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Functional diferential inequalities

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1. Introduction.

Viswanathan [1] showed that certain integral inequalities for ordinary differential equations are generalisations of Gronwall's Lemma. The object of this paper is to prove a theorem on functional differential inequalities that contain in particular the results given in [1].

2. Preliminaries.

Let $\tau > 0$ be a real number, and $C([a, b], R)$ the Banach space of continuous functions mapping the interval $[a, b]$ into R with the topology of uniform convergence. When $[a, b] = [-\tau, 0]$, we let $C = C([-\tau, 0], R)$, and denote the norme of φ in C by $|\varphi|_0 = \sup_{-\tau < \theta < 0} |\varphi(\theta)|$. If x is a function belonging to $C([a - \tau, b], R)$ for any $a \leq b$, then for each fixed $t \in [a, b]$, the symbol x_t will denote an element of the space C defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$. For each element $\varphi \in C$ we define the euclidean norma of φ as follows: $|\varphi|(\theta) = |\varphi(\theta)|$, $-\tau < \theta < 0$. Let $\rho > 0$ be a given constant, and let $C_\rho = \{\varphi \in C_+ : |\varphi|_0 < \rho\}$, where $C_+ = C([-\tau, 0], R_+)$. We consider the scalar functional differential equations

$$\dot{x} = w(t, x_t) \tag{1}$$

where $w \in C(J \times C, R)$.

Definition 1. Let $w \in C(J \times C, R)$. We say that $w(t, \varphi)$ is nondecreasing in φ for each fixed t, $t \in J$, if given $\varphi_1, \varphi_2 \in C$ with $|\varphi_1| < |\varphi_2|$ we have: $w(t, \varphi_1) < w(t, \varphi_2)$.

Definition 2. Let $r(t, t_0, \varphi_0)$ be a solution of (1) on $[t_0, t_0 + a)$. Then $r(t, t_0, \varphi_0)$ is said to be a *maximal solution* of (1) if, for every solution $x(t, t_0, \varphi_0)$ of (1) existing on $[t_0, t_0 + a)$, the inequality

$$x(t, t_0, \varphi_0) \leq r(t, t_0, \varphi_0), \qquad t \in [t_0, t_0 + a)$$

holds. A *minimal solution* may be defined similarly by reversing the above inequality.

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3. Main results.

Theorem 1. *Let* $m \in C([t_0 - \tau, t_0 + a], R_+)$ *satisfying the inequality*

$$m(t) < \varphi_0(0) + \int_{t_0}^{t} w(s, m_s)ds, \qquad t_0 \le t \le t_0 + a$$

where $w \in C([t_0 - \tau, t_0 + a] \times C_+, R_+)$. *Suppose that* $w(t, \psi)$ *nondecreasing in* ψ *for each fixed* t, $t \in [t_0, t_0 + a]$, *and that* $r(t, t_0, \varphi_0)$, *with* $\varphi_0 \in C_\rho$, *is the maximal solution of* (1) *existing foy* $t \ge t_0$.

Then, there exists an $\alpha > 0$ *such that* $m_{t_0} \le \varphi_0$ *implies that* $m(t) \le \le r(t, t_0, \varphi_0)$ *for* $t_0 \le t \le t_0 + \alpha$.

Proof. Define $y \in C([t_0 - \tau, t_0 + a], R)$ as follows:

$$y(t) = \begin{cases} \varphi_0(t - t_0), & t_0 - \tau \le t \le t_0 \\ \varphi_0(0), & t_0 \le t \le t_0 + a \end{cases}$$

Then $w(t, y_t)$ is a continuous function of t on $[t_0, t_0 + a]$ and hence $|w(t, y_t)| < M_1$. We will show that there exists a constant $b \in (0, \rho - \varphi_0(0))$ such that

$$|w(t, \psi) - w(t, y_t)| < 1$$

whenever $t \in [t_0, t_0 + a]$, $\psi \in C_\rho$ and $|\psi - y_t|_0 \le b$. Suppose that this is not true. Then for each $k = 1, 2, \ldots$, there exist $t_k \in [t_0, t_0 + a]$ and $\psi_k \in C_\rho$ such that $|\psi_k - y_{t_k}| < \dfrac{1}{k}$ and

$$|w(t_k, \psi_k) - w(t_k, y_{t_k})| \ge 1$$

Now, choose a subsequence $\{t_{k_p}\}$ such that $\lim_{p \to \infty} t_{k_p} = t_1$ exists, and this leads to a contradiction concerning the continuity of w at (t_1, y_{t_1}). Then it follows that $|w(t, \psi)| \le M = M_1 + 1$ whenever $t \in [t_0, t_0 + a]$, $\psi \in C_\rho$ and $|\psi - y_t|_0 \le b$.

