Extremal eigenvalue problems

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0. Introduction.

This paper is an expanded version of the lectures given by the author at IMPA and the Univeristy of Rio Grande do Sul at Porto Alegre, Brazil. The purpose of this paper is to survey several basic techniques for various classes of extremal eigenvalue problems. A number of important results, to which these methods apply, are listed. Most of the proofs are omitted and references are given to the interested reader.

1. The fundamental variational formula.

Let $T : B \to B$ be a compact operator in a given Banach space B. Let $\{\lambda_m\}_1^\infty$ be the set of all eigenvalues of T each repeated according to its multiplicity. If T has a finite number of nonzero eigenvalues, let'us say m, then we assume that $\lambda_i = 0$, $i = m+1, \ldots,$. Since these eigenvalues are in general complex valued we shall arrange them in certain order, depending on the nature of T. In the most cases the eigenvalues of T are arranged as follows

(1.1) $$|\lambda_1| \geq |\lambda_2| \geq \ldots$$

Usually, T varies in a given class C of compact operators. We are looking for an operator T_0 in C which extremizes specified functional $F(\lambda_1(T), \ldots, \lambda_n(T))$ in C. For example. find $T^* \in C$ such that $|\lambda_n(T_0)| \geq \lambda_n(T)$ for any $T \in C$. The existence of such a T_0 would follow if we can introduce a topology on C such that F is continuous and C is compact with respect to this topology. Assume that an extremal operator T_0 exists. As in the ordinary calculus, T_0 can be characterized by imbedding T_0 in one parameter family $T(\varepsilon)$ such that $T(0) = T_0$ and reducing the original problem on C to some extremal problem for a small complex valued parameter ε near the origin. In most of the cases it sufficies to assume that $T(\varepsilon)$ is an analytic operator valued function of a complex parameter ε in some disc $|\varepsilon| < \rho$. That is, we have a power expansion

(1.2) $$T(\varepsilon) = \sum_{k=0}^{\infty} T_k \varepsilon^k, \quad |\varepsilon| < \rho.$$

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We also assume that each T_k is compact, i. e. $T(\varepsilon)$ is compact. Let $\lambda_0 \neq 0$ be an eigenvalue of T_0. In order to study the function $F(\lambda_1(T(\varepsilon)), \ldots, \lambda_n(T(\varepsilon)))$ we have to know the variation of λ_0 in terms of ε. This is well known by now (e. g. [3, p. 587-588]). Let $E(0)$ be the projection operator associated with the eigenvalue λ_0 of the operator $T(0) = T_0$. That is

$$E(0) = \int_{|z-\lambda_0|=r} (zI - T_0)^{-1}\, dz \tag{1.3}$$

where r is a small positive enough number. Suppose that λ_0 is of a multiplicity m. That is, the dimension of $E(0)B$ is m. Let $E(\varepsilon)$ be the corresponding analytic projection for small $|\varepsilon|$. That is

$$E(\varepsilon) = \int_{|z-\lambda_0|=r} (zI - T(\varepsilon))^{-1}\, dz. \tag{1.4}$$

Then it is known that $E(\varepsilon)B$ has also a finite m-dimensional range. Moreover if $\{x_1, \ldots, x_m\}$ is a basis for $E(0)B$ then $\{E(\varepsilon)x_1, \ldots, E(\varepsilon)x_n\}$ is a basis for $E(\varepsilon)B$. Thus in the neighborhood of λ_0. $T(\varepsilon)$ has m eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_m(\varepsilon)$ (not necessarily distinct) and each $\lambda_k(\varepsilon)$ is given by fractional power series

$$\lambda_k(\varepsilon) = \lambda_0 + \sum_{p=1}^{\infty} \alpha_{kp}\varepsilon^{p/n}. \tag{1.5}$$

To explain the formula (1.5) let us consider the case where B is finite dimensional. That is we can assume that $T(\varepsilon)$ is a matrix $(t_{ij}(\varepsilon))_1^m$ where each $t_{ij}(\varepsilon)$ is analytic function of ε. Then the eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_m(\varepsilon)$ are the roots of the equation

$$d(\lambda, \varepsilon) = \det\, (\lambda\delta_{ij} - t_{ij}(\varepsilon))_1^m = 0. \tag{1.6}$$

Thus each $\lambda_k(\varepsilon)$ is in fact an "algebraic" function and therefore at the branch points $\lambda_k(\varepsilon)$ may have an expansion in terms of $\varepsilon^{1/n}$. In case of the compact operator valued function $T(\varepsilon), \lambda_1(\varepsilon), \ldots, \lambda_m(\varepsilon)$ are given by the equation (1.6) where $t_{ij}(\varepsilon)$ are defined by

$$x_k^*(T(\varepsilon)E(\varepsilon)x_i) = \sum_{j=1}^{m} t_{ij}(\varepsilon)x_k^*(E(\varepsilon)x_j),\ k = 1, \ldots, m. \tag{1.7}$$

Here $\{x_1^*, \ldots, x_m^*\}$ is a basis for $E^*(0)B^*$. In order to characterize the extremal operator T_0 it is sufficient to know α_{k1}, the first nontrivial coefficient in the expansion (1.5). The equations (1.6) and (1.7) show that the actual computation of α_{k1} is complicated. Moreover it may happen that α_{k1} depends not only on T_1 but also on other T_k. If we assume that λ_0 is a pole of order 1 of the resolvent $(\lambda I - T_0)^{-1}$ then the first variation of λ_0 has a relatively simple form.

Theorem 1.1. *Let $T(\varepsilon)$ be given by the series (1.2) converging in the uniform topology. Assume that each T_k, $k = 0, 1, \ldots$ is compact. Let $\lambda_0 \neq 0$ be an eigenvalue of $T(0)$ of a multiplicity m. Assume that λ_0 is a pole of order 1 of the resolvent $(\lambda I - T_0)^{-1}$. That is*

$$E(0)x = \sum_{i=1}^{m} x_i^*(x)x_i, \tag{1.8}$$

$$T_0x_i = \lambda_0 x_i, \qquad T_0^*x_i^* = \lambda_0 x_i^*, \qquad x_i^*(x_j) = \delta_{ij}, \qquad i, j = 1, \ldots, n$$

Then the eigenvalues $\lambda_i(\varepsilon)$ are given by fractional power series

$$\lambda_k(\varepsilon) = \lambda_0 + \omega_k\varepsilon + \sum_{p=n+1}^{\infty} \alpha_{kp}\varepsilon^{p/n} \tag{1.9}$$

of the principal value of $\varepsilon^{1/n}$ in the neighborhood of the origin. The numbers $\omega_1, \ldots, \omega_n$ are the eigenvalues of the matrix $(x_i^*(T_1 x_j))_1^m$.

The proof of this theorem is available in [6].

The assumptions of the theorem hold if T_0 is a symmetric operator in the Hilbert space. Furthermore, if in addition each T_k is symmetric then it is known that each $\lambda_k(\varepsilon)$ is analytic in ε (see [15]). Theorem 1.1 implies that the first variation of λ_0 is linear in ε, i.e.

$$\lambda_k(\varepsilon) = \lambda_0 + \omega_k\varepsilon + \varepsilon\, o\,(\varepsilon), \qquad k = 1, \ldots, m. \tag{1.10}$$

Furthermore ω_k depends only on T_1. On the other hand Theorem 1.1 explains the difficulties in characterizing the extremal operator T_0 as the first variation of λ_0 is not, in general, linear in the first variation — T_1. Only if $m = 1$, i.e. λ_0 is simple (in the algebraic sense), then, indeed, the first variation of λ_0 is linear in the first variation of T_0.

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold. Assume furthermore that $m = 1$. Then in the neighborhood of $\lambda_0 \wedge T(\varepsilon)$ has a simple eigenvalue $\lambda(\varepsilon)$ which is analytic in ε*

$$\lambda(\varepsilon) = \lambda_0 + \omega\varepsilon + \sum_{p=2}^{\infty} \alpha_p \varepsilon^p \tag{1.11}$$

where

$$\omega \doteq x^*(T_1 x), \qquad T_0 x = \lambda_0 x, \qquad T_0^* = \lambda_0 x^*, \qquad x^*(x) = 1 \tag{1.12}$$

2. Some compactness results for the eigenvalues.

The first step in studying extremal eigenvalue problems is to show the existence of the extremal solutions. The following theorem is basic in this area [3, p. 1091].

