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A General Theory of Algebras of Polynomials

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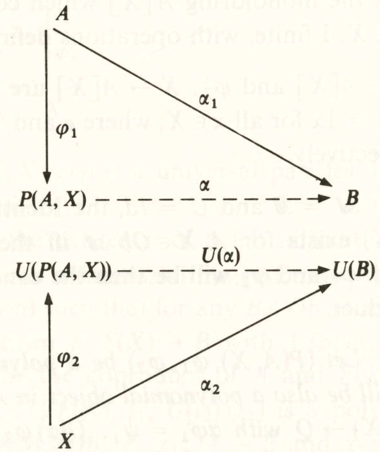
1. Basic Properties of Polynomial Objects and PF-Objects

The purpose of this paper is to give an axiomatical description of the theory of polynomial algebras by categorical means. For notational conventions and elementary results used without further reference we refer to [6] and [7]. The starting point will be the following.

Definition. Let $\mathscr{A}$, $\mathscr{B}$ be categories, $U\colon \mathscr{A} \to \mathscr{B}$ a functor, $A \in Ob\,\mathscr{A}$ and $X \in Ob\,\mathscr{B}$. $(P(A, X), \varphi_1, \varphi_2)$, where $P(A, X) \in Ob\,\mathscr{A}$ and $\varphi_1 : A \to P(A, X)$ and $\varphi_2 : X \to U(P(A, X))$ are morphisms of $\mathscr{A}$ resp. $\mathscr{B}$, will be called a polynomial object in A and X iff for any $B \in Ob\,\mathscr{A}$ and any morphisms $\alpha_1 : A \to B$ and $\alpha_2 : X \to U(B)$ there exists a unique $\alpha\colon P(A, X) \to B$ such that $\alpha_1 = \alpha\varphi_1$ and $\alpha_2 = U(\alpha)\varphi_2$, i.e. the following diagrams are commutative:

(1)



This uniquely determined morphism α will be denotee by $\langle \alpha_1, \alpha_2 \rangle$. By abuse of language and notation we shall call sometimes also $P(A, X)$ alone a polynomial object in A and X.

Examples. 1) Let $\mathscr{A}$ be a variety viewed as a category by defining the morphisms to be the homomorphisms and $\mathscr{B}$ the category $\mathscr{S}$ of sets. Furthermore let $U: \mathscr{A} \to \mathscr{B}$ be the Forgetful Functor which assigns to each algebra A its underlying set $U(A)$. As it is shown in [6] the polynomial object $(P(A, X), \varphi_1, \varphi_2)$ exists for any algebra A in $\mathscr{A}$ and any set X and is essentially unique (see Prop. 1 below). $P(A, X)$ can be obtained by factorizing the wordalgebra $W(A \cup X)$ over $A \cup X$ by the congruence θ which is, roughly spoken, generated by the laws of the variety $\mathscr{A}$ and the identities in A, while φ_1 and φ_2 are the restrictions of the natural mapping v: $W(A \cup X) \to W(A \cup X)/\theta$ to A and X, respectively. It is easy to see that φ_1 and φ_2 are always injective, except the trivial case that $|A| = 1$ and no algebra in the variety $\mathscr{A}$ with more than one element contains a one-element subalgebra.

2) Let $\mathscr{B}$ be the category of all partial algebras of some fixed type and $\mathscr{A}$ a full subcategory consisting only of full algebras with inclusion functor $U: \mathscr{A} \to \mathscr{B}$. We shall give conditions for the existence of polynomial objects later on. Note that the example above can be regarded as a special case of this by taking for X the partial algebra with operations defined nowhere.

3) Let $\mathscr{A}$ be the variety of commutative rings with identity, $\mathscr{B}$ the variety of commutative and U: $\mathscr{A} \to \mathscr{B}$ the Forgetful Functor which "forgets" the additive structure of a ring. Then $(P(A, X), \varphi_1, \varphi_2)$ exists for any A and X, and $P(A, X)$ is just the monoidring $A[X]$ which consists of the formal sums $\sum_{i \in I} a_i x_i$, $a_i \in A$, $x_i \in X$, I finite, with operations defined in the usual way. Furthermore, $\varphi_1: A \to A[X]$ and $\varphi_2: X \to A[X]$ are defined by $\varphi_1(a) = ae$ for all $a \in A$ and $\varphi_2(x) = 1x$ for all $x \in X$, where e and 1 are the identity elements of X and A, respectively.

4) In case that $\mathscr{A} = \mathscr{B}$ and $U = Id$, the Identityfunctor of $\mathscr{A}$, it is easy to see that $P(A, X)$ exists for $A, X \in Ob\, \mathscr{A}$ iff the coproduct $A \amalg X$ exists and both are equal. φ_1 and φ_2 will be then the canonical injections of A and X into the coproduct.

Proposition 1. *Let $(P(A, X), \varphi_1, \varphi_2)$ be a polynomial object in A and X. Then (Q, ψ_1, ψ_2) will be also a polynomial object in A and X iff there is an isomorphism α: $P(A, X) \to Q$ with $\alpha\varphi_1 = \psi_1$, $U(\alpha)\varphi_2 = \psi_2$.*

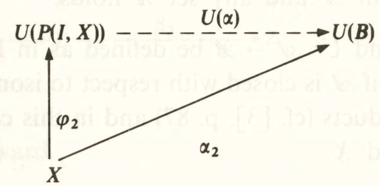
Proof. Suppose (Q, ψ_1, ψ_2) is also a polynomial object in A and X. By way of definition there exist morphisms α: $P(A, X) \to Q$ and β: $Q \to P(A, X)$ such that $\alpha\varphi_1 = \psi_1$, $U(\alpha)\varphi_2 = \psi_2$ and $\beta\psi_1 = \varphi_1$, $U(\beta)\psi_2 = \varphi_2$ which implies $\beta\alpha\varphi_1 = \varphi_1$, $U(\beta\alpha)\varphi_2 = \varphi_2$ and $\alpha\beta\psi_1 = \psi_1$, $U(\alpha\beta)\psi_2 = \psi_2$. By the uniqueness part of the definition of a polynomial object we conclude that $\beta\alpha = 1_{P(A,X)}$ and $\alpha\beta = 1_Q$, i.e. α: $P(A, X) \to Q$ is an isomorphism with $\alpha\varphi_1 = \psi_1$ and $U(\alpha)\varphi_2 = \psi_2$.