Let $\tilde{M} = \sup_{t_0 \le t \le t_0 + a} |w(t, m_t)|$.

Let $\overline{M} = \max\{M, \tilde{M}\}$.

Choose $\alpha = \min\{a, b/_{\overline{M}}\}$.

Let B denote the space of continuous functions from $[t_0 - \tau, t_0 + \alpha]$ into R_+, with the sup-norm and, then, B is a Banach space. Let $S \subset B$ be defined as follows:

$$S = \left\{ x \in B : \begin{array}{lll} \text{(i)} & x(t) = \varphi_0(t - t_0), & t_0 - \tau \le t \le t_0 \\ \text{(ii)} & |x(t_1) - x(t_2)| \le \overline{M}\,|t_1 - t_2|, & t_1, t_2 \in [t_0, t_0 + \alpha] \\ \text{(iii)} & x(t) \ge m(t), & t \in [t_0, t_0 + \alpha] \end{array} \right\}$$

Now, we will show that $S \ne \phi$.

$$\text{Let } x(t) = \begin{cases} \varphi_0(t - t_0), & t_0 - \tau \le t \le t_0 \\ \varphi_0(0) + \displaystyle\int_{t_0}^{t} w(s, m_s)ds, & t_0 \le t \le t_0 + \alpha \end{cases}$$

(i) $x(t) = \varphi_0(t - t_0)$, $t_0 - \tau \le t \le t_0$

(ii) $|x(t_1) - x(t_2)| = \left| \displaystyle\int_{t_0}^{t_1} w(s, m_s)ds - \int_{t_0}^{t_2} w(s, m_s)ds \right| = \left| \int_{t_1}^{t_2} w(s, m_s)ds \right| \le$

$\le \displaystyle\int_{t_1}^{t_2} |w(s, m_s)|ds \le \overline{M}\,|t_1 - t_2|$, $t_1, t_2 \in [t_0, t_0 + \alpha]$.

Under the above conditions it follows that

(iii) $x(t) = \varphi_0(0) + \displaystyle\int_{t_0}^{t} w(s, m_s)ds \ge m(t)$, $t \in [t_0, t_0 + \alpha]$.

Therefore $S \ne \phi$.

Let us show that S is convex.

Let $f, g \in S$. We will show that

$$h(t) = \lambda f(t) + (1 - \lambda)g(t) \in S, \quad 0 \le \lambda \le 1 \quad \text{and} \quad t \in [t_0 - \tau, t_0 + \alpha]$$

(i) $h(t) = \lambda f(t) + (1 - \lambda)g(t) = \lambda\varphi_0(t - t_0) + (1 - \lambda)\varphi_0(t - t_0) =$
$= \varphi_0(t - t_0)$, $t_0 - \tau_0 - \tau \le t \le t_0$

(ii) $|h(t_1) - h(t_2)| = |\lambda f(t_1) + (1 - \lambda)g(t_1) - \lambda f(t_2) - (1 - \lambda)g(t_2)| \le$
$\le \overline{M}\,|t_1 - t_2|$, $t_1, t_2 \in [t_0, t_0 + \alpha]$

(iii) $h(t) = \lambda f(t) + (1-\lambda)g(t) \geq \lambda m(t) + (1-\lambda)m(t) = m(t),\ t \in [t_0, t_0+\alpha]$

Thus, $h(t) \in S$ and, hence S is convex.

S is closed. In fact, if $x_n \in S$ and the sequence $\{x_n\}$ converges uniformly for x on B. Then, $x_n(t)$ converges for each t. To see that $x \in S$, it is enough to prove condition (ii) because conditions (i) and (iii) are trivially satisfyed. As x_n is pointwise convergent, to x, we have that $x_n(t_1) \to x(t_1)$ and $x(t_2 \to x(t_2)$, for all $t_1,\ t_2 \in [t_0, t_0+\alpha]$, and thus

$$|x_n(t_1) - x_n(t_2)| \leq \bar{M}\,|t_1 - t_2|$$

By taking the limit, as $n \to \infty$, it follows that $|x(t_1) - x(t_2)| \leq \bar{M}|t_1 - t_2|$. Hence S is closed.