Theorem 2.1. *Let* $\{T_n\}_1^\infty$ *be a sequence of compact operators in B. Assume that converges to* $T_n \wedge T$ *in the uniform topology. Then there exists an enumeration* $\lambda_m(T_n)$, $m = 1, 2, \ldots$ *of the eigenvalues of* T_n *which satisfies the condition* (1.1) *such that*

$$\lim_{n\to\infty} \lambda_m(T_n) = \lambda_m(T), \qquad m = 1, 2, \ldots. \tag{2.1}$$

However this result is hard to apply. Indeed let C be a bounded set of compact operators $T\colon B \to B$. Suppose we consider a maximum problem

$$\sup_{T\in C} F(\lambda_1(T), \ldots, \lambda_p(T)) = M. \tag{2.2}$$

Let $\{T_n\}_1^\infty$ be the maximizing sequence, i.e.

$$M = \lim_{n\to\infty} F(\lambda_1(T_n), \ldots, \lambda_p(T_n)). \tag{2.3}$$

If we can find a convergent subsequence of $\{T_n\}$ in the strong topology then from Theorem 2.1 we deduce

$$\sup_{T\in C} F(\lambda_1(T), \ldots, \lambda_p(T)) = F(\lambda_1(T_0), \ldots, \lambda_p(T_0)), \qquad T_0 \in C. \tag{2.4}$$

In order to have always a convergent subsequence we need to assume that C is compact set in the strong topology. However, in most interesting applications C is compact in some weaker topology. In case of $B = \mathscr{H}$ ($\mathscr{H}$ a Hilbert space) we have the following result

Theorem 2.2. *Let* $\mathscr{H}$ *be a Hilbert space with an inner product* (x, y). *Assume that* $K : \mathscr{H} \to \mathscr{H}$ *is a compact symmetric positive definite operator.*

Let $Q_n : \mathscr{H} \to \mathscr{H}$ *be a sequence of linear operators which converge weakly to* Q. *That is*

$$(Q_n x, y) \to (Qx, y), \qquad n \to \infty \tag{2.5}$$

for any $x, y \in \mathscr{H}$.

Then there exists an enumeration $\lambda_m(KQ_n)$, $m = 1, 2, \ldots$ *such that*

$$\lim_{n\to\infty} \lambda_m(KQ_n) = \lambda_m(KQ), \qquad m = 1, 2, \ldots. \tag{2.6}$$

The proof of this result can be found in [8]. The proof follows from Theorem 2.1 by noting that the spectrum of KQ and $K^{1/2}QK^{1/2}$ are identical and using the lemma.

Lemma 2.1. *Let* $\mathscr{H}$ *be a Hilbetr space with an inner product* (x, y). *Assume that* $S, T\colon \mathscr{H} \to \mathscr{H}$ *are compact linear operators. Let* $Q_n : \mathscr{H} \to \mathscr{H}$ *be a sequence of linear operators which converge weakly to* Q. *Then* SQ_nT *converges to* SQT *uniformly.*

The results of Theorem 2.2 hold if we replace the assumption that K is positive definite by assumption that K belongs to some C_p class (see for the definition [3, p. 1089]). The proof is given in [6]. Let us give a general setting for which Theorem 2.2 applies. Denote by $L_p(d\sigma)$, $1 \le p \le \infty$, the set of all measurable p-integrable functions with respect to a sigma finite measure σ on $I \subset R^n$. Assume that $K : L_2(d\sigma) \to L_2(d\sigma)$ is a compact integral operator

$$K(f) = \int_I K(x, y) f(y)\, d\sigma. \tag{2.7}$$

Let φ be a bounded function in the class $L_\infty(d\sigma)$. Then φ can be regarded as an operator $\varphi\colon L_2(d\sigma) \to L_2(d\sigma)$

$$\varphi(f) = \varphi f. \tag{2.8}$$

Recall that $\varphi_n \to \varphi$, in w^* topology as $n \to \infty$ if

$$\int_I \varphi_n \psi\, d\sigma \to \int_I \varphi\psi\, d\sigma, \qquad n \to \infty \tag{2.9}$$

for any $\psi \in L_1(d\sigma)$. This convergence is equivalent to the weak convergence of the operators $\varphi_n\colon L_2(d\sigma) \to L_2(d\sigma)$ to the operator φ. It is a standard fact that a unit ball in $L_\infty(d\sigma)$ is w^* compact. In what follows we shall consider the following bounded set C in $L_\infty(d\sigma)$ which is defined by the conditions

(2.10) $$0 \le m(\xi) \le \varphi(\xi) \le M(\xi), \qquad m, M \in L_\infty(d\sigma),$$

(2.11) $$\int_I \varphi \rho_i d\sigma = c_i, \qquad i = 0, 1, \dots, k.$$

It is easy to show that C is compact in w^* topology. Thus if K is positive definite or $K \in C_p$ then for any continuous function $F(x_1, \dots, x_n)$ we have

(2.12) $$\max_C \lambda_1(K\varphi), \dots, \lambda_n(K\varphi)) = F(\lambda_1(K\psi), \dots, \lambda_n(K\psi)).$$

For our applications we shall assume that K is positive definite. Since the spectrum of $K\varphi$ is equivalent to the spectrum of $K^{1/2}\varphi K^{1/2}$ we have thal all nontrivial eigenvalues λ of $K\varphi$ are positive and λ is a pole of order 1 for the resolvent of $K\varphi$. We arrange these eigenvalues in a decreasing order

(2.13) $$\lambda_1(K\varphi) \ge \lambda_2(K\varphi) \ge \dots$$

Suppose that χ is an eigenfunction of $K\varphi$

(2.14) $$\int_I K(x, y)\varphi(y)\chi(y)d\sigma(y) = \lambda\chi(x).$$

Since the conjugate operator of $K\varphi$ is given by

(2.15) $$(K\varphi)^* = \varphi K$$

we immediatly have that

(2.16) $$(K\varphi)^*(\varphi\chi) = \lambda\varphi\chi.$$

Now let χ_i be an eigenfunction corresponding to $\lambda_i = \lambda_i(K\varphi)$

(2.17) $$\int_I K(x, y)\varphi\chi_i(y)d\sigma(y) = \lambda_i\chi(x), \qquad i = 1, \dots.$$

We normalize χ_i by the condition (1.8)

(2.18) $$\int_I \varphi\chi_i\chi_j d\sigma = \sigma_{ij}.$$

Now we can apply Theorem 1.1 in that particular case.

Theorem 2.3. *Let $\varphi \ne 0$ be a nonnegative function in $L_\infty(d\sigma)$. Assume that the operator K: $L_2(d\sigma) \to L_2(d\sigma)$ defined by (2.7) is a compact symmetric positive definite operator. Let $\lambda > 0$ be an eigenvalue of $K\varphi$ of multiplicity m*

(2.19) $$\lambda_p(K\varphi) > \lambda = \lambda_{p+1}(K\varphi) = \dots = \lambda_{p+m}(K\varphi) > \lambda_{p+m+1}(K\varphi).$$

Let

(2.20) $$\varphi(\varepsilon) = \varphi + \varepsilon\varphi_1, \qquad \varphi_1 \in L_\infty(d\sigma).$$

Then for a small positive ε we have the equality

(2.21) $$\lambda_{p+r}(K\varphi(\varepsilon)) = \lambda(1 + \omega_r\varepsilon) + \varepsilon\, o\,(\varepsilon), \qquad r = 1, \dots, m$$

where $\omega_1, \dots, \omega_m$ are the eigenvalues of the symmetric matrix $A = (a_{ij})_1^m$

(2.22) $$a_{ij} = \int_I \varphi_1\chi_{p+i}\chi_{p+j}d\sigma, \qquad i, j = 1, \dots, m.$$

This result will be applied to characterize the extremal functions ψ. We conclude this section with another compactness result for the eigenvalues. Consider a set $C(I)$ composed of all continuous functions on I. Let $K(x, y)$ be a continuous function on $I \times I$ and consider $K_\sigma : C(I) \to C(I)$ where K_σ is defined by (2.7) and σ is a measure in $C^*(I)$. It is known that such a K_σ is compact. It is a standard fact that the unit ball of $C^*(I)$ is compact in ω^* topology. For this topology we have the following compactness result [6].

