_{j=1}^{m-1}\left\langle \partial_t U_{j+\frac{1}{2}} - \partial_t U_{j-\frac{1}{2}}, \left(\frac{U_{j+1} + U_{j-1}}{2}\right)_{xx}\right\rangle\right| \\ \le C\{\sup_{0\le j\ge m-1} \|\partial_t U_{j+\frac{1}{2}}\|^2 + \varepsilon \sup_{0\le j\le m}\|(U_j)_{xx}\|^2 + \\ + \sum_{j=0}^{m-1} \Delta t\,\|(\partial_t U_{j+\frac{1}{2}})_x\|^2\},$$

for any $\varepsilon < 0$ arbitrary.

Formulas (1.8) (ii)-(1.8) (iii) imply bounds for $\|(U_0)_{xx}\|$ and $\|(U_1)_{xx}\|$ in terms of the data so that if we choose ε in (2.10) conveniently, carry it over (2.9) and take lemma 2.1 into consideration we obtain

$$\|(U_m)_{xx}\| < C + C' \sum_{j=0}^{m} \Delta t\,\|(U_j)_{xx}\|^2,$$

where C and C' depend on the data. This inequality, by Gronwall's lemma, implies (2.7).

To conclude this section, we now present a result on the error of the best least squares approximation by elements of $\mathscr{H}_N$. For any function $g \in L^2$, let

$$w_2(g;\delta) \equiv \sup_{|\Delta x|\le\delta}\left[\frac{1}{L}\int_0^L |g(x+\Delta x) - g(x)|^2\,dx\right]^{\frac{1}{2}}$$

denote the *L^2-modulus of continuity*. We remark that $w_2(g;\delta)$ is a non-decreasing function of δ, and that $\lim_{\delta\to 0} w_2(g;\delta) = 0$ for any $g \in L^2$. Hence the following lemma is true.

Lemma 2.3. *Assume $u \in C^1(0, L)$, $u(0) = u(L) = 0$, and that Du is absolutely continuous with $D^2u \in L^2$. Then, for each positive integer N, there exists a trigonometric polynomial $u_N \in \mathscr{H}_N$, namely $u_N(x) = \sum_{j=1}^{N} \langle u, \phi_j\rangle\,\phi_j(x)$, such that*

$$(2.11) \qquad \|D^j(u - u_N)\| \le K\frac{w_2(D^2u;\,1/N)}{N^{2-j}}, \quad \textit{for all} \quad 0 \le j \le 2,$$

where K is a constant.

Proof. We first extend u as an odd function to $[-L, L]$, getting the conditions $D^ju(-L) = D^ju(L)$, $0 \le j \le 1$, and then apply theorem 5 from [1].

3. Convergence

In this section we shall go through the convergence analysis of our algorithm, establishing an uniform bound for the errors associated with the approximations at each time level. The following theorem summarizes the question.

Theorem 3.1. *Suppose the exact solution u of* (1.1) – (1.5) *is twice continuously differentiable in x and four times in t over* (0, L) × (0, T). *Then there exists a constant C, depending on the data and the derivatives* $\partial^{i+j}u/\partial x^i\,\partial t^j$, $i = 0, 1, 2$, $j = 0, 2, 3, 4$, such that,

$$\text{(3.1)}\qquad \sup_{0\le n\le M-1}\{\|\partial_t[u(.,t_{n+\frac{1}{2}})-U_{n+\frac{1}{2}}]\|^2+\|u(.,t_{n+\frac{1}{2}})-U_{n+\frac{1}{2}}\|_1^2\}^{\frac{1}{2}}+ \\ +\left\{\sum_{n=1}^{M-1}\Delta t\,\|[\delta_t(u(.,t_n)-U_n)]_x\|^2\right\}^{\frac{1}{2}}\le C[(\Delta t)^2+N^{-1}].$$