Functions of S are equicontinuous. In order to show this we must show that for each $\varepsilon > 0$ and for each $t \in [t_0 - \tau, t_0 + \alpha]$, there exists a $\delta = \delta(\varepsilon, t_1) > 0$ such that $|t - t_1| < \delta \Rightarrow |x(t) - x(t_1)| < \varepsilon$, for all $x \in S$. For $t_1 \in [t_0 - \tau, t_0]$ we have that for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_1)$ such that $|t - t_1| < \delta \Rightarrow |x(t) - x(t_1)| < \varepsilon$, for all $x \in S$, since $|x(t) - x(t_1)| = |\varphi_0(t-t_0) - \varphi_0(t_1 - t_0)|$ and φ_0 is continuous. For $t_1 \in [t_0, t_0+\alpha]$ we always have that: $|x(t) - x(t_1)| \leq M\,|t - t_1|$ Then, it is enough to take $\delta = \varepsilon/_{\bar{M}}$.

Functions of S are uniformly bounded. In fact. For all $x \in S$, we have:

if $t_0 - \tau \leq t \leq t_0 \Rightarrow x(t) = \varphi_0(t - t_0) \leq \rho$, because $\varphi_0 \in C_\rho$;

if $t_0 \leq t \leq t_0 + \alpha \Rightarrow |x(t)| - |\varphi_0(0)| \leq |x(t) - \varphi_0(0)| = |x(t) - x(t_0)| \leq$

$\leq \bar{M}\,|t - t_0| \leq \bar{M}\,|t_0 + \alpha - t_0| = \bar{M}\alpha \leq \bar{M}\dfrac{b}{\bar{M}} = b;$

then

$$|x(t)| \leq b + \varphi_0(0) < \rho - \varphi_0(0) + \varphi_0(0) = \rho$$

Therefore, from Ascoli's Theorem, S is compact.

Define a mapping T on S as follows: for an element $x \in S$, let

(i) $(Tx)(t) = \varphi_0(t - t_0),\ t_0 - \tau \leq t \leq t_0$

(ii) $(Tx)(t) = \varphi_0(0) + \displaystyle\int_{t_0}^{t} w(s, x_s)ds,\quad t_0 \leq t \leq t_0 + \alpha$

For every $x \in S$ and $t \in [t_0, t_0 + \alpha]$, we have

$$|x(t) - x(t_0)| = |x(t) - \varphi_0(0)| \leq \bar{M}\,|t - t_0| \leq \bar{M}\alpha \leq b.$$

Consequently

$$|x_t - y_t|_0 \leq b \text{ and } x_t \in C_\rho, \text{ because}$$

$$|x_t - y_t|_0 = \sup_{-\tau \leq \theta \leq 0} |x(t+\theta) - y(t+\theta)|.$$

When $t + \theta \leq t_0$ we have that

$$x(t+\theta) = y(t+\theta) \Rightarrow |x(t+\theta) - y(t+\theta)| = 0.$$

Then we are interested only in case $t + \theta \geq t_0$, i.e., $t_0 - t \leq \theta \leq 0$.

Hence $|x_t - y_t|_0 = \displaystyle\sup_{t_0 - t \leq \theta \leq 0} |x(t+\theta) - y(t+\theta)| =$

$= \displaystyle\sup_{t_0 \leq t+\theta \leq t} |x(t+\theta) - y(t+\theta) \leq$

$\leq \displaystyle\sup_{t_0 \leq t+\theta \leq t_0+\alpha} |x(t+\theta) - y(t+\theta)| =$

and, by taking $t + \theta = s$,

$$= \sup_{t_0 \leq s \leq t_0+\alpha} |x(s) - y(s)| = \sup_{t_0 \leq s \leq t_0+\alpha} |x(s) - \varphi_0(0)| \leq b.$$

Let us show that $x_t \in C_\rho$. We have that $|x(t)| \leq \rho$. Then $|x_t|_0 = \displaystyle\sup_{-\tau \leq \theta \leq 0} |x(t+\theta)|$. If $t_0 - \tau \leq t_0 + \theta \leq t_0$, we have $x(t+\theta) = \varphi_0(t+\theta-t_0)$ and, since $\varphi_0 \in C_\rho$, we have that $x_1 \in C_\rho$. If $t_0 \leq t_0 + \theta \leq t_0 + \alpha$, we have that $|x(t+\theta)| < \rho$, hence $x_t \in C_\rho$.

