Conjugation of two finite subnormal subgroups

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Abstract. *A necessary and sufficient condition for two finite subnormal subgroups to be conjugate is presented in this paper.*

The present short note is devoted to the study of conjugacy relationship of finite subnormal subgroups of a group G. That is, given $A = G_{10} \leq \leq \ldots \leq G_{1n} = G$ and $B = G_{20} \leq \ldots \leq G_{2m} = G$, where G_{1i} is a normal subgroup of $G_{1,i+1}$ and G_{2j} is a normal subgroup of $G_{2,j+1}$ for $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$, we seek conditions which guarante the existence of g in G such that $A = g^{-1}Bg$. We present such a condition in Theorem 6. Also we give an example showing that the following necessary condition is not a sufficient condition for two finite subnormal subgroups A and B to be conjugate: Any Sylow subgroup of A is conjugate to a subgroup of B and any Sylow subgroup of B is conjugate to a subgroup of A. However we show in Proposition 9 that the above condition is a sufficient condition if the local structure of A is somewhat restricted. This work resulted from a conversation with Prof. Wielandt on this subject.

Some remarks on the terminology and notation are the following. For any subset S of G, the subnormal closure of S is the intersection of all subnormal subgroups of G containing S. For $g \in G$ we write $S^g = g^{-1}Sg$ and $S^G = \langle S^g \mid g \in G \rangle$. The notation $S \underset{G}{\to} B$ means that there exists $g \in G$ such that $S^g \leq B$.

Let T be any subset of G. Then $C_S(T) = \{s \varepsilon S \mid st = ts$ for all $t \varepsilon T\}$ and $[S, T]$ is the subgroup generated dy $s^{-1}t^{-1}st$, where $s \varepsilon S$ and $t \varepsilon T$. Let π be a set of prime numbers. A π-subgroup of G is a subgroup H of G such that either $H = 1$ or all primes divisors of $|H|$ belongs to π.

For the convenience of the reader we record two known results which will be used.

Lemma 1. *Let G be a finite group and let π be a set of prime numbers. Define πG to be the subgroup of G generated by all π-subgroups of G. Then $G = \pi G$ if and only if the index of each maximal normal subgroup of G contains a prime factor from π.*

Proof. 3.2 of [2].

Lemma 2. *Let M be an abelian normal q-subgroup of the finite group G, where q is a prime. Let P be a subgroup of G of order prime to q. Then $M = C_M(P) \times [M, P]$.*

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Proof. Theorem 2.3 on page 117 of [1].

Lemma 3. *Let A and B be two subnormal subgroups of a group G. Suppose $S \leq A$ and $S^A = A$. Then the subnormal closure of S in G equals A. In particular $S \underset{G}{\to} B$ implies $A \underset{G}{\to} B$ in this case.*

Proof. By the general fact that two conjugate subsets have their subnormal closures also conjugate, the second conclusion follows from the first. Let the subnormal closure of S be T. Since A is a subnormal subgroup, $T \leq A$. If $T \neq A$, then $T^A \underset{+}{\leq} A$. However we have $A = S^A \leq T^A$. Therefore $T = A$ as desired.

Proposition 4. *Let p be a prime and let P be a Sylow p-subgroup of a finite subnormal subgroup A of a group G. Suppose every proper maximal normal subgroup of A has index divisible by p. If $P \underset{G}{\to} B$, where B is a subnormal subgroup of G, then $A \underset{G}{\to} B$.*

Proof. By Lemma 1 we see that $A = P^A$. Now Lemma 3 completes the proof of the proposition.

Lemma 5. *Let A be a finite subnormal subgroup of a group G and p a prime. Suppose B is a subnormal subgroup of G. If $N_A(P) \underset{G}{\to} B$ for some Sylow p-sbgroup P of A, then $A \underset{G}{\to} B$.*

Proof. By Lemma 3 it suffices to show that $A = (N_A(P))^A$. Let $M = (N_A(P))^A$. Since $N_A(P) \leq M$, $N_A(M) = M$. Since M is normal in A, $M = A$. The proof of the lemma is complete.