Proof. We write the equation for the exact solution at $t = t_n$ in the weak finite difference form

$$\text{(3.2)}\qquad \rho_0\left\langle\frac{u(.,t_{n+1})-2u(.,t_n)+u(.,t_{n-1})}{\Delta t^2},\phi\right\rangle+ \\ +a_1\left\langle\frac{1}{4}u_x(.,t_{n+1})+\frac{1}{2}u_x(.,t_n)+\frac{1}{4}u_x(.,t_{n-1}),\phi_x\right\rangle+ \\ +\frac{a_2}{3}\langle[u_x(.,t_n)]^3,\phi_x\rangle+ \\ +a_3\left\langle\frac{u_x(.,t_{n+1})-u_x(.,t_{n-1})}{2\Delta t},\phi_x\right\rangle= \\ =\langle f(.,t_n),\phi\rangle+\langle A_n(.,\Delta t),\phi\rangle,\quad \phi\in H_0^1,\ n=1,\dots,M-1,$$

with the initial conditions

$$\text{(3.3)}\qquad u(x,t_0)=u_0(x),$$

$$\text{(3.4)}\qquad u(x,t_1)=F(x,\Delta t)+B(x,\Delta t),$$

where

$$H_0^1=\text{closure of }C_0^\infty(0,L)\text{ in }H^1,$$

and $A_n(x,\Delta t)=0(\Delta t^2)$, each n, $B(x,\Delta t)=0(\Delta t^3)$, $\Delta t\to 0$, as L^2-valued mappings.

The corresponding equations for the approximations $U_n(x)$ are, from (1.8),

$$\text{(3.5)}\qquad \rho_0\langle\partial_t^2U_n,\psi\rangle+a_1\langle(U_{n,\frac{1}{4}})_x,\psi_x\rangle+ \\ +\frac{a_2}{3}\langle(U_n)_x^3,\psi_x\rangle+a_3\langle(\delta_tU_n)_x,\psi_x\rangle \\ =\langle f_n,\psi\rangle,\psi\in\mathscr{H}_N,\ n=1,2,\dots,M-1,$$

$$\text{(3.6)}\qquad \langle U_0,\psi\rangle=\langle u_0,\psi\rangle,\psi\in\mathscr{H}_N,$$

$$\text{(3.7)}\qquad \langle U_1,\psi\rangle=\langle F(.,\Delta t),\psi\rangle,\ \psi\in\mathscr{H}_N.$$

Hence, choosing in (3.2) $\phi=\psi\in\mathscr{H}_N\subset H_0^1$ and taking the difference between (3.2) and (3.5), (3.3) and (3.6), (3.4) and (3.7), respectively, we obtain a system of equations for the *error functions*

$$e_n(x)=u(x,t_n)-U_n(x),\ n=0,\dots,M.$$

Such are

$$\text{(3.8)}\qquad \rho_p\langle\partial_t^2e_n,\psi\rangle+a_1\langle(e_{n,\frac{1}{4}})_x,\psi_x\rangle+\frac{a_2}{3}\langle(u_x^3(.,t_n)-(U_n)_x^3),\psi_x\rangle+ \\ +a_3\langle(\delta_t\,e_n)_x,\psi_x\rangle=\langle A_n,\psi\rangle,\psi\in\mathscr{H}_N,$$

$$\text{(3.9)}\qquad \langle e_0,\psi\rangle=0,\ \psi\in\mathscr{H}_N,$$

$$\text{(3.10)}\qquad \langle e_1,\psi\rangle=\langle B,\psi\rangle,\ \psi\in\mathscr{H}_N.$$

As a matter of fact, since $\psi_{xx}\in\mathscr{H}_N$, integration by parts leads us to the conditions

$$\text{(3.9)}'\qquad \langle(e_0)_x,\psi_x\rangle=0,\ \psi\in\mathscr{H}_N,$$

$$\text{(3.10)}'\qquad \langle(e_1)_x,\psi_x\rangle=\langle B_x,\psi_x\rangle,\ \psi\in\mathscr{H}_N,$$

which are complementary to conditions (3.9) and (3.10) for the error analysis.