Therefore $w(s, x_s)$ is a continuous function of s and $|w(s, x_s)| \leq M$ for $t_0 \leq s \leq t_0 + a$. Then the mapping T is well-defined on S.

We will show that T is continuous.

For $t_0 - \tau \leq t \leq t_0$, $(Tx)(t) = \varphi_0(t - t_0)$ and, hence T is continuous.

For $t_0 \leq t \leq t_0 + \alpha$, $(Tx)(t) = \varphi_0(0) + \displaystyle\int_{t_0}^{t} w(s, x_s)ds$ and it is still continuous. In fact. Consider the sequence $\{x^n(s)\}_{n=1}^{\infty}$ of functions of S which is uniformly convergent for $x(s) \in S$, on $[t_0 - \tau, t_0 + \alpha]$. Then the sequence $\{x_s^n\}_{n=1}^{\infty}$ converges for x_s on C_ρ, uniformly in $s \in [t_0, t_0 + \alpha]$. Indeed, let the set $V=\{x_s^n\in C_\rho,\ n\in N,\ s\in[t_0, t_0+\alpha]\}$. From the definition of V, we have that V is uniformly bounded. Now, we are going to show that V is equicontinuous.

$$|x_s^n(\theta_1)-x_s^n(\theta_2)| = |x^n(s+\theta_1)-x^n(s+\theta_1)| \le \bar{M}\,|\theta_1-\theta_2|$$

if $s+\theta_1,\ s+\theta_2\in[t_0, t_0+\alpha]$, then for each $\varepsilon>0$, there exists $\delta_1=\delta_1(\varepsilon)$ such that $|\theta_1-\theta_2|<\delta_1 \Rightarrow |x_s^n(\theta_1)-x_s^n(\theta_2)|<\varepsilon$; so that is enough to take $\delta_1=\varepsilon/_{\bar{M}}$;

if $s+\theta_1,\ s+\theta_2\in[t_0-\tau, t_0]$, we have that

$$|x_s^n(\theta_1)-x_s^n(\theta_2)| = |\varphi_0(s+\theta_1-t_0)-\varphi_0(s+\theta_2-t_0)|$$

and since φ_0 is continuous on $[-\tau, 0]$ and hence is uniformly continuous, we have that, for each $\varepsilon>0$, there exists $\delta_2=\delta_2(\varepsilon)$ such that $|\theta_1-\theta_2|< <\delta_2 \Rightarrow |x_s^n(\theta_1)-x_s^n(\theta_2)|<\varepsilon$.

Taking $\delta=\min\{\delta_1,\delta_2\}$, we have that, for each $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)$ such that $|\theta_1-\theta_2|<\delta \Rightarrow |x_s^n(\theta_1)-x_s^n(\theta_2)|<\varepsilon$, for all $x_s^n\in V$. Therefore V is equicontinuous.

Then, by Ascoli's Theorem, $\bar{V}$ is compact.

As $x_s^n\to x_s$, uniformly on $[t_0, t_0+\alpha]$, we have that $x_s\in\bar{V}$.

Then $w(s,\psi)$ is uniformly continuous on $[t_0, t_0+\alpha]\times V$ and, hence, given $\psi_1,\ \psi_2\in\bar{V}$ and $s\in[t_0, t_0+\alpha]$, we have that, for each $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)$ such that $|\psi_1-\psi_2|_0<\delta \Rightarrow |w(s,\psi_1)-w(s,\psi_2)|<\varepsilon$.

As $x_s^n\to x_s$, uniformly in s, $s\in[t_0, t_0+\alpha]$, we have that, for each $\delta>0$, there exists $N=N(\delta)$ such that $n\ge N \Rightarrow |x_s^n-x_s|_0<\delta$. Consequently, for each $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that $n\ge N \Rightarrow |w(s,x_s^n)-w(s,x_s)|< < \dfrac{\varepsilon}{(t_0+\alpha)-t_0}$, i.e., $w(s,x_s^n)$ converges to $w(s,x_s)$ uniformly in $s\in[t_0, t_0+\alpha]$.

Thus,

$$|(Tx^n)(t)-(Tx)(t)| \le \left|\int_{t_0}^{t} |w(s,x_s^n)-w(s,x_s)|\,ds\right| < \int_{t_0}^{t} \frac{\varepsilon}{(t_0+\alpha)-t_0}\,ds < \varepsilon$$

and, hence, $(Tx^n)(t)$ converges uniformly to $(Tx)(t)$, then, T is continuous.