Theorem 6. *Let A and B be two finite subnormal subgroups of a group G. Let p be a prime, P a Sylow p-subgroup of A and Q a Sylow p-subgroup of B. Then A is conjugate to B if and only if $N_A(P) \underset{G}{\to} B$ and $N_B(Q) \underset{G}{\to} A$.*

Proof. Clear from Lemma 5.

Example 7. Let p and q be two different primes. Let $S \cong Z_p$ and $T \cong Z_q$. Let G be the wreath product of $S \times T$ by Z_2, i.e., $G = A_1 \times B_1 \times A_1 \times B_2) \cdot \langle x \rangle$, where $x^2 = 1$, $A_1^x \cong A_2 \cong Z_p$ and $B_1^x = B_2 \cong Z_q$. Let $A = A_1 \times B_1$ and $B = A_1 \times B_2$. Then A and B are two non conjugate finite subnormal subgroups of G. Note that any Sylow subgroup of A is conjugate to a subset of B and vice versa.

Lemma 8. *Let A be a finite group such that every element of A has prime power order. Then $A = S^A$ for some Sylow s-subgroup S of A.*

Proof. We apply induction on $|A|$. Let M be a minimal normal subgroup of A. Then M is either an elementary abelian p-group or a non-abelian simple group. If $M = A$, then the result is clear. Suppose $M \neq A$. Let $\bar{A} = A/M$. By induction we have $\bar{A} = \bar{P}^{\bar{A}}$ for some Sylow p-subgroup $\bar{P}$ of $\bar{A}$. Let P be a Sylow p-subgroup of A such that $PM/M = P$. If M is non-abelian, then M is the unique minimal normal subgroup of A. Hence $M \leq P^A$ and $P^A = A$ in this case. If M is a p-group, then $M \leq P$ and $P^A = A$. Suppose M is an elementary abelian q-group, where $p \neq q$. By Lemma 2 we have $M = C_M(P) \times [M, P]$. Since $q \neq p$, $C_M(P) = 1$. Hence $M = [M, P] \leq P^M \leq P^A$. Therefore $P^A = A$. The proof of the lemma is complete.

Proposition 9. *Let A be a finite subnormal subgroup of a group G such that any element of A has prime power order. Let B be a subnormal subgroup of G. If for every prime p there exists a Sylow p-subgroup P of A such that $P \underset{G}{\to} B$, then $A \underset{G}{\to} B$.*

Proof. This is a consequence of Lemmas 3 and 8.

In the next example we show the following condition is not a sufficient condition for two finite subnormal subgroups A and B to be conjugate. For any two prime divisors p and q of $|A|$, a Hall ${}_{p,q}$-subgroup of A is conjugate to a subset of B and vice versa.

Example 9. Let p, q and r be three distinct primes. Let K be the subgroup of Σ_6 generated by $x = (25)(36)$ and $y = (14)(36)$. Let $V = P_1 \times P_2 \times P_3 \times P_4 \times P_5 \times P_6$, where $P_1 \cong P_4 \cong Z_p$, $P_2 \cong P_5 \cong Z_q$ and $P_3 \cong P_6 \cong Z_r$. Let G be the semi-direct product of V by K such that K acts on $\{P_1, \dots, P_6\}$ in the same way as K acts on $\{1, \dots, 6\}$.

Let $A = P_1 \times P_2 \times P_3$ and $B = P_1 \times P_2 \times P_6$. Then A and B are two non-conjugate finite subnormal subgroups of G. However $P_1 \times P_2 = P_1 \times P_2$, $(P_2 \times P_3)^y = P_2 \times P_6$ and $(P_1 \times P_3)^x = P_1 \times P_6$.

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