$$\langle e_{\frac{1}{2}},\psi\rangle=\frac{1}{2}\langle B,\psi\rangle_1,\ \psi\in\mathscr{H}_N,$$

so that taking

$$\psi(x)=\left[\frac{u(x,0)+u(x,t_1)}{2}-U_{\frac{1}{2}}(x)\right]+\left[\left(\frac{u_N(x,0)+u_N(x,t_1)}{2}\right)- \\ -\left(\frac{u(x,0)+u(x,t_1)}{2}\right)\right]=e_{\frac{1}{2}}(x)+\beta_{\frac{1}{2}}(x)\in\mathscr{H}_N,$$

and applying Cauchy-Schwarz inequality, we arrive at the estimate

$$\|e_{\frac{1}{2}}\|_1^2\le\varepsilon\,\|e_{\frac{1}{2}}\|_1^2+C(\varepsilon)\{\|\beta_{\frac{1}{2}}\|_1^2+\|B\|_1^2\},$$

with ε positive and arbitrary. Choose $\varepsilon = 1/2$ and recall lemma 2.3: there exists a constant C such that

$$\|e_{\frac{1}{2}}\|_1 \leq C\{\Delta t^3 + 1/N\}. \tag{3.11}$$

We shall need this relation later. Now we focus attention on equation (3.8): Taking $\psi = \delta_t e_n - \delta_t \beta_n = \delta_t u(., t_n) - \delta_t U_n - (\delta_t u(., t_n) - \delta_t u_N(., t_n)) \in \mathscr{H}_N$ we obtain

$$\begin{aligned}
&\frac{\rho_0}{2\Delta t}\{\|\partial_t e_{n+\frac{1}{2}}\|^2 - \|\partial_t e_{n-\frac{1}{2}}\|^2\} + \\
&+ \frac{a_1}{2\Delta t}\{\|(e_{n+\frac{1}{2}})_x\|^2 - \|(e_{n-\frac{1}{2}})_x\|^2\} - \\
&- \rho_0 \langle \partial_t^2 e_n, \delta_t \beta_n\rangle - a_1 \langle (e_{n,\frac{1}{4}})_x, (\delta_t \beta_n)_x\rangle + \\
&+ a_3 \|(\delta_t e_n)_x\|^2 - a_3 \langle (\delta_t e_n)_x, (\delta_t \beta_n)_x\rangle \\
&+ \frac{a_2}{3}\langle (e_n)_x [u_x^2(., t_n) + (U_n)_x^2 + u_x(., t_n)(U_n)_x], \alpha >= \\
&= \langle A_n, \delta_t e_n - \delta_t \beta_n\rangle,
\end{aligned} \tag{3.12}$$

where $\alpha = (\delta_t e_n)_x - (\delta_t \beta_n)_x$.

Lemma 2.2 together with formulas (2.1) and (1.6) imply the following estimation for the non-linear term

$$\begin{aligned}
&\left|\frac{a_2}{3}\langle (e_n)_x [u_x^2(., t_n) + (U_n)_x^2 + u_x(., t_n + (U_n)_x], \alpha\rangle\right| \\
&\quad \leq C \|(e_n)_x\| (\|(\delta_t e_n)_x\| + \|(\delta_t \beta_n)_x\|).
\end{aligned} \tag{3.13}$$

Hence, multiplying (3.12) by $2\Delta t$ and summing from 1 to m, $2 \leq m \leq M-1$, we reach our basic relation for the analysis, which is

