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On global solutions of a nonlinear dispersive equation of Sobolev type

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1. Introduction.

In this note we shall study the so-called generalized Benjamin-Bona-Mahony equation

$$(1.1) \qquad u_t + u_x - u_{xxt} + (f(u))_x = g(x, t)$$
$$u(x, 0) = \varphi(x)$$

in $-\infty < x < \infty$, $t \geq 0$. Here the subscripts denote partial derivatives; suitable condition on f, g and φ will be given later. Our main purpose will be to show the existence and uniqueness of global classical solutions of (1.1). Furthermore, we will prove the continuous dependence on initial data. Also, in the last section, we shall study the linear part of (1.1), that is

$$u_t + u_x - u_{xxt} = 0$$

by using the stationary phase method, in order to obtain its asymptotic behavior at $t \to \infty$, which eventually "could" preclude the asymptotic behavior of (1.1) (with $g \equiv 0$).

The equation (1.1) was suggested by T. B. Benjamin in [2]. This note is a more complete version of an invited lecture presented by the authors at the Second Brazilian Seminar in Analysis held at the University of São Paulo (USP) in 1975. In our opinion the problem (1.1) is still of interest, because of its close relationship with the generalized Korteweg-de Vries equation, i.e.

$$u_t + u_{xxx} + (f(u))_x = 0$$

which has been extensively studied in recent years. The final word concerning the properties of solutions of (1.1) is, in our opinion, far from being satisfactory.

Related work which has been done in recent years besides Benjamin-Bona-Mahony's paper, [1], $\left(\text{where they treated the particular case } f(s) = \frac{s^2}{2}\right)$, were concerned mainly with periodic solutions (in x), see [4], [5], [6], [7] among others.

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One last word about the title: it has been customary to call the equation (1.1) a Sobolev type equation (see [8]) because S. Sobolev studied and introduced similar equations for fluid flow problems. We are very grateful to Prof. J. Cooper for his interesting comments about this work.

2. Existence and uniqueness of regular solutions.

Let $T > 0$ and let us consider the space $C^0(\mathbb{R} \times [0, T])$ of all real-valued continuous functions φ: $\mathbb{R} \times [0, T] \to \mathbb{R}$. We denote by $L^\infty(\mathbb{R} \times [0, T])$ the space of all real-valued measurable essentially bounded functions on $\mathbb{R} \times [0, T]$. Let $X(T) = C^0(\mathbb{R} \times [0, T]) \cap L^\infty(\mathbb{R} \times [0, T])$ with the norm $\| \ \|_{X(T)}$ given by $\|\varphi\|_{X(T)} = \sup_{(x,t)\in\mathbb{R}\times[0,T]} |\varphi(x, t)|$, $\varphi \in X(T)$. We also consider the space $Y(T)$ of all real-valued functions h: $\mathbb{R} \times [0, T] \to \mathbb{R}$ such that $h(\cdot, t) \in L^2(\mathbb{R})$ (i.e. the space of square integrable functions) for each $t \in [0, T]$ and such that the map $t \to h(\cdot, t)$ is continuons in $[0, T]$. The norm in $Y(T)$ is given naturally by

$$\|h\|_{Y(T)} = \sup_{t\in[0,T]} \|h(\cdot, t)\|_{L^2(\mathbb{R})} \cdot$$

Obviously, $X(T)$ and $Y(T)$ are Banach spaces. Observe that whenever $g \in Y(T)$ then $F(g) \in X(T)$, where

$$F(g)(x, t) = \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} e^{-|x-y|} g(y, s) dy ds. \tag{2.1}$$

Let us denote by $G(x - y) = \frac{1}{2} \operatorname{sign}(x - y) e^{-|x-y|}$, where

$$\operatorname{sign}(s) = \begin{cases} 1 & \text{if } s > 0 \\ -1 & \text{if } s < 0 \end{cases}$$

and let f be a real-valued function which satisfies the following condition: there exists a continuous function $M(t)$, non-negative, such that for $u, v \in X(T)$ we have

$$|f(u(x, t)) - f(v(x, t))| \leq M(t) \ |u(x, t) - v(x, t)|.$$

We consider now the map S: $X(T) \to X(T)$ given by

$$(Su)(x, t) = u_0(x) + Au(x, t) + F(g)(x, t) \tag{2.2}$$

where u_0 is a (fixed), uniformly bounded, continuous function on $\mathbb{R}$ and

$$Au(x, t) = \int_0^t \int_{-\infty}^{\infty} G(x - y) [u(y, r) + f(u(y, r))] dy dr.$$