Theorem 2.4. *Let $K(x, y)$ be a continuous bounded function on $I \times I$. Let σ_n and σ belong to $C^*(I)$. Assume that $\sigma_n \to \sigma$ in w^* topology. Then*

(2.23) $$\lim_{n\to\infty} \lambda_m(K_{\sigma_n}) = \lambda_m(K_\sigma), \qquad m = 1, \dots, .$$

This result could be applied in establishing the existence of optimal solution for the minimal critical mass of nuclear reactor [19]. In the original paper this was proved by a different argument. The proof of the theorem follows by considering the Fredholm determinants corresponding to K_{σ_n}.

3. The Krein results and its extensions.

Consider a vibrating string. That is we have an eigenvalue problem for a second order differential operator

(3.1) $$u'' + \mu\varphi(x)u = 0,$$

(3.2) $$u(0) = u(\ell) = 0.$$

Here $\sqrt{\mu}$ are the frequencies of the harmonic oscillation of the string under unit tension and fixed at its end point. Then function $\varphi(x) \geq 0$ represents the density of the string at the point x. M. Krein in his classical work [16] investigated the maximum and the minimum of the eigenvalues

(3.3) $$0 < \mu_1(\varphi) < \mu_2(\varphi) < \ldots$$

of (3.1) in a set C defined by the conditions

(3.4) $$0 \leq m \leq \varphi \leq M < \infty,$$

(3.5) $$\int_0^\ell \varphi(x)\,dx = W.$$

He assumed that m and M are constants. An equivalent formulation is achieved by writing (3.1) in integral form (2.14) where $K(x, y)$ is the Green's function of $-\frac{d^2}{dx^2}$ coupled with the boundary conditions (3.2) and

(3.6) $$d\sigma = dx,\ I = [0, \ell] \text{ and } \lambda = \mu^{-1}$$

From the results of the previous section we deduce the existence of the extremal functions φ^* and φ^{**} such that

(3.7) $$\min_{\varphi\in C} \mu_p(\varphi) = \mu_p(\varphi^*),\ \max_{\varphi\in C} \mu_p(\varphi) = \mu_p(\varphi^{**}).$$

Let φ and ψ be two arbitrary functions in C. Since C is convex we have that

(3.8) $$\varphi(\varepsilon) = \varphi + \varepsilon(\psi - \varphi)$$

belongs to C for any $0 \leq \varepsilon \leq 1$. Let u_p be an eigenfunction corresponding to the p-th eigenvalue $\mu_p(\varphi)$. As all the eigenvalues of (3.1) are simple from (2.22) we get a simple variation formula for $\mu_p(\varphi(\varepsilon))$ (recall that $\mu = \lambda^{-1}$)

(3.9) $$\mu_p(\varphi(\varepsilon)) = \mu_p(\varphi)\left(1 - \varepsilon\int_0^\ell (\psi - \varphi)\,u_p^2\,dx\right) + \varepsilon\, o(\varepsilon), \qquad 0 < \varepsilon.$$

Taking φ to be equal φ^* and φ^{**} from the extremality of φ^* and φ^{**} and the formula (3.9) we get.

Theorem 3.1. *Let φ^* and φ^{**} solve the extremum problems (3.7). Suppose that*

(3.10) $$v_p'' + \mu_p(\varphi^*)\varphi^* v_p = 0, \quad w_p'' + \mu_p(\varphi^{**})\varphi^{**}\,w_p = 0.$$

Then for any $\psi\in C$ we have the inequalities

(3.11) $$\int_0^\ell \psi\, v_p^2\,dx \leq \int_0^\ell \varphi^*\, v_p^2\,dx,$$
$$\int_0^\ell \psi\, w_p^2\,dx \geq \int_0^\ell \varphi^{**}\, w_p^2\,dx$$

The inequalities (3.11) show that φ^* and φ^{**} solve also certain extremal linear problems on C. The classical lemma of Neyman and Pearson characterizes the extremal solutions of the linear problems on C (see for example [12]). We state this lemma in a quite general setting which will be needed later. Let ν be a nonnegative measure on a closed set I in R^n. Denote by C the following subset in $L_\infty(d\nu)$

(3.12) $$0 \leq \varphi \leq 1$$

(3.13) $$\int_I \varphi\,d\nu = k$$

Let $\omega \geq 0$ belong to $L_1(d\nu)$. Define

(3.14) $$X(\omega, \alpha) = \{\xi \mid \omega(\xi) > \alpha\},\ \ Y(\omega, \alpha) = \{\xi \mid \omega(\xi) < \alpha\}, Z(\omega, \alpha) = \{\xi \mid \omega(\xi) = \alpha\}.$$

Let $V \subset I$ be a ν-measurable set. By $\text{mes}_\nu(V)$ we denote the ν-measure of V

(3.15) $$\text{mes}_\nu(V) = \int_V d\nu.$$

Lemma 3.1. (Neyman-Pearson). *Let C be the set of functions defined by* (3.12) *and* (3.13). *Consider a maximum problem*

(3.16) $$\max_{\varphi \in C} \int \varphi\omega\, d\nu$$

where $\omega \in L_1(d\nu)$. Let α^* be a real number defined by the conditions

(3.17) $$\text{mes}_\nu(X(\omega, \alpha^*)) \leq k, \quad \text{mes}_\nu(X(\omega, \alpha^*) \cup Z(\omega, \alpha^*)) \geq k.$$

Then a function ψ in C satisfies the condition

(3.18) $$\max_{\varphi \in C} \int_I \varphi\omega\, d\nu = \int_I \psi\omega\, d\nu$$

if and only if ψ is of the form

(3.19) $$\psi(\xi) = 1, \quad \xi \in X(\omega, \alpha^*), \quad \psi(\xi) = 0, \quad \xi \in Y(\omega, \alpha^*)$$

almost every where with respect to the measure ν.

Note that if $\text{mes}_\nu(X(\omega, \alpha^*)) = k_0 < k$ then the solution to the problem (3.18) is not unique. In fact on the set $Z(\omega, \alpha^*) = Z$ the values of ψ are arbitrary except for the condition $\int_Z \psi\, d\nu = k - k_0$. See [6] for a proof of the lemma. In order to apply the lemma of Neyman-Pearson to inequalities (3.11) we just have to observe that any φ satisfying (3.4) and (3.5) can be written in the form

(3.20) $$\varphi = m + \theta(M - m), \qquad 0 \leq \theta \leq 1$$

(3.21) $$\int_0^{\ell} \theta\, dx = (W - m\ell)/M - m.$$

Thus we are lead to the problem of finding the number of solutions to the equation.