Conversely, assume this to be true. Then for any $B \in Ob\, \mathscr{A}$ and any morphisms γ_1: $A \to B$ and γ_2: $X \to U(B)$ there exists a morphism γ: $P(A, X) \to B$ with $\gamma\varphi_1 = \gamma_1$ and $U(\gamma)\varphi_2 = \gamma_2$ which implies $\gamma\alpha^{-1}\psi_1 = \gamma_1$ and $U(\gamma\alpha^{-1})\psi_2 = \gamma_2$. If there is another morphism δ: $Q \to B$ with $\delta\psi_1 = \gamma_1$ and $U(\delta)\psi_2 = \gamma_2$ then we have $\delta\alpha\varphi_1 = \gamma_1$ and $U(\delta\alpha)\varphi_2 = \gamma_2$ which implies $\delta\alpha = \gamma$ and therefore $\delta = \gamma\alpha^{-1}$.

Theorem 2. *Let $\mathscr{A}$ be a category with finite coproducts. Then $(P(A, X), \varphi_1, \varphi_2)$ exists for all $A \in Ob\, \mathscr{A}$ and $X \in Ob\, \mathscr{B}$ iff U has a left adjoint.*

Proof. Suppose $(P(A, X), \varphi_1\ \varphi_2)$ exists for all $A \in Ob\, \mathscr{A}$ and $X \in Ob\, \mathscr{B}$. Since $\mathscr{A}$ is a category with finite coproducts, it has an initial object I. If we substitute A by I in the diagramm (1) defining the universal property of a polynomial object then the upper triangle will be commutative for each choice of α. Thus for any $B \in Ob\, \mathscr{A}$ and any morphism $\alpha_2: X \to U(B)$ there is a unique morphism α: $P(I, X) \to B$ such that the following diagramm is commutative

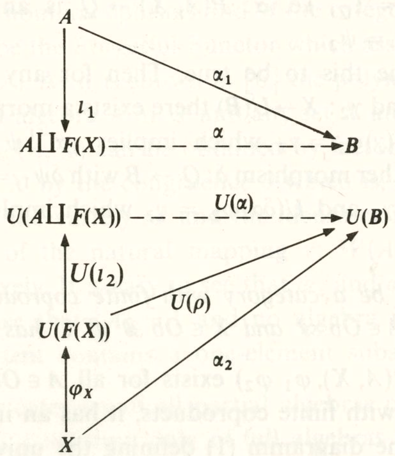
(2)



But this means that $(P(I, X), \varphi_2)$ is a universal pair for U which implies the existence of a left adjoint of U.

Conversely assume the existence of a left adjoint of U which will be denoted by F. Then there is a functorial morphism $\varphi = (\varphi_X)_{X \in Ob\, \mathscr{B}} : 1_{\mathscr{B}} \to UF$ (the front of the adjunction) such that for any $B \in Ob\, \mathscr{A}$ and any α: $X \to U(B)$ there is a unique morphism ρ: $F(X) \to B$ with $U(\rho)\varphi_X = \alpha$. For arbitrary $A \in Ob\, \mathscr{A}$ let $A \amalg F(X)$ be the coproduct of A and $F(X)$ with injections ι_1 and ι_2. We assert that $(A \amalg F(X),\ \iota_1,\ U(\iota_2)\varphi_X)$ is a polynomial object in A and X. To prove this let $B \in Ob\, \mathscr{A}$, α_1: $A \to B$ and α_2: $X \to U(B)$ be arbitrary. If we choose ρ to be the unique morphism with $U(\rho)\varphi_X = \alpha_2$ and α to be the unique morphism with $\alpha\iota_1 = \alpha_1$ and $\alpha\iota_2 = \rho$ then one easily checks that α is the unique morphism with $\alpha\iota_1 = \alpha_1$ and $U(\alpha)\, U(\iota_2)\varphi_X = \alpha_2$, i.e. $(A \amalg F(X), \iota_1, U(\iota_2)\varphi_X)$ has the universal property of a polynomial object in A and X.

(3)



Examples. 1) Let $\mathscr{A}$ be a variety, $\mathscr{B}$ the category of sets and $U: \mathscr{A} \to \mathscr{B}$ the corresponding Forgetful Functor. It is wellknown that U has a left adjoint, namely the Free Object Functor F, and accordingly $P(A, X) \cong A \amalg F(X)$ for any algebra A in $\mathscr{A}$ and any set X holds.

2) Let $\mathscr{A}$, $\mathscr{B}$ and $U: \mathscr{A} \to \mathscr{B}$ be defined as in Example 2) above. Then U has a left adjoint if $\mathscr{A}$ is closed with respect to isomorphic images, subalgebras and direct products (cf. [3], p. 87) and in this case $(P(A, X), \varphi_1, \varphi_2)$ will exist for any A and X.

3) Let $\mathscr{A}$, $\mathscr{B}$ and $U: \mathscr{A} \to \mathscr{B}$ be defined as in Example 3) above. Then U has a left adjoint, namely the functor $F: \mathscr{B} \to \mathscr{A}$ which assigns to each monoid H the monoidring $Z[H]$ and to each morphism $\alpha: H \to H'$ its unique extension $\alpha: Z[H] \to Z[H']$ to a unitary ring-homomorphism. We conclude from Theorem 2 that $A[X] \cong A \amalg Z[X]$.