Theorem 1.2. *Let $\rho > 0$, then there exists a small $T_0 > 0$ such that, whenever $g \in Y(T_0)$ and a uniformly continuous function u_0 on $\mathbb{R}$ are given, with $\|F(g)\|_{X(T_0)}$ and $\sup_{x\in\mathbb{R}} |u_0(x)|$ small enough, the map S (given by (2.2)) has a unique fixed point on $X(T_0)$.*

Proof. Let us denote by $\delta = \sup_{x\in\mathbb{R}} |u_0(x)|$ and let us take $T_0 > 0$ small enough such that $\delta + T_0(1 + M(T_0))\rho < \rho$. This is possible because $M(t)$ is continuous. Thus, if we take $g \in Y(T_0)$ such that

$$\|F(g)\|_{X(T_0)} \leq \rho - \delta - T_0(1 + M(T_0))\rho$$

then we claim that S (in (2.2)) is a contraction map which takes $\{\varphi \in X(T_0), \|\varphi\|_{X(T_0)} \leq \rho\}$ into itself. In fact, a straightforward calculation shows that if $\varphi_1, \varphi_2 \in X(T_0)$, then

$$\|S(\varphi_1) - S(\varphi_2)\|_{X(T_0)} = \|A\varphi_1 - A\varphi_2\|_{X(T_0)} \leq T_0(1 + M(T_0))\|\varphi_1 - \varphi_2\|_{X(T_0)} \tag{2.3}$$

because

$$\int_{-\infty}^{\infty} |G(x - y)| dy = 1.$$

Furthermore, if we take $\varphi_2 \equiv 0$ and φ_1 such that $\|\varphi_1\|_{X(T_0)} \leq \rho$ in (2.3), it follows that

$$\|S(\varphi_1)\|_{X(T_0)} \leq \delta + \|F(g)\|_{X(T_0)} + T_0(1 + M(T_0))\|\varphi_1\|_{X(T_0)} \leq \rho \cdot$$

which completes the proof, because $T_0(1 + M(T_0)) < 1$.

Remarks.

1) The unique fixed point $u \in X(T_0)$ of S in the above theorem is (by the standard theorem on contraction maps) the limit of the sequence

$$\{u_m, m = 1, 2, \ldots\} \quad \text{where} \quad u_0(x, t) \equiv u_0(x),$$

$$u_m(x, t) = Au_{m-1}(x, t) + F(g)(x, t).$$

2) Theorem 1.2 shows that there exists a unique (local) solution $u \in X(T_0)$ of the integral equation

$$u(x,t) = u_0(x) + \int_0^t \int_{-\infty}^{\infty} G(x-y)\left[u + f(u)\right] dy dr + \tag{2.4}$$

$$+ \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} e^{-|x-y|} g(y,r) dy dr$$

3) The procedure above could be generalized in order to consider the case in which $f = f(u, u_x, t)$.

Theorem 2.2. *Let $u_0(x)$ and g remain the same as in Theorem 1.2. Let us suppose also that $g(x,\cdot)$ is a continuous function on $[0, T_0]$. Then the (unique) fixed point of S obtained in Theorem 1.2 satisfies*

$$\operatorname*{Lim}_{|x|\to\infty} |u(x,t)| = 0 \quad \text{if} \quad \operatorname*{Lim}_{|x|\to\infty} |u_0(x)| = 0.$$

Proof. Observe that $F(g)(x,t) = \int_0^t (R*g)(x,r)dr$ where $R(x) = \frac{1}{2}e^{-|x|}$ and $*$ denotes spatial convolution. Thus, $F(g)$ is the inverse Fourier transform of $(1+\xi^2)^{-1}\int_0^t \hat{g}(\xi,r)dr$ (where $\hat{g}$ denotes the Fourier transform in ξ of g). Since $\int_0^t \hat{g}(\xi,r)dr$ belongs to $L^1(\mathbb{R})$, then $(1+\xi^2)^{-1}\int_0^t \hat{g}(\xi,r)dr$ belongs to $L^1(\mathbb{R})$, for each $0 \le t \le T_0$. Thus, the Riemann-Lebesgue theorem implies that $\operatorname*{Lim}_{|x|\to\infty} |F(g)(x,t)| = 0$ for each $0 \le t \le T_0$. Because of this fact and remark 1 above, in order to prove the theorem it is enough to show that $\operatorname*{Lim}_{|x|\to\infty} |Au_m(x,t)| = 0$, $m = 0, 1, \ldots$, for each $t, 0 \le t \le T_0$. Also, because of the definition of the operator A, it will be enough to show by induction that

$$\operatorname*{Lim}_{|x|\to\infty} \left| \int_{-\infty}^{\infty} G(x-y) u_m(y,s) dy \right| = 0 \tag{2.5}$$

and

$$\operatorname*{Lim}_{|x|\to\infty} \left| \int_{-\infty}^{\infty} G(x-y) f(u_m(y,s)) dy \right| = 0. \tag{2.6}$$

