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On structural stability of pairings of vector fields and functions

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Introdution.

1. In this paper we study manifolds on which both a vector field and a function are defined. The case where there is only a vector field is extensely studied in the theory of dynamical systems. The motivation for adding to this structure a functions comes from general system theory: the function can be considered as a "read out function".

We consider various definitions of structural stability and investigate whether they are dense, and if not, there are examples which are structurally stable at all.

In this work we restrict ourselves to the case of C^∞ two dimensional orientable compact manifold, without boundary.

We shall refer to a pair (X, f) where X is a vector field and f is a real function (both defined on the same manifold) as field-function.

2. In Section 1 we give preliminaries, definitions and establish the notation. The main definition says that two field-functions (X, f) and (Y, g) on a manifold M are equivalent if there exists a homeomorphism $h: M \to M$, which is a conjugacy between X and Y, mapping level curves of f in level curves of g.

In section 2 we characterize the local structural stability and prove its "genericity". A singularity of (X, f) is a point p in M which satisfies $X(f)(p) = 0$. In this section the generic singularities are studied. It is convenient to note that if (X, f) has only generic singularities then the set of all singularities is an imbedded submanifold of M of codimension one; in general this set is not a submanifold of M.

In Section 3 we show the existence of a structurally stable field-function on S^2. We construct a non trivial example of a structurally stable field-function (X, f) on S^2, where f is the height function and X is a small pertubation of Grad f.

Section 4 contains some necessary conditions for a field-function to be structurally stable.

In Secton 5 we investigate the qualitative behavior of a pair of diffeomorphisms on an interval. The main result obtained in this section has an interesting application in Section 6.

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In Section 6 we show that the set of the struturally stable field-functions is never dense.

Finally in Section 7 we define the concept of weak structural stability and its "genericity" is shown. This proof is a variation of the one given for the stability of the Morse Function.

Some results concerning structural stability of vector fields and singularity of functions are assumed.

Is wish to thank F. Takens for being the source of many ideas developed in this paper. Finally I want to thank the Department of Mathematics of the University of Groningen for the hospitality they offered me during the time I prepared this paper.

1. Preliminaries.

Consider M a C^∞ two dimensional orientable compact manifold without boundary.

Let $X^r = X^r(M)$ be the space of the C^r vector fields on M with the C^r-topology and $F^{r+1} = F^{r+1}(M)$ be the space of C^{r+1} real valued function with the C^{r+1}-topology. We topologize $W = X^r \times F^{r+1}$ with the natural product topology; we shall always assume the $r > 1$.

We will fix on M a Riemannian metric of class C^∞.

Definition (1.1). Two field-functions (X, f), (Y, g) in W are said to be conjugate (or topologically equivalent) if there exists a homeomorphism $h: M \to M$ mapping trajectories of X onto trajectories of Y and level curves of f onto level curves of g.

Definition (1.2). A field-function $(X, f) \in W$ is structurally stable (in W) if it has a neighborhood B(in W) such that (X, f) is conjugated to every $(Y, g) \in B$. We will denote by Σ the subset of W consisting of the structurally stable (X, f).

Definition (1.3). Let $p,q \in M$. We say that $(X, f) \in W$ at p is equivalent to $(Y, g) \in W$ at q if there exist neighborhoods U of p and V of q, in M, and a homeomorphism $h: U \to V$ which maps trajectories of $X_{|U}$ onto trajectories of $Y_{|V}$ and level curves of $f_{|U}$ onto level curves of $g_{|V}$. From this, the Local Structural Stability in W is given in a natural way. Denote by Σ^l the subset of W consisting of the locally structurally stable (X, f).