4) Let $\mathscr{A} = \mathscr{B}$ and U the Identity Functor on $\mathscr{A}$. Clearly U has a left adjoint, namely U itself, and Theorem 2 yields the trivial result $P(A, X) \cong$ $\cong A \amalg X$.

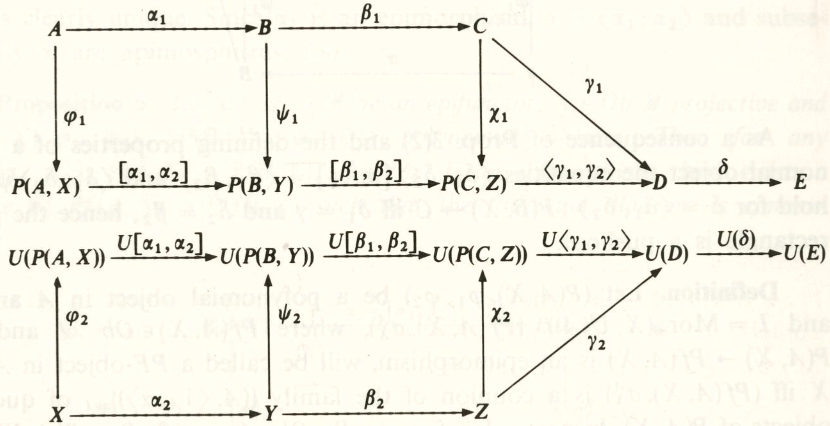
Definition. Let $(P(A, X), \varphi_1, \varphi_2)$, $(P(B, Y), \psi_1, \psi_2)$ be polynomial objects and the morphisms $\alpha_1: A \to B$ and $\alpha_2: X \to Y$ arbitrary. Then the unique morphism $\alpha: P(A, X) \to P(B, Y)$ with $\alpha\varphi_1 = \psi_1\alpha_1$ and $U(\alpha)\varphi_2 = \psi_2\alpha_2$ will be denoted by $[\alpha_1, \alpha_2]$.

For purely technical reasons we state the following.

Proposition 3. *Let $(P(A, X), \varphi_1, \varphi_2)$ $(P(B, Y), \psi_1, \psi_2)$, $(P(C, Z), \chi_1, \chi_2)$ be polynomial objects and the morphisms $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta$ in the diagram (4) below arbitrary. Then the following equalities hold*

(1) $[\beta_1, \beta_2][\alpha_1, \alpha_2] = [\beta_1\alpha_1, \beta_2\alpha_2]$
(2) $\langle\gamma_1, \gamma_2\rangle[\beta_1, \beta_2] = \langle\gamma_1\beta_1, \gamma_2\beta_2\rangle$
(3) $\delta\langle\gamma_1, \gamma_2\rangle = \langle\delta\gamma_1, U(\delta)\gamma_2\rangle$

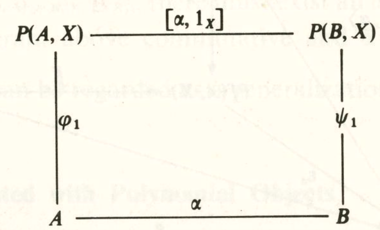
(4)



Proof. Straightforward.

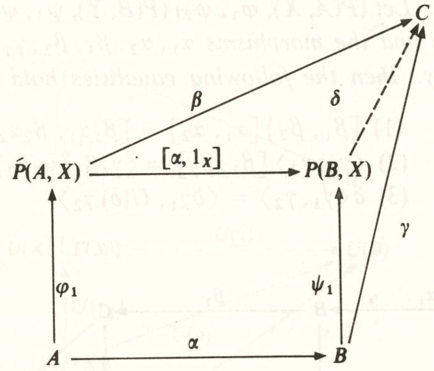
Proposition 4. *Let $(P(A, X), \varphi_1, \varphi_2)$, $(P(B, X), \psi_1, \psi_2)$ be polynomial objects and $\alpha: A \to B$ an arbitrary morphism. Then the diagram (5) below is a pushout*

(5)



Proof. It is clear by definition of $[\alpha, 1_X]$ that (5) is commutative. Suppose there is a $C \in Ob\,\mathscr{A}$ and morphisms $\beta = \langle\beta_1, \beta_2\rangle: P(A, X) \to C$ and $\gamma: B \to C$ such that $\beta\varphi_1 = \gamma\alpha$.

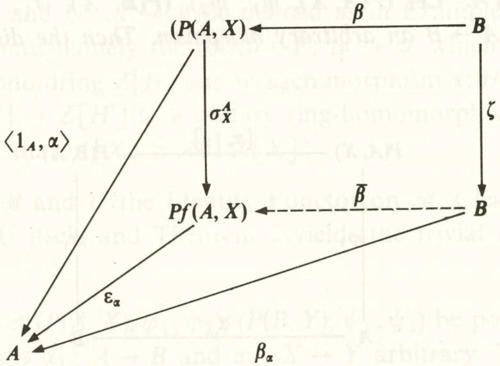
(6)



As a consequence of Prop. 3(2) and the defining properties of a polynomial object the equalities $\langle\delta_1, \delta_2\rangle[\alpha, 1_X] = \langle\beta_1, \beta_2\rangle$ and $\langle\delta_1, \delta_2\rangle\psi_1 = \gamma$ hold for $\delta = \langle\delta_1, \delta_2\rangle : P(B, X) \to C$ iff $\delta_1 = \gamma$ and $\delta_2 = \beta_2$, hence the given rectangle is a pushout.