Let $m = 0$. We fix $z_0 \in \mathbb{R}$, then by dividing the integral (2.5) into two parts (one from $-\infty$ to z_0 and the other from z_0 to $+\infty$) and taking $x \ge z_0$ we obtain

$$\left| \int_{-\infty}^{\infty} G(x-y) u_0(y) dy \right| \le \frac{e^{-x}}{2} \int_{-\infty}^{z_0} e^y |u_0(y)| dy + \sup_{y \ge z_0} |u_0(y)|.$$

Let $\varepsilon > 0$, by choosing z_0 sufficiently large the term $\sup_{y \ge z_0} |u_0(y)|$ can be made less than $\varepsilon/2$. After this, we take x large enough so that

$$\frac{e^{-x}}{2} \int_{-\infty}^{z_0} e^y |u_0(y)| dy < \varepsilon/2.$$

This shows (2.5) for $m = 0$. Similar discussion shows (2.6) when $m = 0$. (Because we can assume $f(0) = 0$ without loss of generality.) Now, the rest of the proof is standard and can be complete by induction by repeated use of the Lebesgue convergence dominated theorem.

Theorem 3.2. *Let $u_0(x)$ be a twice continuously differentiable function which is also uniformly bounded in R. Suppose that f is a continuously differentiable function and $g \in Y(T_0)$, $g(x,\cdot)$ is a continuous function on $[0, T_0]$ (where $T_0 > 0$ is obtained as in Theorem 1.2) then the solution u of (2.4) in $\mathbb{R} \times [0, T_0]$ is a pointwise solution of the Cauchy problem (1.1) with $u(x,0) = u_0(x)$.*

Proof. By dividing the range of integration of (2.4) at $y = x$ and because the right side (of (2.4)) has a partial derivative (in x), it follows that

$$u_x(x,t) = \frac{d}{dx} u_0(x) + \int_0^t (u + f(u)) dr + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} e^{-|x-y|} (u + f(u)) dy dr +$$

$$+ \int_0^t \int_{-\infty}^{\infty} G(x-y) g(y,r) dy dr.$$

Now, by noticing that

$$\operatorname*{Lim}_{|y|\to\infty} e^{-|x-y|}\left[u(y,r) + f(u(y,r))\right] = 0$$

for each $x \in \mathbb{R}$, $0 < r < T_0$, because of Theorem 2.2 and the Lipschitz condition satisfied by f (there is no loss of generality to assume that $f(0) = 0$). Thus, by a similar procedure as above we obtain

$$u_{xx}(x,t) = \frac{d^2}{dx^2}u_0(x) + \int_0^t \frac{\partial}{\partial x}(u + f(u))dr + u(x,t) - u_0(x) - \int_0^t g(x,r)\,dr$$

therefore

$$u_{xxt} = u_x + (f(u))_x + u_t - g(x,t)$$

which proves the theorem.

Now we shall obtain a priori estimatives which will allow us to repeat our argument as many times as we wish. By multiplying equation (1.1) by u and integrating (in space) we get

$$\text{(2.8)} \qquad \int_{-\infty}^{\infty} u(f(u))_x dx = -\int_{-\infty}^{\infty} (h(u(x,t)))_x dx = 0$$

because $\int_{-\infty}^{\infty} u(f(u))_x dx = -\int_{-\infty}^{\infty} u_x f(u) dx$. Thus, from (2.7), (2.8) and the Schwarz inequality we get

$$\text{(2.7)} \qquad \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}[u^2 + u_x^2]dx + \int_{-\infty}^{\infty} u(f(u))_x dx = \int_{-\infty}^{\infty} u\,g(x,t)\,dx.$$

Since $u_0(x)$ vanish as $|x| \to \infty$ and it is twice continuously differentiable, then by Theorem 2.2 and integration by parts we obtain

$$-\int_{-\infty}^{\infty} uu_{xxt}dx = \int_{-\infty}^{\infty} u_x u_{xt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2(x,t)dx.$$

Also, if we denote by $h(r) = \int_{-\infty}^{r} f(y)dy$, then it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(u^2 + u_x^2)dx \le \|u\|_{L^2}\|g\|_{Y(T_0)} \le \int_{-\infty}^{\infty}(u^2 + u_x^2)dx + \|g\|^2_{Y(T_0)}.$$

So, by using Gronwall's inequality, we obtain

$$\text{(2.9)} \qquad \int_{-\infty}^{\infty}(u^2 + u_x^2)dx \le C$$

for all $0 \le t \le T_0$. The positive constant C depends only on g, u_0 and T_0. Then (2.9), together with the standard imbedding theorem $H^1(\mathbb{R}) \subset C^0(\mathbb{R}) \cap \cap L^2(\mathbb{R})$ [where $H^1(\mathbb{R})$ is the Sobolev space of functions $f \in L^2(\mathbb{R})$ whose first derivatives, in the sense of distributions, also belongs to $L^2(\mathbb{R})$], shows that our argument can be repeated as many times as we wish.