Consider $(X, f) \in W$ and $p \in M$. The following notation will be used in the text:

i) $X(f)(p)$ is the derivative of f along X at p;

ii) $L_f(p)$ is the level curve of f passing by p;

iii) $\gamma_X(p)$ is the trajectory of X passing by p;

iv) $\phi_X(x, t)$ is the solution of $\dot{x} = X(x)$ satisfying $\phi_X(x, 0) = x$; $\gamma_X(x) = \{\phi_X(x, t)/t \in \mathbb{R}\}$;

v) Df_p is the derivative of f at p;

vi) For a subset S of M, ∂S is the boundary of S, $f_{|S}$ is the restriction of f to S and $M - S$ is the set of points $q \in M$ such that $q \notin S$;

vii) If C is an oriented curve in M then $(\overset{\frown}{a\ b})_C$ (resp. $[\overset{\frown}{a\ b}]_C$) is the open (respec. closed) are of C with extremes a and b, oriented from a to b.

Definition (1.4). A point $p \in M$ is a regular point of $(X, f) \in W$ if $X(f)(p) \neq 0$. If $X(f)(p) = 0$ then p is a critical point or a singularity of (X, f). The critical set of (X, f) (denoted $C(X, f)$) is the set of the critical points $p \in M$ of (X, f).

Definition (1.5). A point $p \in C(X, f)$ is said to be a critical point of (X, f) of type:

I – if i) $X(p) = 0$; ii) p is a hyperbolic critical point of X; iii) the eigenvalues of DX_p are distinct; iv) p is a regular point of f; v) the eigenspaces of DX_p are transversal to $L_f(p)$ at p.

II — if i) $X(p) \neq 0$; ii) p is a non-degenerate critical point of f; iii) $X(X(f))(p) \neq 0$.

III — if i) $X(p) \neq 0$; ii) p is a regular point of f; iii) $X(X, f))(p) \neq 0$.

IV — if i) $X(p) \neq 0$; ii) p is a regular point of f; iii) $D_p(X(f)) \neq 0$; iv) $X(X(f))(p) = 0$ but $X(X(X(f)))(p) \neq 0$.

We will refer to the critical point of (X, f) of type J as G_J-singularity of $X,f)$, $J =$ I, II, III and IV.

Definition (1.6). A point $p \in M$ is said to be a generic point of (X, f) if either it is a regular point of (X, f) or it is a G_J-singularity of (X, f), $J =$ I, II, III, and IV.

Remark (1.7). Let p be a non degenerate note of X (this means that if λ_1, λ_2 are the eigenvalues of DX_p, with $X(p) = 0$, then $\lambda_1, \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$). We shall refer to a strong trajectory of X at p as that trajectory of X tangent to the eigenspace of DX_p associated to the eigenvalue of larger absolute value.

2. Local theory.

We have the following result:

Theorem 2.

a) *If $C(X, f)$ contains only generic critical points (i.e. G_J-singularities, $J = I, II, III$ and IV) then it is a C^k imbedded sub-manifold of M of dimension one;*

b) $(X, f) \in \Sigma^l$ *(locally structurally stable field-function) if and only if any point of M is a generic point of (X, f);*

c) Σ^l *is open dense in W.*

The proof of Part *a)* is essentially a consequense of the Implicit Function Theorem.

Assuming 2.*b*) is true, then 2.*c*) follows immediately.

Finally 2.*b*) is shown by using a direct and tedious calculation.

We now give a list of generic singularities for a field function (X, f); the broken lines represent the level curves of f, continuous lines the trajectories of X and thick continuous lines represent the critical set of (X, f)):

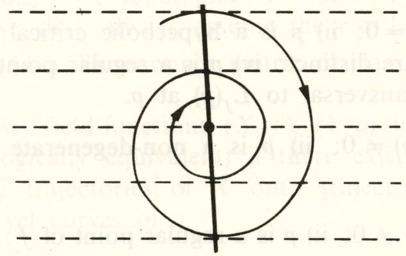

Figure 1 — Example of a G_I-singularity.