Definition. Let $(P(A, X), \varphi_1, \varphi_2)$ be a polynomial object in A and X and $I = \mathrm{Mor}_{\mathscr{B}}(X, U(A))$. $(Pf(A, X), \sigma_X^A)$, where $Pf(A, X) \in Ob\ \mathscr{A}$ and σ_X^A: $P(A, X) \to Pf(A, X)$ is an epimorphism, will be called a PF-object in A and X iff $(Pf(A, X), \sigma_X^A)$ is a counion of the family $((A, \langle 1_A, \alpha\rangle))_{\alpha\in I}$ of quotient objects of $P(A, X)$. In particular, for any $B \in Ob\ \mathscr{A}$, any $\beta : B \to P(A, X)$, any epimorphism $\zeta : B \to \overline{B}$ and any family $(\beta_\alpha)_{\alpha\in I}$ with $\beta_\alpha\zeta = \langle 1_A, \alpha\rangle\beta$ for all $\alpha \in I$ there exists a (necessarily unique) $\overline{\beta} : \overline{B} \to Pf(A, X)$ with $\overline{\beta}\zeta = \sigma_X^A\beta$ (see diagram (7)). By abuse of language we shall call sometimes also $Pf(A, X)$ a PF-object.

(7)



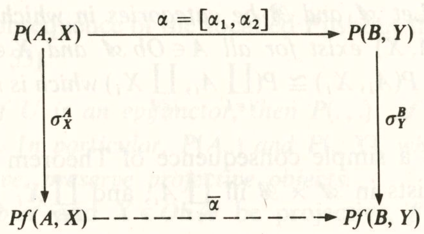
Example. Let $\mathscr{A}$ be a variety, $\mathscr{B}$ the category of sets, $U: \mathscr{A} \to \mathscr{B}$ the corresponding Forgetful Functor and X a set of k elements. Then $Pf(A, X)$ is just the algebra $P_k(A)$ of k-ary polynomial functions as defined in [6].

Proposition 5. *Let $U: \mathscr{A} \to \mathscr{B}$ be an epifunctor, $X \in Ob\ \mathscr{B}$ projective and $(P(A, X), \varphi_1, \varphi_2)$ a polynomial object in A and X. Then for any $B \in Ob\ \mathscr{A}$ and any $\alpha = \langle\alpha_1, \alpha_2\rangle : P(A, X) \to B$, where α_1 is an epimorphism, there exists a unique epimorphism α': $Pf(A, X) \to B$ such that $\alpha'\sigma_X^A = \alpha$.*

Proof. Since $U(\alpha_1)$ is an epimorphism and X is projective, there exists an $\widetilde{\alpha}_2 : X \to U(A)$ such that $\alpha_2 = U(\alpha_1)\widetilde{\alpha}_2$. Using Prop. 3(3) and the defining property of a PF-object we get $\alpha = \langle\alpha_1, \alpha_2\rangle = \alpha_1\langle 1_A, \widetilde{\alpha}_2\rangle = \alpha_1\overline{\alpha_2}\sigma_X^A$ for some $\overline{\alpha_2} : Pf(A, X) \to A$. Thus $\alpha' = \alpha_1\overline{\alpha_2}$ satisfies the condition claimed above and is clearly unique. Since α_1 is an epimorphism, $\alpha = \langle\alpha_1, \alpha_2\rangle$ and subsequently α' are apimosphisms, too.

Proposition 6. *Let $U: \mathscr{A} \to \mathscr{B}$ be an epifunctor, $X \in Ob\ \mathscr{B}$ projective and $(P(A, X), \varphi_1, \varphi_2)$, $(P(B, Y), \psi_1, \psi_2)$ polynomial objects. Then for any $\alpha = [\alpha_1, \alpha_2]$: $P(A, X) \to P(B, Y)$, where α_1 is an epimorphism, there exists a unique $\overline{\alpha}$: $Pf(A, X) \to Pf(B, Y)$ such that the following diagram is commutative*

(8)



Proof. By Prop. (3.2) and Prop. 5 $\langle 1_{\mathscr{B}}, \beta\rangle[\alpha_1, \beta\alpha_2]$ can be factored through σ_X^A for all $\beta \in \mathrm{Mor}_{\mathscr{B}}(Y, U(B))$. Since $(Pf(B, Y), \sigma_Y^B)$ is a counion of the family $((B, \langle 1_B, \beta\rangle)_{\beta\in Mor_{\mathscr{B}}(Y, U(B))}$, there must exist an α: $Pf(A, X) \to Pf(B, Y)$ which makes the diagram above commutative and $\overline{\alpha}$ is clearly unique.

Remark. Prop. 6 can be regarded as a generalization of Prop. 3.31 in [6].

2. Some Functors related with Polynomial Objects

Since a polynomial object has been defined via a universal property, a pair of adjoint functors arises from it in a natural way. These are the functors $P: \mathscr{A} \times \mathscr{B} \to \mathscr{A}$ and $Q: \mathscr{A} \to \mathscr{A} \times \mathscr{B}$, where P assigns to each pair $(A, X) \in Ob\ \mathscr{A} \times \mathscr{B}$ the polynomial object $P(A, X)$ and to each pair $(\alpha_1, \alpha_2) \in$

$\in \mathrm{Mor}_{\mathscr{A}}(A, B) \times \mathrm{Mor}_{\mathscr{B}}(X, Y)$ the morphism $\alpha = [\alpha_1, \alpha_2]: P(A, X) \to P(B, Y)$ and Q is defined by $Q(A) = (A, U(A))$ for all $A \in Ob\,\mathscr{A}$ and $Q(\alpha) = (\alpha, U(\alpha))$ for all morphisms α in $\mathscr{A}$.

Of course, we are mainly interested in the functor P and its properties. Since it is a bifunctor, we get two related functors $P(A,.)$ and $P(., X)$ by fixing the first resp. second variable. If there is a need to stress the dependence of P on two variables we shall write $P(.,.)$ for P.

Proposition 7. *If $\mathscr{A}$ and $\mathscr{B}$ are additive categories and $U: \mathscr{A} \to \mathscr{B}$ is and additive functor, then $P(.,.)$ is also an additive functor.*

Proof. Straightforward.