(3.22) $$u^2 = \alpha,$$

where u is a solution of (3.1) and $\varphi > 0$. It is easy to show that if

(3.23) $$u(\xi) = u'(\eta) = 0$$

and the interval $\xi < x < \eta$ (or $\eta < x < \xi$) does not contain zeros of u or u' then $u^2(x)$ is strictly monotonic in this interval. As $u_n(x)$ (the n-th eigenfunction of (3.1)) has exactly $n+1$ zeros in $[0, \ell]$ and u_n' has exactly n zeros in $[0, \ell]$ the equation (3.22) for a positive α has at most $2n$ solutions. So φ^* and φ^{**} satisfy the equation

(3.24) $$(M - \psi)(\psi - m) = 0$$

except at $2n$ points at most. So $\varphi^* = M$ and $\varphi^{**} = m$ on at most on n intervals. In case of the equation (3.1) the problems (3.7) can be reduced to the appropriate problems involving just the first eigenvalue. Indeed let

(3.25) $$\min_{\varphi \in C} \mu_1(\varphi) = \alpha(M, m, W, \ell), \qquad \max_{\varphi \in C} \mu_1(\varphi) = \beta(M, m, W, \ell)$$

where M, m, w, ℓ are the parameters entering in (3.3) and (3.4). Then Krein proved

(3.26) $$\min_{\varphi \in C} \mu_p(\varphi) = \alpha(M, m, W/p, \ell/p), \qquad \max_{\varphi \in C} \mu_p(\varphi) = \beta(M, m, W/p, \ell/p).$$

The reason for these equalitites is due to the fact that the p-th eigenvalue of (3.1) can be characterized by the first eigenvalue of the equation (3.1) with appropriate *b.c.*

Theorem 3.2. *Let φ be a nonnegative bounded function. Denote by $\mu(x_0, x_1)\,(x_0 < x_1)$ the first eigenvalue of the equation* (3.1) *coupled with b.c. $u(x_0) = u(x_1) = 0$. Then for any $n+1$ points*

(3.27) $$x_0 = 0 < x_1 < \ldots < x_n = \ell$$

we have the inequalities

(3.28) $$\max_{1 \leq i \leq n} \mu(x_{i-1}, x_i) \geq \mu_n \geq \min_{1 \leq i \leq n} \mu(x_{i-1}, x_i)$$

where μ_n is the n-th eigenvalue of (3.1).

The theorem above can be deduced from the min-max and max-min characterization of λ_n. The Krein's result (3.26) tells in particular that $\varphi^* = M$ and $\varphi^{**} = m$ exactly on n disjoint intervals. In order to determine $\alpha(M, m, W, \ell)$ and $\beta(M, m, W, \ell)$ we have to solve some transcendental equation. If one assumes that $m = 0$ then it is possible to have a closed solution (Krein)

(3.29) $$\beta(M, O, W, \ell) = \frac{\pi^2}{W^2} M,$$

(3.30) $$\alpha(M, O, W, \ell) = \frac{4M}{W^2} \chi\left(\frac{W}{M\ell}\right)$$

where $\chi(t)$ $(0 \le t \le 1)$ is defined as the smallest positive root of the equation

(3.31) $$\sqrt{\chi}\ \mathrm{tg}\sqrt{\chi} = t/1 - t.$$

It is possible to extend Krein's results to higher order differential operator L of a special class. Let L be of the form

(3.32) $$L = \frac{1}{\omega_{n+1}}(x)\ D_n\ D_{n-1} \dots D_1, \qquad \omega_{n+1} > 0$$

and

(3.33) $$D_k = \frac{d}{dx}\ \frac{1}{\omega_k}(x), \qquad \omega_k(x) > 0, \qquad \omega_k \in C^{n+1-k}(0, \ell).$$

Assume furthermore that

(3.34) $$n = 2m, \qquad \omega_{2m+2-k} = \omega_k, \qquad k = 1, \dots, m.$$

Then L is a self adjoint differential operator.

If one considers conjugate boundary conditions, e.g.

(3.35) $$f^{(i)}(0) = f^{(i)}(\ell) = 0, \qquad i = 0, \dots, m-1$$

then the corresponding Green's function $G(x, y)$ to $(-1)^m\ L$ coupled with (3.35) is a symmetric oscillating kernel (see [13] for the spectral properties of the oscillating kernels).

In particular all the eigenvalues of the equation

(3.36) $$(-1)^m L u = \mu\varphi u, \qquad u^{(i)}(0) = u^{(i)}(\ell) = 0, \qquad i = 0, \dots m-1$$

are positive and distinct, i.e. (3.3) holds. Furthermore the corresponding p-th eigenfunction u_p of (3.36) vanishes exactly p-1 times in $(0, \ell)$. Of course, we assume that $\varphi \ge 0$. Let C be a nonempty set in $L_\infty(dx)$ of the form

(3.37) $$0 < m(x) \le \varphi(x) \le M(x), \qquad m(x) < M(x), \qquad 0 \le x \le \ell,$$

(3.38) $$\int_0^\ell \varphi(x)\omega_1^2(x)\,dx = W.$$

According to the previous section we have extremal solutions to the problems (3.7). Representing any φ which satisfy (3.37) in the form (3.20) we obtain that θ satisfies the conditions

(3.39) $$0 \le \theta \le 1, \int_0^\ell \theta\,dv = W^1, \ dv = (M(x) - m(x))\,\omega_1^2(x)\,dx.$$

Let v_p and w_p be the corresponding eigenfunctions for $\mu_p(\varphi^*)$ and $\mu_p(\varphi^{**})$ respectively. Using the variational formula (3.9) one obtains

(3.40) $$\begin{aligned} &\int_0^\ell \theta(v_p^2/\omega_1^2)\,dv \le \int_0^\ell \theta^*(v_p^2/\omega_1^2)\,dv, \\ &\int_0^\ell \theta(w_p^2/\omega_1^2)\,dv \ge \int_0^\ell \theta^{**}(w_p^2/\omega_1^2)\,dv, \end{aligned}$$

where

(3.41) $$\varphi^* = m + \theta^*(M - m), \qquad \varphi^{**} = m + \theta^{**}(M - m).$$

Thus, to apply the lemma of Neyman-Pearson one has to consider the equation $u_p^2 = \alpha^2\omega_1^2$, i.e. $u_p = \pm\alpha\omega_1$. Using the fact that u_p has exactly $p-1$ zeros in $(0, \ell)$ from the Rolle theorem we obtain.

Theorem 3.3. *Consider an eigenvalue problem* (3.36) *where L is defined by* (3.32)-(3.34). *Let C be a set of functions given by* (3.37)-(3.38). *Let* φ^* *and* φ^{**} *be extremal solutions to the problems* (3.7). *Then* φ^* *and* φ^{**} *satisfy* (3.24) *except at* $2(2m + p - 1)$ *points in* $[0, \ell]$. *That is* $\varphi^*(x) = M(x)$ *and* $\varphi^*(x) = m(x)$ *on at most* $2m + p - 1$ *disjoint intervals in* $[0, \ell]$.

The detailed proof of this theorem is given in [6].

4. The convoy principle and its applications.

As we saw in the previous sections, a possible characterization of an extremal solutions to eigenvalue problems by use of the variational formula (Theorem 1.1) is only possible when the eigenvalues involved are simples. Then (1.12) shows that a first variation of the eigenvalue is linear in the first variation of the operator. In general case, for example for partial differential equations, the eigenvalues may have a multiplicity of a high order. Moreover, in some examples (e.g. [5]) the maximal eigenvalue has the maximal multiplicity. All these facts point out that these problems should be attacked by another methods. This can be done in case of compact symmetric operators using a max-min characterization of the eigenvalues of symmetric operators.

Let A, $K : \mathscr{H} \to \mathscr{H}$ be linear symmetric operators in a Hilbert space $\mathscr{H}$ with an inner product (x, y). We assume that K is positive definite. In many applications we have to study the spectrum of the operator KA. For example if one studies the frequencies of a vibrating string (3.1), then one rewrites (3.1) in an integral form (2.14) which is of the form KA. Here K corresponds to the Green's function of the appropriated differential equation and Af is a multiplication of f by a density $\varphi(x)$. Assume that KA is compact. This assumption holds if K or A are compact. Since $AK = (KA)^*$, AK is also compact and has the same spectrum as KA. If we introduce a new inner product $\langle x, y \rangle$ in $\mathscr{H}$

$$\langle x, y \rangle = (Kx, y) \tag{4.1}$$

then the operator AK is symmetric with respect to $\langle x, y \rangle$. Thus the spectrum of KA is real and all the nonzero poles of the resolvent of KA are of order 1. We arrange the nonnegative eigenvalues of KA in a decreasing order

$$\lambda_1(KA) \geq \lambda_2(KA) \geq \ldots \geq \lambda_n(KA) \geq \ldots . \tag{4.2}$$

If KA has only $n(\geq 0)$ positive eigenvalues then assume that $\lambda_j(KA) = 0$ for $j > n$. If $\mathscr{H}$ is finite dimensional (n-dimensional) then just assume (4.2). Assume for simplicity of exposition that $\lambda_1(KA) > 0$, $i = 1, 2, \ldots$. Then, there exists a sequence of eigenvectors χ_i such that

$$AK\chi_i = \lambda_i(KA)\chi_i, \qquad (K\chi_i, \chi_j) = \delta_{ij}, \qquad i, j = 1, 2, \ldots, . \tag{4.3}$$

We now state a max-mim principle due to Pólya and Schiffer [20].