$$\begin{aligned}
&\rho_0 \|\partial_t e_{m+\frac{1}{2}}\|^2 + a_1 \|(e_{m+\frac{1}{2}})_x\|^2 - \rho_0 \|\partial_t e_{\frac{1}{2}}\|^2 - a_1 \|(e_{\frac{1}{2}})_x\|^2 + \\
&\qquad + 2a_3 \sum_{j=1}^{m} \Delta t \|(\delta_t e_j)_x\|^2 \\
&\leq \varepsilon \sum_{j=1}^{m} \Delta t \|(\delta_t e_j)_x\|^2 + \\
&+ C(\varepsilon)\left[\sum_{j=1}^{m-1} \Delta t \|(e_{j+\frac{1}{2}})_x\|^2 + \sum_{j=1}^{m} \Delta t \|(\delta_t \beta_j)_x\|^2\right] + \\
&+ 2\rho_0 \sum_{j=1}^{m} \langle \partial_t^2 e_j, \delta_t \beta_j\rangle \Delta t + 2a_1 \sum_{j=1}^{m} \Delta t \langle (e_{j,\frac{1}{4}})_x, (\delta_t \beta_j)_x\rangle + \\
&+ 2a_3 \sum_{j=1}^{m} \Delta t \langle (\delta_t e_j)_x, (\delta_t \beta_j)_x\rangle + 2\sum_{j=1}^{m} \Delta t \langle A_j, \delta_t e_j - \delta_t \beta_j\rangle,
\end{aligned} \tag{3.14}$$

for any $\varepsilon > 0$.

The terms in the right hand side of (3.14) still remaining to be estimated can be bounded in the following way:

$$\begin{aligned}
&\left|2a_1 \sum_{j=1}^{m} \Delta t \langle (e_{j,\frac{1}{4}})_x, (\delta_t \beta_j)_x\rangle\right| \\
&\quad \leq C\left[\sum_{j=0}^{m} \Delta t \|(e_{j+\frac{1}{2}})_x\|^2 + \sum_{j=1}^{m} \Delta t \|(\delta_t \beta_j)_x\|^2\right];
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\left|2a_3 \sum_{j=1}^{m} \Delta t \langle (\delta_t e_j)_x, (\delta_t \beta_j)_x\right| \\
&\quad \leq \varepsilon \sum_{j=1}^{m} \Delta t \|(\delta_t e_j)_x\|^2 + C(\varepsilon) \sum_{j=1}^{m} \Delta t \|(\delta_t \beta_j)_x\|^2;
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\left|2 \sum_{j=1}^{m} \Delta t \langle A_j, \delta_t e_j - \delta_t \beta_j\rangle\right| \\
&\quad \leq C\left[(\Delta t)^4 + \sum_{j=0}^{m-1} \Delta t \|\partial_t e_{j+\frac{1}{2}}\|^2 + \sum_{j=1}^{m} \Delta t \|\delta_t \beta_j\|^2\right];
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\left|2\rho_0 \sum_{j=1}^{m} \langle \partial_t^2 e_j, \delta_t \beta_j\rangle \Delta t\right| = \\
&= \left|\rho_0 \sum_{j=1}^{m} \langle \partial_t e_{j+\frac{1}{2}} - \partial_t e_{j-\frac{1}{2}}, \partial_t \beta_{j+\frac{1}{2}} + \partial_t \beta_{j-\frac{1}{2}}\rangle\right| \\
&\leq \rho_0 \left|\sum_{j=1}^{m} \langle \partial_t e_{j+\frac{1}{2}} - \partial_t e_{j-\frac{1}{2}}, \partial_t \beta_{j+\frac{1}{2}}\rangle\right| + \\
&+ \rho_0 \left|\sum_{j=1}^{m} \langle \partial_t e_{j+\frac{1}{2}} - \partial_t e_{j-\frac{1}{2}}, \partial_t \beta_{j-\frac{1}{2}}\rangle\right| \\
&= \rho_0 \left|\sum_{j=1}^{m-1} \left\langle \partial_t e_{j+\frac{1}{2}}, \frac{\partial_t \beta_{j+\frac{3}{2}} - \partial_t \beta_{j+\frac{1}{2}}}{\Delta t}\right\rangle \Delta t + \right. \\
&\left. + \langle \partial_t e_{\frac{1}{2}}, \partial_t \beta_{\frac{3}{2}}\rangle - \langle \partial_t e_{m+\frac{1}{2}}, \partial_t \beta_{m+\frac{1}{2}}\rangle\right| + \\
&+ \rho_0 \left| - \sum_{j=2}^{m+1} \Delta t \left\langle \partial_t e_{j-\frac{1}{2}}, \frac{\partial_t \beta_{j-\frac{1}{2}} - \partial_t \beta_{j-\frac{3}{2}}}{\Delta t}\right\rangle + \right. \\
&\left. + \langle \delta_t e_{m+\frac{1}{2}}, \partial_t \beta_{m+\frac{1}{2}}\rangle - \langle \partial_t e_{\frac{1}{2}}, \partial_t \beta_{\frac{1}{2}}\rangle\right| \\
&\leq \varepsilon \|\partial_t e_{m+\frac{1}{2}}\|^2 + C(\varepsilon)\left[\sup_{1 \leq j \leq M-1} \|\delta_t \beta_j\|^2 + \right. \\
&\left. + \|\partial_t e_{\frac{1}{2}}\|^2 + \sum_{j=1}^{m-1} \Delta t \|\partial_t e_{j+\frac{1}{2}}\|^2 + \sum_{j=1}^{m} \Delta t \|\partial_t^2 \beta_j\|^2\right]
\end{aligned} \tag{3.18}$$