Theorem 8. *$P(.,.)$ preserves colimits. In particular, P preserves coproducts, pushouts, coequalizers, cokernels and epimorphisms.*

Proof. This is an immediate consequence of the fact that P has a right adjoint, namely Q (cf. [7], p. 67).

Corollary 9. *Let $\mathscr{A}$ and $\mathscr{B}$ be categories in which coproducts and polynomial objects $P(A, X)$ exist for all $A \in Ob\,\mathscr{A}$ and $X \in Ob\,\mathscr{B}$. Then there is an isomorphism $\coprod_{i\in I} P(A_i, X_i) \cong P(\coprod_{i\in I} A_i, \coprod_{i\in I} X_i)$ which is natural in all variables.*

Proof. This is a simple consequence of Theorem 8 by the observation that $\coprod_{iI\in} (A_i, X_i)$ exists in $\mathscr{A} \times \mathscr{B}$ iff $\coprod_{i\in I} A_i$, and $\coprod_{i\in I} X_i$ exists in $\mathscr{A}$ and $\mathscr{B}$, respectively.

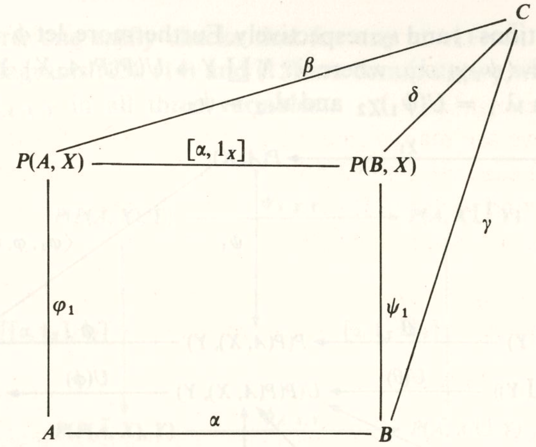
Corollary 10 *$P(A,.)$ and $P(., X)$ preserve pushouts, coequalizers and epimorphisms for any $A \in Ob\,\mathscr{A}$ and $X \in Ob\,\mathscr{B}$.*

Proof. Let $I_A: \mathscr{B} \to \mathscr{A} \times \mathscr{B}$ be the functor defined by $I_A(X) = (A, X)$ for all $X \in Ob\,\mathscr{B}$ and $I_A(\alpha) = (1_A, \alpha)$ for all morphisms α in $\mathscr{B}$. It is easy to see that I_A preserves pushouts, coequalizers and epimorphisms. Hence, applying Theorem 8, $P(A,.) = P(.,.)\,I_A$ also does. The proof for $P(.,X)$ is similar.

Proposition 11. *Let $(P(A, X), \varphi_1, \varphi_2)$, $(P(B, X), \psi_1, \psi_2)$ be polynomial objects and $\alpha: A \to B$ a monomorphism. Then $P(\alpha, X) = [\alpha, 1_X]: P(A, X) \to$ $\to P(B, X)$ is a monomorphism iff there is a $C \in Ob\,\mathscr{A}$, a monomorphism $\beta: P(A, X) \to C$ and a $\gamma: B \to C$ such that $\beta\varphi_1 = \gamma\alpha$.*

Proof. As a consequence of Prop. 4 there must be a $\delta: P(B, X) \to C$ which makes the diagram (9) commutative.

(9)



Since $\beta = \delta[\alpha_X]$ is a monomorphism, $[\alpha, 1_x]$ must be a monomorphism, too.

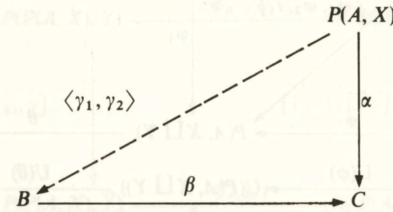
Theorem 12. *If $\mathscr{A}$ is an abelian category, then $P(., X)$ is a monofunctor for all $X \in Ob\,\mathscr{B}$.*

Proof. This is a consequence of the socalled Pushout theorem for abelian categories (cf. [3], p. 53).

Proposition 13. *If U is an epifunctor, then $P(.,.)$: $\mathscr{A} \times \mathscr{B} \to \mathscr{A}$ preserves projectives objects. In particular, $P(A,.)$ and $P(., X)$, where $A \in Ob\,\mathscr{A}$ and $X \in Ob\,\mathscr{B}$ are projective, preserve projective objects.*

Proof. Let $A \in Ob\,\mathscr{A}$ and $X \in Ob\,\mathscr{B}$ be projective, $\beta: B \to C$ and epimorphism and $\alpha = \langle \alpha_1, \alpha_2 \rangle: P(A, X) \to C$ an arbitrary morphism. Since A and X are projective and $U(\beta)$ is an epimorphism, there exist morphisms $\gamma_1: A \to B$ and $\gamma_2: X \to U(B)$ such that $\beta\gamma_1 = \alpha_1$ and $U(\beta)\,\gamma_2 = \alpha_2$. An application of Prop. 2(3) yields $\beta\langle \gamma_1, \gamma_2 \rangle = \langle \alpha_1, \alpha_2 \rangle$ (see diagram (10)).