Theorem 4.1. (The convoy principle). *Let S be an n-dimensional subspace of $\mathscr{H}$. Assume that $S = [x_1, \ldots, x_n]$ where*

$$(Kx_i, x_j) = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{4.4}$$

Let A and K be symmetric operators, such that K is positive definite and KA is compact. Denote by $A(x_1, \ldots, x_n)$ the following symmetric matrix

$$A(x_1, \ldots, x_n) = (KAKx_i, x_j)_1^n. \tag{4.5}$$

Let $\lambda_i(A, S)$, $i = 1, \ldots, n$ be the eigenvalues of the matrix $A(x_1, \ldots, x_n)$ arranged in a decreasing order. Then

$$\lambda_i(KA) \geq \lambda_i(A, S), \qquad i = 1, \ldots, n. \tag{4.6}$$

If $S = [\chi_1, \ldots, \chi_n]$ then the equality sign holds for $i = 1, \ldots, n$. Furthermore if

$$\lambda_i(KA) = \lambda_i(A, S) \tag{4.7}$$

for some i then S contains an eigenvector of AK which corresponds to the eigenvalue $\lambda_i(KA)$.

A detailed proof Theorem 4.1 can be found in [5].

In fact the convoy principle states that $\lambda_n(KA)$ has the following max-min characterization

$$\lambda_n(KA) = \max_S \min_{x \in S} \frac{(KAKx, x)}{(Kx, x)} \tag{4.8}$$

where S ranges over all possible n dimensional subspaces in $\mathscr{H}$. There is also a min-max characterization of $\lambda_n(KA)$ (e.g. [2])

$$\lambda_n(KA) = \min_T \max_{x \in T} \frac{(KAKx, x)}{(Kx, x)} \tag{4.9}$$

where T ranges over all subspaces in $\mathscr{H}$ of co-dimension $n - 1$. For our purposes the convoy principle is indispensable. Because of the two following reasons.

First, when we consider a maximal problem

$$\max_{A \in C} F(\lambda_1(KA), \ldots, \lambda_n(KA)) \tag{4.10}$$

on some set C of symmetric operators, then we can interchange the maximum over set C with the maximum over the n-dimensional subspaces S entering in the characterization of $\lambda_n(KA)$ (4.8).

Secondly, the minimum problem appearing in (4.8) over the set S is a finite dimensional problem. This to compare with the characterization (4.9) in which the minimum and the maximum are infinite dimensional problems.

Let C be a convex set of linear bounded symmetric nonnegative definite operators. Assume furthermore that we can introduce a locally convex linear topology in C such that C is compact with respect to this topology. (This is always true if $\mathscr{H}$ is finite dimensional and C is bounded). Denote by ε the set of extremal points of C. By the Krein-Milman theorem we have $C = \text{cl co}(\varepsilon)$. Define

$$(4.11) \qquad H_k(\varepsilon) = \left\{ \begin{array}{l} A \mid A = \sum_{i=1}^{k} \alpha_i A_i, \quad A_i \in \varepsilon \\ \alpha_i \geq 0, \quad i = 1, \dots, k, \quad \sum_{i=1}^{k} \alpha_i = 1 \end{array} \right\}$$

The main result in [5] can be stated as follows.

Theorem 4.2. *Let K be a compact positive definite operator. Consider the maximum problem (4.10) where A is restricted by the conditions $f_i(A) = \alpha_i$, $i = 1, \dots, q$ where each f_i is a linear bounded functional on C. Assume that $F: \mathbb{R}^n \to \mathbb{R}$, F continuous and $F(x_1, \dots, x_n)$ is an increasing function of its arguments. Then this maximum is achieved for A^* such that A^* belongs to $H_{n(n+1)/2+q}(\varepsilon)$. In case that $F(x_1, \dots, x_n) = x_n$ and A^* satisfies the inequality $\lambda_{n-1}(KA^*) > \lambda_n(KA^*)$ then A^* belongs $H_{n+q}(\varepsilon)$.*

In case of $\dim \mathscr{H} < \infty$ we just may assume that C is bounded set of symmetric operators. For $q = 0$ the bounds $n(n+1)/2$ and n are sharp. Since Theorem 4.2 is stated in a different form then in [5] we outline a proof of the theorem for a general F. From the compactness assumptions we have

$$(4.12) \qquad \max F(\lambda_1(KA), \dots, \lambda_n(KA)) = F(\lambda_1(KB), \dots, \lambda_n(KB))$$
$$A \in C, \qquad f_i(A) = \alpha_i, \qquad i = 1, \dots, q.$$

So

$$(4.13) \qquad BKy_i = \lambda_i(KB)y_i, \qquad (Ky_i, y_j) = \delta_{ij}, \qquad i,j = 1, \dots, n.$$

Consider a following subset C' of C

$$(4.14) \qquad \begin{array}{l} f_k(A) = \alpha_k, \qquad k = 1, \dots, q \\ (KAKy_i, y_j) = \lambda_i(KB)\delta_{ij}, \quad i,j = 1, \dots, n, \quad i+j < 2n. \end{array}$$

In view of (4.13) C' is not empty. Thus $C' = \text{cl co}(\varepsilon')$ where ε' is a set of the extreme points of C'. Clearly

$$(4.15) \qquad \varepsilon' \subset H_{n(n+1)/2+q}.$$

Let A^* solve the maximal problem

$$(4.16) \qquad \max_{A \in C'} (KAKy_n, y_n) = (KA^*Ky_n, y_n)$$

and $A^* \in \varepsilon' \subset H_{n(n+1)/2+q}(\varepsilon)$.

Let $S = [y_1, \dots, y_n]$. Then from (4.14) and (4.16) $A^*(y_1, \dots, y_n)$ is diagonal and

$$(4.17) \qquad \lambda_i(A^*, S) \geq \lambda_2(KB), \qquad i = 1, \dots, n.$$

Thus from the convoy principle we deduce

$$(4.18) \qquad \lambda_i(KA^*) \geq \lambda_i(KB), \qquad i = 1, \dots, n.$$

As $F(x_1, \dots, x_n)$ increases we have

$$(4.19) \qquad F(\lambda_1(KA^*), \dots, \lambda_n(KA^*)) \geq F(\lambda_1(KB), \dots, \lambda_n(KB)).$$

Now (4.12) implies the equality sign in (4.19) and this establishes Theorem 4.2 for a general F. In case that $F = x_n$ the proof is more delicate. In case that $K = (k_{ij})_1^m$ is an oscillating matrix; C is a set of nonnegative definite diagonal matrices having a trace of magnitude 1; $F = x_n$; $q = 0$; Theorem 4.2 was proved first by P. Nowosad [17]. It is worth to remark that in this case his result is more general then Theorem 4.2 since K is allowed to be nonsymmetric. See Karlin [14] for a refinement of Nowosad results. Consider the maximum problem (2.12) where K is a compact symmetric positive definite operator of the form (2.7) and C is given by (2.10) and (2.11). In section 2 we already demonstrated that the extremal solution exists. Assume that σ is nonatomic. That is if $\sigma(X) > 0$ then there exists two sets in X, $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \phi$ such that $\sigma(X_i) > 0$ for $i = 1,2$. Then we can improve Theorem 4.2 in this case.

Theorem 4.3. *Let σ be a nonnegative nonatomic measure on* I. *Let C be a nonempty set defined by the condition (2.10) and (2.11). Assume that the operator K defined by (2.7) is compact symmetric and positive definite. Consider the extremal problem $\max_C F(\lambda_1(K\varphi), \dots, \lambda_n(K\varphi))$ where F is a con-*

tinuous function on $\mathbb{R}^n_+$. *If F is an increasing function of each of its arguments then this maximum is achieved for some function* φ^* *satisfying* (3.24) *and* (2.11).