In the above inequalities $\varepsilon > 0$ is, as always, to be chosen conveniently. If we do this and collect (3.15)-(3.18) into (3.14) we get

$$\|\partial_t e_{m+\frac{1}{2}}\|^2 + \|(e_{m+\frac{1}{2}})_x\|^2 + \sum_{j=1}^{m} \Delta t \,\|(\delta_t e_j)_x\|^2 \le$$

$$\text{(3.19)} \qquad \langle C_1 \Big[\partial_t e_{\frac{1}{2}}\|^2 + \|(e_{\frac{1}{2}})_x\|^2 + \sup_{1\le j\le M-1} \|\delta_t \beta_j\|^2 +$$

$$+ \sum_{j=1}^{m} \Delta t \,\|\delta_t \beta_j\|_1^2 + \sum_{j=1}^{m-1} \Delta t \,\|\partial_t^2 \beta_j\|^2 + (\Delta t)^4\Big] +$$

$$+ C_2 \left[\sum_{j=0}^{m-1} \Delta t (\|\partial_t e_{j+\frac{1}{2}}\|^2 + \|(e_{j+\frac{1}{2}})_x\|^2) \right].$$

Applying to (3.19) the discrete version of Gronwall's lemma, we can conclude that

$$\text{(3.20)} \qquad \|\partial_t e_{m+\frac{1}{2}}\| + \|e_{m+\frac{1}{2}}\|_1 + \left[\sum_{j=1}^{m} \Delta t \,\|(\delta_t e_j)_x\|^2 \right]^{\frac{1}{2}}$$

$$\le C\{(\Delta t)^2 + \|\partial_t e_{\frac{1}{2}}\| + \|(e_{\frac{1}{2}})_x\| +$$

$$+ \sup_{1\le j\le M-1} \|\delta_t \beta_j\| + \left[\sum_{j=1}^{M-1} \Delta t \,\|\delta_t \beta_j\|_1^2 \right]^{\frac{1}{2}} + \left[\sum_{j=1}^{M-1} \Delta t \,\|\partial_t^2 \beta_j\|^2 \right]^{\frac{1}{2}} \Big\}.$$

By lemma 2.3.

$$\sup_{1\le j\le M-1} \|\delta_t \beta_j\| = 0\left(\frac{1}{N^2}\right), \quad \left[\sum_{j=1}^{M-1} \Delta t \,\|\partial_t^2 \beta_j\|^2 \right]^{\frac{1}{2}} = 0\left(\frac{1}{N^2}\right),$$

$$\left[\sum_{j=1}^{M-1} \Delta t \,\|\delta_t \beta_j\|_1^2 \right]^{\frac{1}{2}} = 0\left(\frac{1}{N}\right),$$

as $N \to \infty$. On the other hand, from (3.11),

$$\|\partial_t e_{\frac{1}{2}}\| = 0\left(\frac{1}{N^2} + \Delta t^3\right)$$

$$\|(e_{\frac{1}{2}})_x\| = 0\left(\frac{1}{N} + \Delta t^3\right),$$

as $\Delta t \to 0$, $N \to \infty$.