(10)

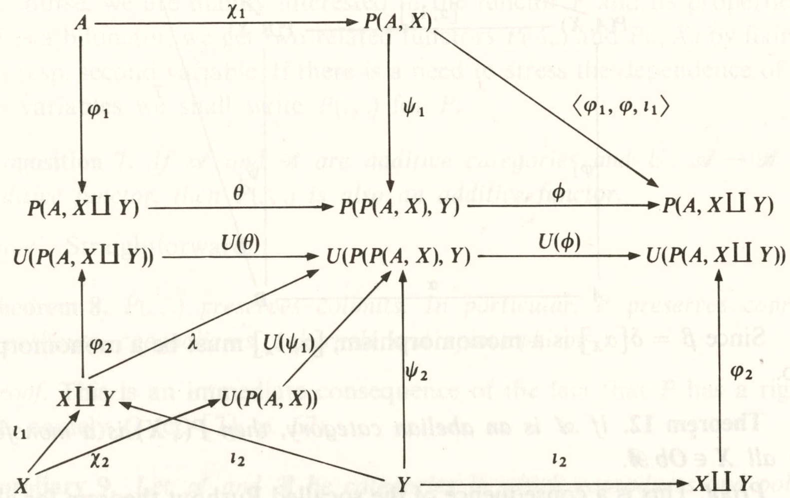


Proposition 14. *Suppose $P(P(A, X), Y)$ and $P(A, X \coprod Y)$ both exists. Then there is an isomorphism $\phi = \phi_{A,X,Y}: P(P(A, X), Y) \to P(A, X \coprod Y)$ which is natural in all variables.*

Proof. Let $(P(A, X \coprod Y), \varphi_1, \varphi_2)$, $(P(P(A, X), Y), \psi_1, \psi_2)$, $(P(A, X), \chi_1, \chi_2)$ be the corresponding polynomial objects and $X \coprod Y$ the coproduct of X

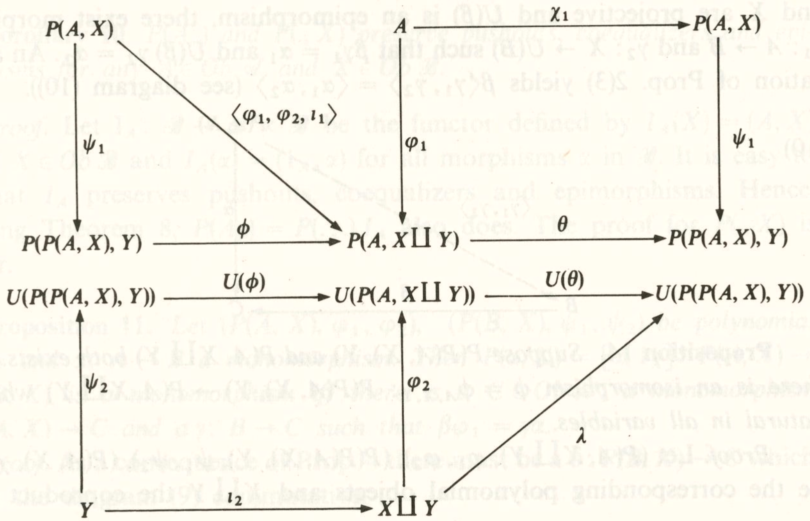
and Y with injections ι_1 and ι_2, respectively. Furthermore, let $\phi = \langle\langle\varphi_1, \varphi_2, \iota_1\rangle, \varphi_2, \iota_2\rangle$ and $\theta = \langle\psi_1\chi_1, \lambda\rangle$, where $\lambda: X\coprod Y \to U(P(P(A, X), Y))$ is the unique morphism with $\lambda\iota_1 = U(\psi_1)\chi_2$ and $\lambda\iota_2 = \psi_2$.

(11)



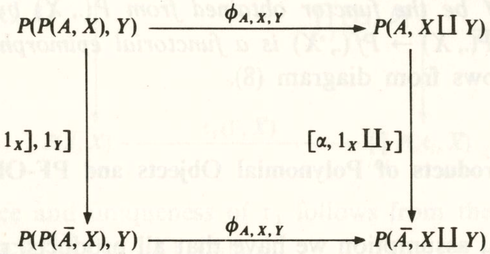
From diagram (11) we conclude that $\phi\theta\varphi_1 = \phi\psi_1\chi_1 = \langle\varphi_1, \varphi_2, \iota_1\rangle\chi_1 = \varphi_1$, $U(\phi\theta)\,\varphi\iota_1\iota_2 = U(\phi)\,U(\theta)\,\varphi\iota_2\iota_1 = U(\phi)\,\lambda\iota_1 = U(\phi)\,U(\psi_1)\,\chi_2 = U(\phi\psi_1)\,\chi_2 = U(\langle\varphi_1, \varphi\iota_2\iota_1\rangle)\,\chi_2 = \varphi\iota_2\iota_1$ and $U(\phi\theta)\,\varphi\iota_2\iota_2 = U(\phi)\,U(\theta)\,\varphi\iota_2\iota_2 = U(\phi)\,\lambda\iota_2 = U(\phi)\,\psi_2 = \varphi\iota_2\iota_2$. By the uniqueness part of the definitions of a polynomial object and of a coproduct we get $\phi\theta = 1_{P(A, X\coprod Y)}$. In a similar way one proves $\theta\phi = 1_{P(P(A,X),Y)}$ using diagram (12).

(12)

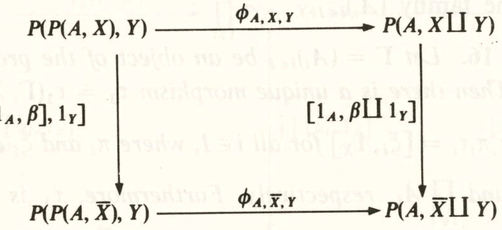


Furthermore, one easily checks that for any $\alpha: A \to \bar{A}$, $\beta: X \to \bar{X}$ and $\gamma: Y \to \bar{Y}$ the diagrams (13), (14) and (15) are commutative which proves the naturality of $\phi_{A,X,Y}$ in all three variables.