We claim that $TS\subset S$. Indeed, let $x\in S$ such that

(i) $(Tx)(t)=\varphi_0(t-t_0),\ t_0-\tau\le t\le t_0$

(ii) $|(Tx)(t_1)-(Tx)(t_2)| = \left|\int_{t_1}^{t_2} w(s,x_s)\,ds\right| \le$

$$\le \left|\int_{t_1}^{t_2} |w(s,x_s)|\,ds\right| \le \bar{M}\,|t_1-t_2|,\quad t_1,t_2\in[t_0, t_0+\alpha]$$

(iii) $(Tx)(t)=\varphi_0(0)+\int_{t_0}^{t} w(s,x_s)\,ds \ge \varphi_0(0)+\int_{t_0}^{t} w(s,m_s)\,ds \ge m(t)$,

$$t\in|t_0, t_0+\alpha|$$

The inequality $\varphi_0(0)+\int_{t_0}^{t} w(s,x_s)\,ds \ge \varphi_0(0)+\int_{t_0}^{t} w(s,m_s)\,ds$ holds, because $x(s)\ge m(s)$, $s\in[t_0, t_0+\alpha]$. In fact. As $x_s(\theta)=x(s+\theta)$, if $s+\theta\le t_0$, and, from hypoteses, $m_{t_0}\le\varphi_0$, it follows that $x(s+\theta)=\varphi_0(s+\theta-t_0)\ge m_{t_0}(s+\theta)$. If $s+\theta\ge t_0$ it follows $x(s+\theta)\ge m(s+\theta)=m_s(\theta)$. Thus, T maps S into S.

As the hypoteses of the Schauder fixed point theorem hold, thus there is a function $x\in S$ such that:

(i) $(Tx)(t)=x(t)=\varphi_0(t-t_0),\ t_0-\tau\le t\le t_0$

(ii) $(Tx)(t)=x(t)=\varphi_0(0)+\int_{t_0}^{t} w(s,x_s)\,ds,\ t_0\le t\le t_0+\alpha.$

Since $x\in S$, the integrand in the foregoing equation is a continuous function of s. Thus, for $t_0\le t<t_0+\alpha$, we can differentiate to obtain

$$\dot{x}(t)=w(t,x_t),\quad t_0\le t<t_0+\alpha.$$

As $r(t,t_0,\varphi_0)$ is the maximal solution of the above equation, we have $x(t)\le r(t)$, $t_0\le t<t_0+\alpha$.

As $x(t)\ge m(t)$, $t_0\le t\le t_0+\alpha$, we have that

$$m(t)\le r(t,t_0,\varphi_0),\quad t_0\le t<t_0+\alpha$$

This completes the proof of the theorem.

Corollary 1. *Let* $m \in C([t_0 - \tau, \infty), R_+)$ *satisfying the inequality*

$$m(t) \leq \varphi_0(0) + \int_{t_0}^{t} w(s, m_s) ds, \; t \geq t_0$$

where $w \in C([t_0 - \tau, \infty) \times C_+, R_+)$. *Suppose that* $w(t, \psi)$ *is nondecreasing in* ψ *for each fixed t,* $t \in [t_0, \infty)$, *and that* $r(t, t_0, \varphi_0)$, *with* $\varphi_0 \in C_\rho$, *is the maximal solution of (1) existing for* $t \geq t_0$.

Then, $m_{t_0} \leq \varphi_0$ *implies that*

$$m(t) \leq r(t, t_0, \varphi_0) \text{ for } t \geq t_0.$$

Proof. Suppose that the set

$$Z = \{t \geq t_0; \; m(t) > r(t, t_0, \varphi_0)\}$$

is not empty. Let $t_1 = \inf Z$. Then $m(t_1) = r(t_1, t_0, \varphi_0)$. Therefore $t_1 \notin Z$. For $t \leq t_1$, we have that $m(t) \leq r(t, t_0, \varphi_0)$ and so $m_{t_1} \leq r_{t_1}(t_0, \varphi_0)$. As r is increasing and taking $\varphi_1 = r_{t_1}(t_0, \varphi_0)$ we have

$$m(t) \leq \varphi_0(0) + \int_{t_0}^{t} w(s, m_s) ds \leq r(t_1, t_0, \varphi_0) + \int_{t_0}^{t} w(s, m_s) ds =$$

$$= \varphi_1(0) + \int_{t_0}^{t} w(s, m_s) ds.$$

By using Theorem 1 in t_1, we have that there exists an α_1 such that for $t \in [t_1, t_1 + \alpha]$, $m(t) \leq r(t, t_0, \varphi_0)$. This leads to a contradiction. Then $Z = \phi$, and $m(t) \leq r(t, t_0, \varphi_0)$, for $t \geq t_0$.