The proof of the theorem follows from the arguments given for the proof of Theorem 4.2 and the following lemma.

Lemma 4.1. *Let* σ *be a nonnegative nonatomic measure on I. Consider a set C given by the conditions* (2.10) *and* (2.11). *Assume furthermore that this set is not empty. Then the set of its extreme points is exactly the set of all functions satisfying the equation* (3.24) *almost everywhere with respect to* σ, *together with the conditions* (2.11).

See [8] for the proof of these results. The assumption of Theorem 4.3 that σ is nonatomic can not be removed. Indeed if we allow σ to be atomic, then we are in the situation described by Theorem 4.2. As we pointed out we have extremal solutions which are not extreme point. Thus Theorem 4.3 generalizes the results of section 3 for the problem $\min \mu_n(\varphi)$ (recall $\lambda = \mu^{-1}$). However as Theorem 4.3 is very general more results are needed to characterizé φ^* completely. Even in the simplest cases to characterize φ^* completely is a hard variational problem. Let us apply Theorem 4.3 to vibrating membranes. That is, we have an eigenvalue problem

$$\Delta u + \mu\varphi u = 0, \tag{4.20}$$

in a bounded connected domain $I \subset \mathbb{R}^n$ with a smooth boundary ∂I. Here Δ is the Laplacian $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Assume the Dirichlet boundary conditions

$$u(\partial I) = 0. \tag{4.21}$$

The function $\varphi(\xi)$ is the density of the membrane at the point ξ. The condition (2.10) means that the density φ is bounded from below and from above. Choose $d\sigma$ to be the ordinary Lebesgue measure and let $\rho_0 = 1$. Then the first condition of (2.11) means that the membrane has a fixed mass C_0. The other constraints (2.11) can be interpreted also in physical terms (e.g. [16]). From Theorem 4.3 we deduce

Theorem 4.4. *Consider the membrane* (4.20) *on a bounded connected domain* $I \subset \mathbb{R}^n$ *with a smooth boundary* ∂I. *Assume that the membrane has a fixed total mass* c_0 ($\rho_0 = 1$) *and the density* φ *satisfies the conditions* (2.10) *and* (2.11). *Let F be a continuous function on* $\mathbb{R}^n_+$ *increasing with respect to its arguments. Consider the extremal problem* $\min F(\mu_1(\varphi), \dots, \mu_n(\varphi))$ *in this class of membranes. Then the minimum is achieved for some density* φ^* *in the specified class such that* φ^* *satisfies the equation* (3.24).

In case that I is a ball in $\mathbb{R}^n$ using the Schwarz symmetrization and the classical fact that this symmetrization decreases the integral $\int |\nabla u|^2 dx$ ([21]) one can solve the problem $\min \mu_1(\varphi)$, $\varphi \in C$.

Theorem 4.5. *Let I be a ball in* $\mathbb{R}^n$

$$B(R) = \{x \mid x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad r^2 = \sum_{i=1}^{n} x_i^2 \le R^2\}. \tag{4.22}$$

Let M, m and ω *be function of r. Assume furthermore that M(r) and m(r) are nonnegative and decreasing and* $\omega(r)$ *is positive and increasing for* $0 \le r \le R$. *Let C be a nonempty set of functions defined by* (2.10) *and*

$$\int_I \varphi(\xi)\omega(\xi)\,d\xi = W > 0. \tag{4.23}$$

Then $\min_{\varphi \in C} \mu_1(\varphi) = \mu_1(\varphi^*)$ *where* φ^* *depends only on r,* $\varphi^*(r)$ *is a decreasing function and* φ^* *satisfies* (3.24) *and* (4.23).

For $n = 2$, $M(r) = M$, $m(r) = m$, $\omega(r) = 1$ this theorem was announced by Krein [16], without proof.

The case $n = 2$ and $\omega(r) = 1$ was proved by Nowosad [18].

See [6] for a general case.

We conclude this section with the following result. As we saw in Section 2 when the eigenvalues are not simple the variational formula becomes nonlinear and it is virtually impossible to use the variational method in characterizing the extremal solutions with multiple eigenvalues. However, for a very special functions $F(x_1, \dots, x_p)$ one can have an analogous result which was obtained in Section 3 by using the first variation for simple eigenvalues.

Let $F \in C^{(1)}(\mathbb{R}^n_+)$. We say that ∇F preserves order in $\mathbb{R}^n_+$ if for any $x_1 \ge x_2 \ge \dots \ge x_n \ge 0$ we have $\frac{\partial F}{\partial x_1} \ge \frac{\partial F}{\partial x_2} \ge \dots \ge \frac{\partial F}{\partial x_n} \ge 0$.

Theorem 4.6. *Let* $K\colon L_2\,(d\sigma) \to L_2\,(d\sigma)$ *be a compact positive definite operator of the form* (2.7). *Assume that* $F \in C^{(1)}(\mathbb{R}^n_+)$ *and* ∇F *preserves order in* $\mathbb{R}^n_+$. *Let* ψ *solve the problem* (2.12). *Let* χ_i, $i = 1, \dots, n$ *be the eigenfunctions os* $K\psi$ *satisfying the condition* (2.17) *and* (2.18). *Then* ψ *solves also the linear problem*

$$\sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(\lambda_1(K\psi), \dots, \lambda_n(K\psi)) \int_I \varphi \chi_j^2 d\sigma \le \tag{4.24}$$
$$\le \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(\lambda_1(K\psi), \dots, \lambda_n(K\psi)) \int \psi \chi_j^2 d\sigma$$

for any $\varphi \in C$.

This follows from the following useful lemma which extends the Ky-Fan result [4].

Theorem 4.7. *Let the assumptions of Theorem* 4.1 *holds. Let*

$$\omega_1 \ge \omega_2 \ge \dots \ge \omega_n \ge 0. \textit{ Then}$$

Then

$$\sum_{i=1}^{n} \omega_i (KAKx_i, x_i) \le \sum_{i=1}^{n} \omega_i \lambda_i(KA). \tag{4.25}$$

If the equality sign holds in (4.25) *then* $[x_1, \dots, x_i] = [\chi_1, \dots, \chi_i]$ *if* $\omega_i > \omega_{i+1}$ *for* $i = 1, \dots, n-1$.

See [7] for a proof of this theorem.

5. Eigenvalue problems for measures.

In this section we consider special eigenvalue problems defined on a class of non-negative measures. We characterize the extremal solution in terms of a generalized splines which solves an interesting interpolation problem at its knots. Furthermore we give a fairly explicit way to compute the maximum in question. Proofs of the results appearing in this section are given in [6]. Let $K(x, y)$ be a continuous kernel on $I \times I$, where $I = [a, b]$. Denote by C the set of all non-negative finite measures in $C^*(I)$ normalized by the condition

$$\int_I \rho^2(x) d\sigma = 1, \tag{5.1}$$

where $\rho(x) > 0$ is a continuous function on I. Assume that $K(x, y)$ is a symmetric and positive definite kernel. Let K_σ be an operator defined by (2.7). For any non-negative meassure we have that the spectrum of K_σ is non-negative

$$\lambda_1(\sigma) \ge \lambda_2(\sigma) \ge \dots \ge 0. \tag{5.2}$$

According to Theorem 2.4 each $\lambda_n(\sigma)$ is a continuous functional on C in w^* topology. Clearly, C is a compact set with respect to this topology. The set of the extreme points of C are the measures $\delta(x - \zeta)/\rho^2(\xi)$ where $\delta(x - \zeta)$ is the Dirac measure. According to Theorem 4.2 (see for details [5]) we have

$$\max_{\sigma \in C} \lambda_p(\sigma) = \lambda_p(\sigma^*), \tag{5.3}$$

where

$$\sigma^* = \sum_{i=1}^{k} \alpha_i^* \delta(x - \zeta_i^*), \ a \le \zeta_1^* < \zeta_2^* \dots < \zeta_k^* \le b, \tag{5.4}$$

$$\alpha_i^* > 0 \ i = 1, \dots, k, \quad \sum_{i=1}^{k} \alpha_i^* \rho^2(\zeta_i^*) = 1, \qquad k \le p(p+1)/2.$$

With any measure $\sigma = \sum_{i=1}^{k} \alpha_i \delta(x - \zeta_i) \in C^*(I)$ we associate a general spline

$$f(x) = \sum_{i=1}^{k} a_i K(x, \zeta_i), \ a \le \zeta_1 < \zeta_2 < \dots < \zeta_k \le b. \tag{5.5}$$

The points ζ_i are called the knots of the spline $f(x)$. Let K be a strictly sign consistent kernel of order $n(SSC_n)$.