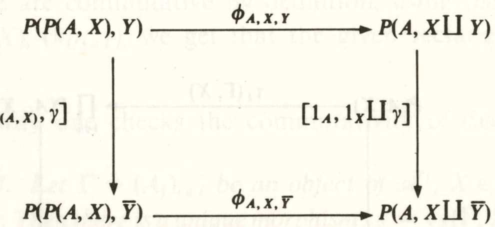
(13)



(14)



(15)



Let $\bar{\mathscr{A}}$ be the subcategory of $\mathscr{A}$, where $Ob\,\bar{\mathscr{A}} = Ob\,\mathscr{A}$ and the morphisms of $\bar{\mathscr{A}}$ are the epimorphisms of $\mathscr{A}$. If U is an epifunctor and $X \in Ob\,\mathscr{B}$ projective, then Prop. 6 can be used to define a functor $Pf(.,X): \bar{\mathscr{A}} \to \mathscr{A}$, which assigns to each $A \in Ob\,\bar{\mathscr{A}}$ the PF-object $Pf(A, X)$ and to each $\alpha\colon A \to B$ the unique morphism $\alpha\colon Pf(A, X) \to Pf(B, X)$ with $\bar{\alpha}\sigma_X^A = \sigma_X^B[\alpha, 1_X]$.

For the sake of reference we state.

Proposition 15. *Let U be an epifunctor, $X \in Ob\,\mathscr{B}$ be projective and $\bar{P}(.,X)\colon \bar{\mathscr{A}} \to \mathscr{A}$ be the functor obtained from $P(.,X)$ by restriction. Then $\sigma = (\sigma_X^A)_{A \in ob\mathscr{A}}\colon \bar{P}(.,X) \to Pf(.,X)$ is a functorial epimorphism.*

Proof. Follows from diagram (8).

3. On Direct Products of Polynomial Objects and PF-Objects

As a general assumption we have that all products, polynomial objects and *Pf*-objects occurring in this section exist. A product of a family of objects $(A_i)_{i \in I}$ will be denoted by $(\prod_{i \in I} A_i, (\xi_i)_{i \in I})$, where the ξ_i are the projections of the product. By abuse of language we shall call sometimes also $\prod_{i \in I'} A_i$ alone a product of the family $(A_i)_{i \in I}$.

Proposition 16. *Let $\Gamma = (A_i)_{i \in I}$ be an object of the product-category $\mathscr{A}^I$ and $X \in Ob\,\mathscr{B}$. Then there is a unique morphism $\tau_1 = \tau_1(\Gamma, X)\colon P(\prod_{i \in I} A_i, X) \to \prod_{i \in I} P(A_i, X)$ with $\pi_i\tau_i = [\xi_i, 1_X]$ for all $i \in I$, where π_i and ξ_i are the projections of $\prod_{i \in I} P(A_i, X)$ and $\prod_{i \in I} A_i$, respectively. Furthermore, τ_1 is natural in Γ and X, i.e. the following diagrams are commutative for each choice of $\bar{\Gamma} = (\bar{A}_{i \in I}$, $\alpha = (\alpha_i)_{i \in I}{}^J\colon \Gamma \to \bar{\Gamma}$ and $\beta\colon X \to \bar{X}$ (for notational convenience we have set $A = \prod_{i \in I} A_i$ and $\bar{A} = \prod_{i \in I} \bar{A}_i$).*

(16)

$$\begin{array}{ccc} P(A, X) & \xrightarrow{\tau_1(\Gamma, X)} & \prod_{i \in I} P(A_i, X) \\ \Big\downarrow{\scriptstyle [\prod_{i \in I} \alpha_i, 1_X]} & & \Big\downarrow{\scriptstyle \prod_{i \in I} [\alpha_i, 1_X]} \\ P(\bar{A}, X) & \xrightarrow{\tau_1(\bar{\Gamma}, X)} & \prod_{i \in I} P(\bar{A}_i, X) \end{array}$$

(17)

$$\begin{array}{ccc} P(A, X) & \xrightarrow{\tau_1(\Gamma, X)} & \prod_{i \in I} P(A_i, X) \\ \Big\downarrow{\scriptstyle [1_A, \beta]} & & \Big\downarrow{\scriptstyle \prod_{i \in I} [1_{A_i}, \beta]} \\ P(A, \bar{X}) & \xrightarrow{\tau_1(\Gamma, \bar{X})} & \prod_{i \in I} P(A_i, \bar{X}) \end{array}$$

Proof. Existence and uniqueness of τ_1 follows from the universal property of the product $(\prod_{i \in I} P(A_i, X), (\pi_i)_{i \in I})$.

To prove the commutativity of the diagram (16), we consider the following diagram

(18)

$$\begin{array}{ccccc} P(A, X) & \xrightarrow{\tau_1(\Gamma, X)} & \prod_{i \in I} P(A_i, X) & \xrightarrow{\pi_i} & P(A_i, X) \\ \Big\downarrow{\scriptstyle [\prod_{i \in I} \alpha_i, 1_X]} & & \Big\downarrow{\scriptstyle \prod_{i \in I} [\alpha_i, 1_X]} & & \Big\downarrow{\scriptstyle [\alpha_i, 1_X]} \\ P(\bar{A}, X) & \xrightarrow{\tau_1(\bar{\Gamma}, X)} & \prod_{i \in I} P(\bar{A}_i, X) & \xrightarrow{\bar{\pi}_i} & P(\bar{A}_i, X) \end{array}$$

where $\bar{\pi}_i$ are the projections of $\prod_{i \in I} P(\bar{A}_i, X)$. Since the right rectangle and the outer rectangle are commutative by definition, using the universal property of $(\prod_{i \in I} P(\bar{A}_i, X), (\bar{\pi}_i)_{i \in I})$, we get that the given rectangle is also commutative.

In a similar way one checks the commutativity of diagram (17).