Corollary 2. *Let* $m \in C([t_0 - \tau, t_0 + a], R_+)$ *satisfying the inequality*

$$m(t) \leq \varphi_0(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s) ds, \; t_0 \leq t \leq t_0 + a$$

where $w \in C([t_0 - \tau, t_0 + a] \times C_+, R_+)$, $\beta(t) > 0$ *and continuous. Suppose that* $w(t, \psi)$ *is nondecreasing in* ψ *for each fixed t,* $t \in [t_0, t_0 + a]$, *and that* $r(t, t_0, \varphi_0)$, *with* $\varphi_0 \in C_\rho$, *is the maximal solution of* $\dot{r} = w(t, r_t + \beta_t)$ *existing for* $t \geq t_0$.

Then, there exists an $\alpha > 0$ *such that* $m_{t_0} \leq \varphi_0 + \beta_{t_0}$ *implies that* $m(t) \leq \leq \beta(t) + r(t, t_0, \varphi_0)$ *for* $t_0 \leq t \leq t_0 + \alpha$.

Proof. Defining $p(t) = m(t) - \beta(t)$, it follows $m(t) = p(t) + \beta(t)$ which implies $m_t = p_t + \beta_t$. So, the inequality becomes

$$p(t) \leq \varphi_0(0) + \int_{t_0}^{t} w(s, p_s + \beta_s) ds,$$

where $p_{t_0} = m_{t_0} - \beta_{t_0} \leq \varphi_0 + \beta_{t_0} - \beta_{t_0} = \varphi_0$.

By applying Theorem 1, we obtain that there exists an $\alpha > 0$ such that $p(t) \leq r(t, t_0, \varphi_0)$ for $t_0 \leq t \leq t_0 + \alpha$.

Therefore, there exists an $\alpha > 0$ such that

$$m(t) \leq \beta(t) + r(t, t_0, \varphi_0) \text{ for } t_0 \leq t \leq t_0 + \alpha.$$

Corollary 3. *Let* $m \in C([t_0 - \tau, \infty), R_+)$ *satisfying the inequality*

$$m(t) \leq \varphi_0(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s) ds, \; t \geq t_0$$

where $w \in C([t_0 - \tau, \infty) \times C_+, R_+)$, $\beta(t) > 0$ *and continuous. Suppose that* $w(t, \psi)$ *is nondecreasing in* ψ *for each fixed t,* $t \in [t_0 - \tau, \infty)$ *and that* $r(t, t_0, \varphi_0)$, *with* $\varphi_0 \in C_\rho$, *is the maximal solution of* $\dot{r} = w(t, r_t + \beta_t)$ *existing for* $t \geq t_0$.

Then $m_{t_0} \leq \varphi_0 + \beta_{t_0}$ *implies that*

$$m(t) \leq \beta(t) + r(t, t_0, \varphi_0) \text{ for } t \geq t_0.$$

Proof. Suppose that the set

$$Z = \{t \geq t_0; \; m(t) > \beta(t) + r(t, t_0, \varphi_0)\}$$

is not empty. Let $t_1 = \inf Z$. Then $m(t_1) = \beta(t_1) + r(t_1, t_0, \varphi_0)$. Therefore $t_1 \notin Z$. For $t \leq t_1$, we have that $m(t) \leq \beta(t) + r(t, t_0, \varphi_0)$ and so $m_{t_1} \leq \beta_{t_1} + + r_{t_1}(t_0, \varphi_0) = \beta_{t_1} + \varphi_1$, where $\varphi_1 = r_{t_1}(t_0, \varphi_0)$. As r is increasing, we have

$$m(t) \leq \varphi_0(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s) ds \leq$$

$$\leq r(t_1, t_0, \varphi_0) + \beta(t) + \int_{t_0}^{t} w(s, m_s) ds =$$

$$= \varphi_1(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s) ds.$$

By using Theorem 1 in t_1, we have that there exists an $\alpha_1 > 0$ such that for $t \in [t_1, t_1 + \alpha_1]$, $m(t) \leq \beta(t) + r(t, t_0, \varphi_0)$. This leads to a contradiction. Then $Z = \phi$, and $m(t) \leq \beta(t) + r(t, t_0, \varphi_0)$, for $t \geq t_0$.