Hence (3.20) implies (3.1), and our theorem is proved.

4. Numerical Results

We performed numerical experiments with algorithm (1.10)-(1.12), taking several different relations between inertial, elastic and dissipation coefficients. In general we took $N = 15$, $\rho_0 = 1$, $f = 0$, $L = 1$ and $\Delta t = 40^{-1}$ sec.

The first computation was done with $u_0(x) = 2/5\, x(1 - x)$, $u_1(x) \equiv 0$, $M = 480$, $a_1 = a_2 = a_3 = 1$. The result was a rapid decay to zero in $T_A = 120\,\Delta t$ seconds. Fig. 1 shows the motion of the mid-point of the bar: it just goes back to the rest configuration. In Figs. 2, 3 and 4 we have the same situation with diminishing viscosity coefficients $a_3 = 0.2$, 0.1, 0.05, respectively. We can see that oscilations do appear, by virtue of the increase in the elastic force, with a damping directly proportional to a_3. The decay to zero times are $T_A = 138\,\Delta t$, $T_A = 254\,\Delta t$ and $T_A = 422\,\Delta t$, respectively.

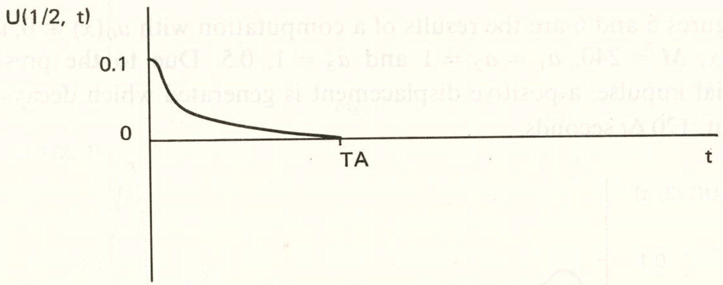

Fig. 1

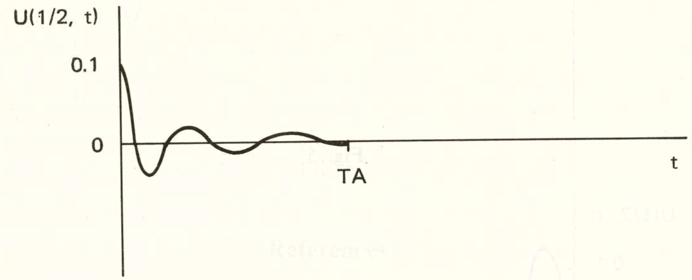

Fig. 2

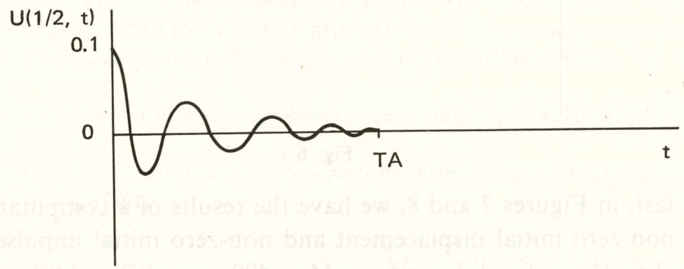

Fig. 3

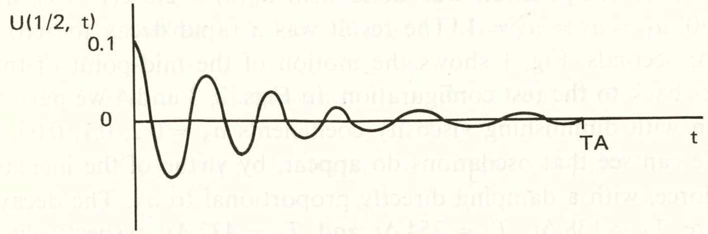

Fig. 4

Figures 5 and 6 are the results of a computation with $u_0(x) \equiv 0$, $u_1(x) = \sin \pi x$, $M = 240$, $a_1 = a_2 = 1$ and $a_3 = 1$, 0.5. Due to the presence of the initial impulse, a positive displacement is generated which decays to rest in about $120\, \Delta t$ seconds.