That is

$$\det(K(x_i\, y_j))_1^n \ge 0, \quad a < {}^{x_1 < x_2 < \dots < x_n}_{y_1 < y_2 < \dots < y_n} < b, \tag{5.6}$$
$$\det(K(x_i, y_j))_1^n > 0, \ a < x_1, y_1 < x_2, y_2 < \dots < x_n, y_n < b.$$

It is known [13] that the Green's function of the differential operator $(-1)^m L$ given by the equalities (3.32)-(3.35) is a SSC_n kernel for $n = 1, 2, \dots,$. If K is the Green's function corresponding to a differential operator M then we can easily obtain from the spline (5.5) the corresponding measure σ by the equality

$$Mf = \sum_{i=1}^{k} \alpha_i \delta(x - \zeta_i) = \sigma. \tag{5.7}$$

Using the classical Jentzsch theorem for n-th compound kernel we deduce that

$$\lambda_n(\sigma) > \lambda_{n+1}(\sigma) \tag{5.8}$$

for a non-negative σ such that $\lambda_n(\sigma) > 0$. From Theorem 4.2 we deduce that $k = p$ in the equality (5.4) if K is a symmetric postive definite and SSC_{p-1} kernel. Let v_p be the p-th eigenfunction of $K_{\sigma*}$. From (2.7) we obtain that v_p is a spline of the form.

$$v_p(x) = \sum_{i=1}^{p} \beta_i K(x, \zeta_i^*). \tag{5.9}$$

Using the analogous to the variation (3.8) we deduce the first inequality in (3.11) where $\psi\, dx = d\sigma \in C$ and $\varphi^* dx = d\sigma^*$. With the help of the lemma of Neyman-Pearson we characterize the properties of the spline v_p. Altogether we obtained.

Theorem 5.1. *Let $K(x, y)$ be a continuous symmetric and positive definite kernel on $I \times I$. Let C be a set of non-negative measures in $C^*(I)$ normalized by the condition (5.1), where ρ is a positive continuous function on I. If K is a strictly sign consistent kernel of order $p - 1$ then there exists a measure σ^* of the form (5.4) with $k = p$ such that σ^* solves the problem (5.3). Let v_p be the p-th eigenfunction of the operator $K_{\sigma*}$. Then v_p has the following properties.*

$$v_p(\zeta_i^*)/\rho(\zeta_i^*) = (-1)^{i-1}, \quad i = 1, \ldots, p, \tag{5.10}$$

$$|v_p(x)/\rho(x)| \le 1, \quad a \le x \le b.$$

It is possible to characterize $\lambda_p(\sigma^*)$ and the points ζ_i^*, $i = 1, \ldots, p$, in a more explicit way. Let $a \le \xi_1 < \ldots < \xi_p \le b$. Denote by $(t_{ij}(\xi_1, \ldots, \xi_p))_1^n$ the inverse matrix of $(K(\xi_i, \xi_j))_1^n$.

Theorem 5.2. *Let the assumptions of Theorem 5.1 hold. Then*

$$\begin{aligned} \lambda_p(\sigma^*) &= \left(\sum_{i,j=1}^{p} |t_{ij}(\xi_1^*, \ldots, \xi_p^*)| \, \rho(\xi_i^*)\, \rho(\xi_j^*) \right)^{-1} \\ &= \max_{a \le \xi_1 < \cdots < \xi_p \le b} \left(\sum_{i,j=1}^{p} |t_{ij}(\xi_1, \ldots, \xi_p)| \, \rho(\xi_i)\, \rho(\xi_j) \right)^{-1}. \end{aligned} \tag{5.11}$$

For a special type kernel which is the Green's function of second order differential operator it is possible to compute explicitly the maximum (5.3).

Theorem 5.3. *Let $\varphi(x)$ and $\psi(x)$ be positive continuous functions on* I. *Assume furthermore that $\varphi(x)/\psi(x)$ strictly increases on I. Let $K(x, y)$ be a kernel of the form*

$$K(x, y) = \begin{cases} \varphi(x)\psi(y), & \text{for} \quad a \le x \le y \le b, \\ \varphi(y)\psi(x), & \text{for} \quad a \le y \le x \le b. \end{cases}$$

Then

$$\max_{\sigma \in C} \lambda_p(\sigma) = \left\{ p + \frac{2(p-1)}{\left[\dfrac{\varphi(b)\,\psi(a)}{\psi(b)\,\varphi(a)}\right]^{1/2(p-1)} - 1} \right\}^{-1}, \tag{5.13}$$

where $\rho^2(x) = \varphi(x)\,\psi(x)$. For $p \ge 2$ the r ximum is obtain for a measure σ^* of the form (5.4) $k = p$. The points ζ_i^*, $i = 1, \ldots, p$, are the unique solutions of the equations

$$\varphi(\zeta_i^*)/\psi(\zeta_i^*) = \left[\frac{\varphi(b)\,\psi(a)}{\psi(b)\,\varphi(a)}\right]^{1/p-1} \varphi(\zeta_{i-1}^*)/\psi(\zeta_{i-1}^*), \; i = 2, \ldots, p-1 \tag{5.14}$$

with $\zeta_1^* = a$ and $\zeta_p^* = b$. For $p = 1$ ζ_1^* is an arbitrary in I.

Note that the kernel $K(x, y)$ which satisfy the assumptions of Theorem 5.3 is symmetric positive definite and SSC_n kernel for any n (e.g. [13]).

6. Compact Riemannian manifolds.

Let $\mathscr{M}$ be a compact smooth (C^∞) n dimensional real Riemannian manifold ($n \ge 2$). We refer to [1] for the definitions and properties of Riemannian manifolds needed in this section. Denote the points of $\mathscr{M}$ by $x = (x^1, \ldots, x^n)$. Let dx stand for the Grassmannian product $|dx^1 \wedge \ldots \wedge dx^n|$. The metric is given by the matrix $G(x) = (g_{ij}(x))_1^n$. Denote by g the determinant of G. The volume element dV is given is given by $\sqrt{g}\, dx$. Let $L^2(\mathscr{M})$ be the set of measurable square integrable functions f such that

$$\int_{\mathscr{M}} f^2\, dV < \infty.$$

Thus $L_2(\mathscr{M})$ is the usual Hilbert space with the inner product $(f, g) = \int_{\mathscr{M}} fg\, dV$. Denote by $C^\infty(\mathscr{M})$ the set of smooth functions defined on $\mathscr{M}$.