Theorem 2. *Let $m \in C([t_0 - \tau, t_0 + a], R_+)$ satistying the inequanty*

$$m(t) \geq \varphi_0(0) + \int_{t_0}^{t} w(s, m_s)\,ds, \quad t_0 \leq t \leq t_0 + a$$

where $w \in C([t_0 - \tau, t_0 + a] \times C_+, R_+)$. Suppose that $w(t, \psi)$ is nondecreasing in ψ for each fixed t, $t \in [t_0, t_0 + a]$, and that $r(t, t_0, \varphi_0)$, with $\varphi_0 \in C_\rho$, is the minimal solution of (1) *existing for $t \geq t_0$.*

Then, there exists an $\alpha > 0$ such that $m_{t_0} \geq \varphi_0$ implies that

$$m(t) \geq r(t, t_0, \varphi_0) \text{ for } t_0 \leq t \leq t_0 + \alpha.$$

The proof is similar to the one given in Theorem 1.

Corollary 1. *Let $m \in C([t_0 - \tau, \infty), R_+)$ satisfying the inequality*

$$m(t) \geq \varphi_0(0) + \int_{t_0}^{t} w(s, m_s)\,ds, \quad t \geq t_0$$

where $w \in C([t_0 - \tau, \infty) \times C_+, R_+)$. Suppose that $w(t, \psi)$ is nondecreasing in ψ for each fixed t, $t \in [t_0, \infty)$, and that $r(t, t_0, \varphi_0)$, with $\varphi_0 \in C_\rho$, is the minimal solution of (1) *existing for $t \geq t_0$.*

Then $m_{t_0} \geq \varphi_0$ implies that

$$m(t) \geq r(t, t_0, \varphi_0) \text{ for } t \geq t_0$$

The proof is similar to the one given in Corollary 1 of Theorem 1.

Corollary 2. *Let $w \in C([t_0 - \tau, t_0 + a], R_+)$ satisfying the inequality*

$$m(t) \geq \varphi_0(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s)\,ds, \quad t_0 \leq t \leq t_0 + a$$

where $w \in C([t_0 - \tau, t_0 + a] \times C_+, R_+)$, $\beta(t) > 0$ and continuous. Suppose that $\omega(t, \psi)$ is nondecreasing in ψ for each fixed t, $t \in [t_0 - \tau, t_0 + a]$, and that $r(t, t_0, \varphi_0)$, with $\varphi_0 \in C_\rho$, is the minimal solution of $\dot{r} = w(t, r_t + \beta_t)$ and that $r(t, t_0, \varphi_0)$, with $\varphi_0 \in C_\rho$, is the minimal solution of $\dot{r} = w(t, r_t + \beta_t)$ existing for $t \geq t_0$.

Then, there exists an $\alpha > 0$ such that $m_{t_0} \geq \varphi_0 + \beta_{t_0}$ implies that

$$m(t) \geq \beta(t) + r(t, t_0, \varphi_0) \text{ for } t_0 \leq t \leq t_0 + \alpha.$$

The proof is similar to the one given in Corollary 2 of Theorem 1.

Corollary 3. *Let $m \in C([t_0 - \tau, \infty), R_+)$ satisfying the inequality*

$$m(t) \geq \varphi_0(0) + \beta(t) + \int_{t_0}^{t} w(s, m_s)\,ds, \quad t \geq t_0$$

where $w \in C([t_0 - \tau, \infty) \times C_+, R_+)$, $\beta(t) > 0$ and continuous. Suppose that $w(t, \psi)$ is nondecreasing in ψ for each fixed t, $t \in [t_0 - \tau, \infty)$ and that $r(t, t_0, \varphi_0)$, with $\varphi_0 \in C_\rho$, is the minimal solution of $\dot{r} = w(t, r_t + \beta_t)$ existing for $t \geq t_0$.

Then $m_{t_0} \geq \varphi_0 + \beta_{t_0}$ implies that

$$m(t) \geq \beta(t) + r(t, t_0, \varphi_0) \text{ for } t \geq t_0.$$

The proof is similar to the one given in Corollary 3 of Theorem 1.

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