Proposition 17. *Let $\Gamma = (A_i)_{i \in I}$ be an object of $\bar{\mathscr{A}}^I$, $X \in Ob\,\mathscr{B}$ projective and U an epifunctor. Then there is a unique morphism $\tau_2 = \tau_2(\Gamma, X)\colon Pf(\prod_{i \in I} A_i, X) \to \prod_{i \in I} Pf(A_i, X)$ with $\pi_i\tau_2 = Pf(\xi_i, X)$ for all $i \in I$, where π_i and ξ_i are defined as in Prop. 16. Furthermore, τ_2 is natural in Γ and X.*

Proof. The proof is essentially the same as the proof of Prop. 16 since only functorial properties of $P(.,X)$ had been used. Note that Γ has to be regarded as an object of $\mathscr{A}^I$, that is the naturality w.r.t. Γ holds only for families of epimorphisms.

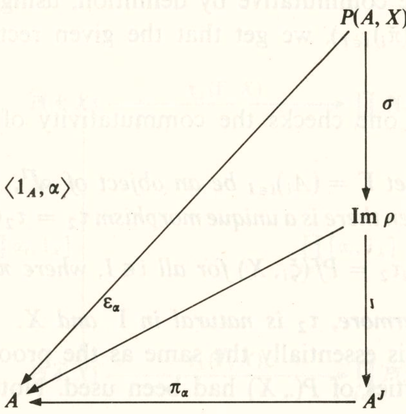
Example. We list some consequences of Prop. 16 and Prop. 17 for the classical case, where $\mathscr{A}$ is a variety, $\mathscr{B}$ the category of sets and $U: \mathscr{A} \to \mathscr{B}$ the corresponding Forgetful Functor. For an application of Prop. 17, U must be an epifunctor, which is valid iff the epimorphisms of the variety $\mathscr{A}$ are just the surjective homomorphisms.

Suppose $\tau_1: P(\prod_{i\in I} A_i, X) \to \prod_{i\in I} P(A_i, X)$ is an epimorphism. Then using the naturality of τ_1 with respect to X, we get that τ_1 remains an epimorphism, when X is replaced by a subset $\overline{X}$ (since there is an epimorphism $\beta: X \to \overline{X}$ and $\prod_{i\in I} [1_{A_i}, \beta]$ is an epimorphism). If τ_1 is a monomorphism and $P(.,X)$ preserves monomorphisms, then we conclude from Proposition 16 that τ_1 reamins a monomorphism, when A_i is replaced by a subalgebra $\overline{A}_i$ for all $i \in I$. For τ_2 the first assertion is also valid, whereas the second becomes trivial by the fact that τ_2 is always a monomorphism (cf. [6], ch. 3, Prop. 3.53).

Proposition 18. *Let $\mathscr{A}$ be a complete abelian category, $(P(A,X), \varphi_1, \varphi_2)$ a polynomial object in A and X and $J = Mor_{\mathscr{B}}(X, U(A))$. Then the PF-object $(Pf(A,X), \sigma_X^A)$ exists and the family $(\varepsilon_\alpha)_{\alpha\in J}$ (see diagram (7)) has the following property: If for any morphisms γ, δ in $\mathscr{A}$ the equalities $\varepsilon_\alpha\gamma = \varepsilon_\alpha\delta$ hold for all $\alpha \in J$, then $\gamma = \delta$.*

Proof. Let $\rho: P(A,X) \to A^J$ be the unique morphism with $\pi_\alpha{}^\rho = \langle 1_A, \alpha\rangle$ for all $\alpha \in J$, where the π_α are the projections of A^J, and $\rho = \iota\sigma$ the factorization of ρ by its image. Then it is easy to see that $(\operatorname{Im}\rho, \sigma)$ is a PF-object in A and X (see diagram (19)).

(19)

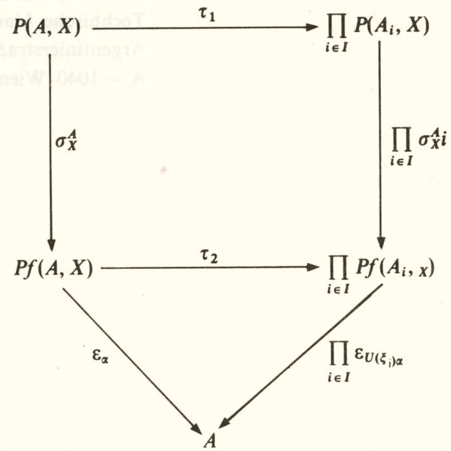


Since $\varepsilon_\alpha = \pi_\alpha^2$ for all $\alpha \in J$ and ι is a monomorphism, the stated property of $(\varepsilon_\alpha)_{\alpha\in J}$ is an easy consequence of the corresponding property of $(\pi_\alpha)_{\alpha\in J}$.

Proposition 19. *Let $\mathscr{A}$ be a complete abelian category,* U: $\mathscr{A} \to \mathscr{B}$ *an epifunctor and $X \in Ob\,\mathscr{B}$ projective. Then for any family $(A_i)_{i\in I}$ of objects in $\mathscr{A}$ $\tau_2: Pf(\prod_{i\in I} A_i, X) \to \prod_{i\in I} Pf(A_i, X)$ is a monomorphism.*

Proof. Let $(A, (\xi_i)_{i\in I})$ denote the product of the family $(A_i)_{i\in I}$. We assert the commutativity of the following diagram (20)

(20)



The proof of this assertion is done by a simple diagram chasing using the universal property of $(A, (\xi_i)_{i\in I})$ and the fact that σ_X^A is an epimorphism.

Suppose now $\tau_2\gamma = \tau_2\delta$ for some morphisms γ and δ. This implies $\varepsilon_\alpha\gamma = \varepsilon_\alpha\delta$ for all $\alpha \in \operatorname{Mor}_{\mathscr{B}}(X, U(A))$ by the commutativity of diagram (20) and subsequently $\gamma = \delta$ by Prop. 18.

Remark. Clearly, the hypotheses of Prop. 19 can be weakened in may ways, but it seems to be rather difficult to find necessary and sufficient conditions for τ_2 to be a monomorphism.

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