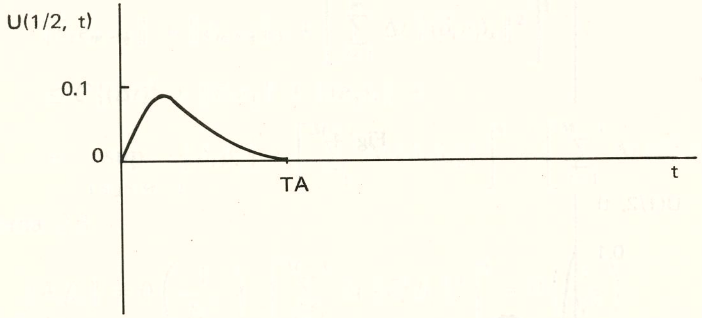

Fig. 5

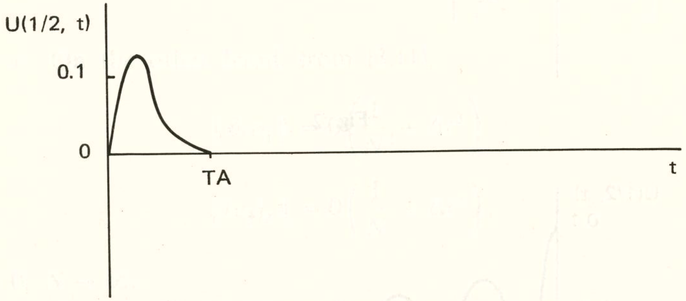

Fig. 6

At last, in Figures 7 and 8, we have the results of a computation for the case of non-zero initial displacement and non-zero initial impulse. We took $u_0(x) = 2/5\, x(1-x)$, $u_1(x) = \sin \pi x$, $M = 400$, $a_3 = 0.2$ and 0.1, respectively. Damped oscilations occur with decay to rest times $T_A = 162\, \Delta t$ and $T_A = 357\, \Delta t$.

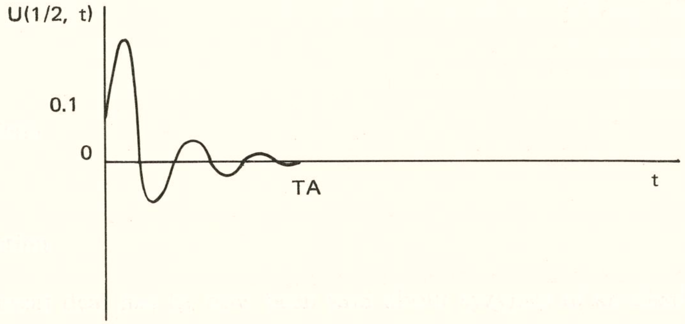

Fig. 7

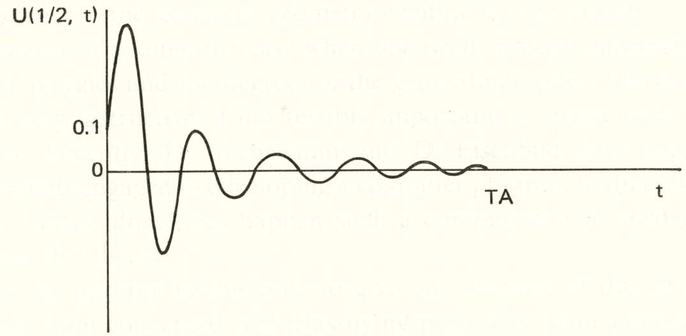

Fig. 8

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