Let $T > 0$. The uniqueness of the solution (on $\mathbb{R} \times [0,T]$) obtained above, may be shown as follows: suppose that u and v are solutions of (1.1) with the same initial data at $t = 0$. Thus, we consider $w(x,t) = u(x,t) - v(x,t)$, therefore, w satisfies the equation

$$\text{(2.10)} \qquad w_t + w_x - w_{xxt} + w(f(u) - f(v))_x = 0$$

with $w(x,0) = 0$. Multiplying (2.10) by w and integrating we obtain

$$\text{(2.11)} \qquad \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(w^2 + w_x^2)dx = \int_{-\infty}^{\infty} w(f(u) - f(v))dx$$

because $\int_{-\infty}^{\infty}(w^2)_x dx = 0$. Integration by parts of (2.11) gives us

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(w^2 + w_x^2)dx = \int_{-\infty}^{\infty} w_x(f(u) - f(v))dx \le M(t)\int_{-\infty}^{\infty}(w^2 + w_x^2)dx.$$

Therefore, by Gronwall's inequality we get $w \equiv 0$ on $\mathbb{R} \times [0,T]$, because $w(x,0) = 0$.

3. On the continuous dependence on the initial data.

Let $T > 0$, fixed but otherwise arbitrary. Let $u = u(x,t)$ and $u_n = u_n(x,t)$ be the C^∞ — solutions of (1.1) with initial data $u(x,0) = \varphi_0(x)$ and $u_n(x,0) = \varphi_n(x)$ respectively, where φ_0, $\varphi_n(x)$ belong to $C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and they vanish as $|x| \to \infty$. Let $w_n(x,t) = u(x,t) - u_n(x,t)$. Thus w_n satisfies the equation

$$\text{(3.1)} \qquad \frac{\partial}{\partial t}w_n + \frac{\partial}{\partial x}w_n - \frac{\partial^3}{\partial x^2 \partial t}w_n + \frac{\partial}{\partial x}(f(u) - f(u_n)) = 0$$

with $w_n(x,0) = \varphi_0(x) - \varphi_n(x)$. If we multiply (3.1) by w_n and integrate, we find according to the same reason used in the end of the last section, that

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left[w_n^2+\left(\frac{\partial}{\partial x}w_n\right)^2\right]dx=\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}w_n\right)[f(u)-f(v)]dx\leq$$

$$\leq M(t)\int_{-\infty}^{\infty}\left[w_n^2+\left(\frac{\partial}{\partial x}w_n\right)^2\right]dx.$$

So,

$$\int_{-\infty}^{\infty}\left[w_n^2+\left(\frac{\partial}{\partial x}w_n\right)^2\right]dx\leq$$

$$\leq 2\left[\int_{-\infty}^{\infty}\left[(\varphi_n(x))^2+\left(\frac{\partial}{\partial x}\varphi_n(x)\right)^2\right]dx\right]\ \exp\left(\int_0^t M(s)dx\right).$$

Therefore, if $\varphi_n(x)$ converges to $\varphi_0(x)$ as $n\to\infty$, in the norm of $H^1(\mathbb{R})$, it follows that u_n will converge to u as $n\to\infty$ in the same norm, for $0\leq t\leq T$.

4. Asymptotic behavior of the linear part of (1.1).

In this section we analyse the linear part of (1.1) that is, we consider the problem

$$\text{(4.1)}\qquad \begin{aligned} &v_t+v_x-v_{xxt}=0\\ &v(x,0)=\varphi(x)\end{aligned}$$

in $-\infty<x<\infty$, $t\geq 0$. Here the initial condition $\varphi(x)$ is assumed to belong to the Schwartz space of rapidly decreasing functions on $\mathbb{R}$. By taking the Fourier transform (in x) of $v(x,t)$ in (4.1) we see that

$$\hat{v}(\xi,t)=\exp(i\xi t(1+\xi^2)^{-1})\,\hat{\varphi}(\xi).$$

So, we may write

$$\text{(4.2)}\qquad v(x,t)=R(x,t)*\varphi(x)$$

where $*$ denotes spatial convolution and the source function $R(x,t)$ is given by

$$\text{(4.3)}\qquad R(x-y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\xi(x-y)+i\xi t(1+\xi^2)^{-1})d\xi$$