Let Δ be the Laplacian corresponding to the metric G. Consider the eigenvalue problem

$$\Delta u + \mu u = 0 \tag{6.1}$$

on $\mathscr{M}$. It is known

$$\Delta u_k + \mu_k u_k = 0, \qquad k = 0, 1, \ldots \tag{6.2}$$

(6.3) $$\int_{\mathcal{M}} u_i u_j \, dV = \delta_{ij}, \qquad i,j = 0, 1, \dots$$

(6.4) $$0 = \mu_0 < \mu_1 \le \mu_2 \le \dots .$$

The function $u_1, \dots$ are smooth and $\{u_k\}_0^\infty$ is an orthonormal basis in $L_2(\mathcal{M})$. Clearly $u_0 = const.$ The eigenvalues μ_k can be characterized by the Rayleigh ratio

(6.5) $$\int_{\mathcal{M}} \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \, dV \Big/ \int_{\mathcal{M}} u^2 \, dV.$$

Here $G^{-1} = (g^{ij})_1^n$. Consider a new metric on $\mathcal{M}$ which is given by the matrix $\hat{G} = (\hat{g}_{ij})_1^n$. Assume that this metric is conformal to the old metric G. So

(6.6) $$\hat{g}_{ij}(x) = \varphi^2(x)\, g_{ij}(x).$$

Here $\varphi(x)$ is a nonnegative bounded function, i.e. $\varphi \in L_\infty(\mathcal{M})$. Suppose first that φ is smooth and positive. Denote by $\hat{\Delta}$ the corresponding Laplacian to the metric $\hat{G}$. Consider the eigenvalue problem

(6.7) $$\hat{\Delta} u + \mu u = 0.$$

Denote by $\{\mu_k(\varphi)\}_0^\infty$ the corresponding eigenvalues of $\hat{\Delta}$. The eigenvalues $\mu_k(\varphi)$ are given by the Rayleigh ratio

(6.8) $$\int_{\mathcal{M}} \varphi^{n-2} \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \, dV \Big/ \int_{\mathcal{M}} \varphi^n u^2 \, dV.$$

As in Section 4 $\mu_p(\varphi)$ can be characterized by the Convoy Principle.

Theorem 6.1. (The Convoy Principle). *Let $f_0, \dots, f_p$ be smooth functions. Assume that*

(6.9) $$\int_{\mathcal{M}} f_i f_j \varphi^n \, dV = \delta_{ij}, \qquad i,j = 0, 1, \dots, p.$$

Let $A(\varphi, f_0, \dots, f_p)$ be the matrix with the entries

(6.10) $$a_{ij} = \int_{\mathcal{M}} \varphi^{n-2} \left(\sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \frac{\partial f_i}{\partial x^\alpha} \frac{\partial f_j}{\partial x^\beta} \right) dV,$$

$i,j = 0, \dots, p$. Denote by $\mu_0(\varphi, f_0, \dots, f_p) \le \dots \le \mu_p(\varphi, f_0, \dots, f_p)$ the eigenvalues of $A(\varphi(f_0, \dots, f_p))$ arranged in the increasing order. Then

(6.11) $$\mu_p(\varphi) = \min_{f_0, \dots, f_p} \mu_p(\varphi, f_0, \dots, f_p).$$

The minimum is achieved for the eigenfunctions $u_0, \dots, u_p$ of (6.7).

Since the Rayleigh ratio is defined for any nonnegative bounded measurable function $\varphi (\neq 0)$ let $\mu_p(\varphi)$ be given by (6.11) (it may be needed to replace min by inf). Let $0 \le m < M$ be functions in $L_\infty(\mathcal{M})$. Denote by C the set of functions φ satisfying the inequality $m \le \varphi \le M$ and normalized by the condition

(6.12) $$\int_{\mathcal{M}} \varphi^n \, dV = W.$$

This set of functions has the following geometric meaning. Assume that M and m are positive and constant. Then the condition $m \le \varphi \le M$ means that the metrics G and $\hat{G}$ are equivalent. That is

(6.13) $$md(x, y) \le \hat{d}(x, y) \le Md(x, y)$$

where $d(x, y)$ and $\hat{d}(x, y)$ are the distances between the points x and y according to the metrics G and $\hat{G}$ respectively. The condition (6.12) means that the new manifold $\hat{\mathcal{M}}$ has a fixed volume W. Let $F(\xi_1, \dots, \xi_p)$ be a continuous function on R_+^p. Consider a minimal problem

(6.14) $$\min_{C} F(\mu_1(\varphi), \dots, \mu_p(\varphi)).$$

We conjecture

Conjecture 1. *There exists an extremal function φ^* in C such that the minimum (6.14) is attained at φ^*.*

In [9] we prove

Theorem 6.2. *Let $\mathcal{M}$ be a compact smooth Riemannian manifold of dimension $n \ge 2$. Let C be a nonempty set of functions defined by (3.4) and (6.12). Let $F(\xi_1, \dots, \xi_p)$ be a continuous increasing function of its arguments in $\mathbb{R}^p$. Then*

(6.15) $$\inf_{C} F(\mu_1(\varphi), \dots, \mu_p(\varphi)) = \inf_{C*} F(\mu_1(\psi), \dots, \mu_p(\psi))$$

where C^* is the set of functions satisfying (3.24) and (6.12).

Let us consider a special case in which we can solve at least one nontrivial problem of the form (6.15). Consider a two dimensional compact Riemannian manifold, i.e. $n = 2$. As in the Rayleigh ratio $\varphi^{n-2} = 1$ we have that $\mu_p(\varphi)$ are the eigenvalues of the equation

$$\Delta u + \mu\varphi^2 u = 0 \tag{6.16}$$

where Δ is the original Laplacian.

Let $\mathscr{M}$ be the unit sphere S^2

$$S^2 = \{x \mid x = (x^1, x^2, x^3), \sum_{i=1}^{3} (x^i)^2 = 1\}. \tag{6.17}$$

Assume that $0 \le m < M$ are constants. Consider the problem $\min \mu_1(\varphi)$, $\varphi \in C$. As in the case of the circular membrane to characterize the minimal φ^* we have use the symmetrization principle (e.g. [10]).

Theorem 6.3. *Let S^2 be the unit sphere of the form (6.17). Let $M > m \ge 0$ be constants. Denote by C a nonempty set of measurable functions on S^2 satisfying the conditions (3.4) and (6.12). Consider the problem $\min \mu_1(\varphi)$ on C, where $\mu_1(\varphi)$ is the first nontrivial eigenvalue of (6.16). Then this minimum is achieved for a function $\varphi^* = \varphi^*(x_3)$ of the form*

$$\begin{aligned} &\varphi^*(x_3) = M \text{ for } -1 \le x_3 \le h_1,\ h_2 \le x_3 \le 1, \\ &\varphi^*(x_3) = m \text{ for } h_1 < x_3 < h_2. \end{aligned} \tag{6.18}$$

The eigenvalue $\mu_1(\varphi^*)$ is the first nontrivial eigenvalue of the problem

$$\frac{d}{dt}\left[(1-t^2)\frac{du}{dt}\right] + \mu\varphi^*(t)^2 u = 0 \tag{6.19}$$

$$u'(-1) = u'(1) = 0 \tag{6.20}$$

The difference $h_2 - h_1$ is determined by the equation (6.12)

$$2\pi\{m^2(h_2 - h_1) + M^2[2 - (h_2 - h_1)]\} = W. \tag{6.21}$$

Furthermore, the corresponding solution u of (6.19) has to satisfy either the condition

$$u(h_2) = -u(h_1) \tag{6.22}$$

if

$$-1 < h_1 \le h_2 < 1 \tag{6.23}$$

or the condition

$$0 < u(-1) \le -u(h_2) \tag{6.24}$$

if

$$h_1 = -1. \tag{6.25}$$

(Note that $\varphi^*(-x_3)$ is also extremal thus if (6.23) does not hold we may assume (6.25)).

Unfortunately, Theorem 6.3 does not characterize $\varphi^*(t)$ completely, and we have to determine explicitly the value of h_1. We conjecture.

Conjecture 2. *Let the assumptions of Theorem 6.3 hold. Then the extremal function φ^* given by (6.18) is an even function of x_3, i.e. $h_2 = -h_1$.*

Note that if φ^* is even then the corresponding eigenfunction u is odd and the condition (6.22) trivially holds. It can be shown that this conjecture holds for $m = 0$. See for details [9]. In conclusion, let us recall the result due to Hersch [11]

$$\max_{\varphi \in C} \mu_1(\varphi) = \mu_1(\varphi^{**}) = 8\pi/W. \tag{6.26}$$

Here C is the set considered in Theorem 6.3 and φ^{**} is a constant function equal to $(W/4\pi)^{1/2}$. Thus φ^{**} is not an extreme point in C, contrary to the Krein result for the vibrating string.

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