Thus, if we call $h(\xi)=\xi(1+\xi^2)^{-1}$, then we may rewrite (4.2) as

$$\text{(4.4)}\qquad v(x,t)=\int_{-\infty}^{\infty}\exp(ih(\xi)t)\,\hat{\varphi}(\xi)d\xi.$$

Therefore, (4.4) may suggest to us that in order to obtain some information about the asymptotic behavior of the solution v (of (4.1)) we could use the method of stationary phase. This method tell us, in particular, that the major contribution of the value of the integral (4.4) arises from a neighborhood of the points $\pm\infty$ and from a neighborhood of those points ξ such that the derivative of h vanishes, i.e. $h'(\xi)=0$. In our case, those ξ's are $\xi=\pm 1$. We divide the integral (4.4) into two parts, being the first integral (for a fixed large $r>1$)

$$\int_{-r}^{r}\exp(ih(\xi)t)\,\hat{\varphi}(\xi)d\xi$$

which behaves for large t like

$$\text{(4.5)}\qquad \left(\frac{2\pi}{t\,|h''(\xi_0)|}\right)^{1/2}\hat{\varphi}(\xi_0)\exp\left[ith(\xi_0)+i\frac{\pi}{4}\right]$$

where $\xi_0=\pm 1$, therefore

$$h''(\xi_0)=\begin{cases}1 & \text{if}\quad \xi_0=1\\ -1 & \text{if}\quad \xi_0=-1\end{cases},$$

that is, in absolute value, (4.5) behaves like

$$\frac{\text{Constant}}{|t|^{1/2}}$$

for large t. Here the constant depends on $\hat{\varphi}$ (and r). For the second integral, i.e. $\int_{|\xi|>r}\exp(ih(\xi)t)\,\hat{\varphi}(\xi)d\xi$ we could approximate $\xi(1+\xi^2)^{-1}$ by ξ^{-1} for ξ large, thus

$$\int_{|\xi|>r}\exp(it\xi^{-1})\,\hat{\varphi}(\xi)d\xi=-\frac{1}{it}\int_{|\xi|>r}\left[\frac{d}{d\xi}\exp(it\xi^{-1})\right]\xi^2\,\hat{\varphi}(\xi)d\xi$$

$$\text{(4.6)}\qquad =\frac{1}{it}\int_{|\xi|>r}\exp(it\xi^{-1})\frac{d}{d\xi}(\xi^2\,\hat{\varphi}(\xi))d\xi+0(|t|^{-1}).$$

From the above discussion we find that the solution $v(x, t)$ behaves, for large t like

$$\frac{\text{Constant}}{|t|^{1/2}} + \frac{\text{Constant}}{|t|}.$$

Another observation about the solutions of (4.1) with smooth initial data vanishing at infinity, is the following: if we consider the bilinear form $B(u, v)$ given by

(4.7) $$B(u, v)(t) = \int_{-\infty}^{\infty} [uv + u_x v_x] dx$$

where u and v are C^∞-solutions of (4.1) with initial data at $t = 0$, $u(x, 0)$ and $v(x, 0)$ $(\in C^\infty(\mathbb{R}))$ vanishing at infinity, then $B(u, v)(t)$ is independent of t, that is, leaves invariant the "free" solutions of (4.1). In fact, by using the equation (4.1) and integrating by parts (4.7) we get

$$\frac{d}{dt} B(u, v)(t) = \int_{-\infty}^{\infty} (uv_t + u_t v + u_{xt} v_x + u_x v_{xt}) dx =$$

$$= \int_{-\infty}^{\infty} [u(-v_x + v_{xxt}) + v(-u_x + u_{xxt})] dx +$$

$$+ \int_{-\infty}^{\infty} (u_{xt} v_x + u_x v_{xt}) dx =$$

$$= -\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(uv) dx + \int_{-\infty}^{\infty} (uv_{xxt} + vu_{xxt}) dx +$$

$$+ \int_{-\infty}^{\infty} (u_{xt} v_x + u_x v_{xt}) dx = 0.$$

This completes the proof that $B(u, v)(t)$ is independent of t. Furthermore, it can easily be checked that B is continuous if we introduce the norm $|||u|||^2 = \|u(\cdot, t)\|_{L^2}^2 + \|u_x(\cdot, t)\|_{L^2}^2$.

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