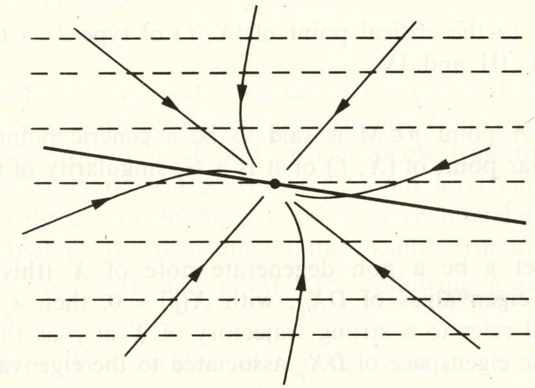

Figure 2 — Example of a G_I-singularity.

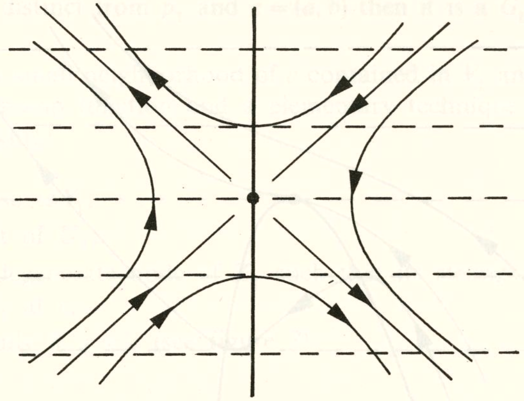

Figure 3 — Example of a G_I-singularity.

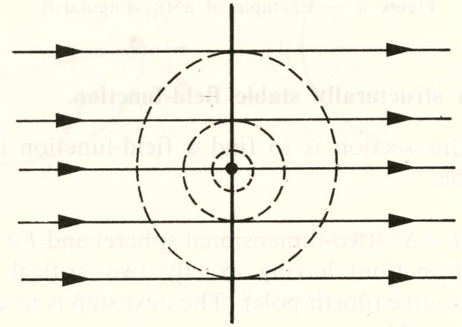

Figure 4 — Example of a G_{II}-singularity.

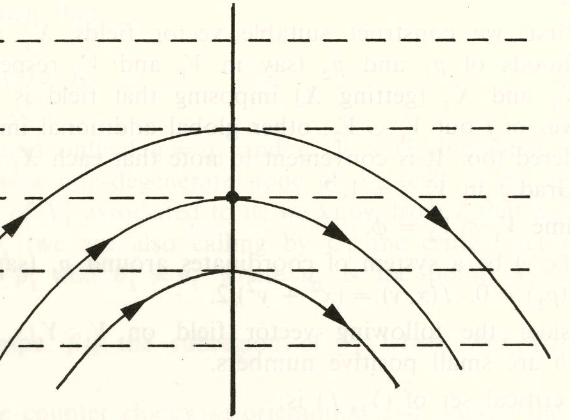

Figure 5 — Example of a G_{III}-singularity.

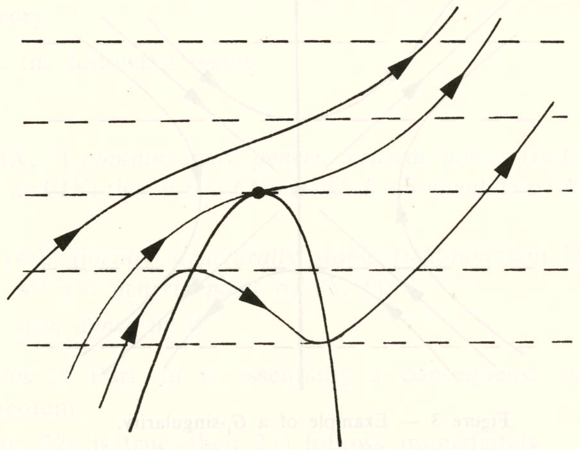

Figure 6 — Example of a G_{IV}-singularity.

3. Existence of a structurally stable field-function.

Our goal in this section is to find a field-function (X, f) in W which is structurally stable.

Example (3.1). $M = S^2$ (two-dimensional sphere) and $F : M \to \mathbb{R}$ a C^r-Morse function (Height Function) having exactly two critical points: a sink p_1 (south pole) and a source (north pole). The next step is to construct a suitable C^r-vector field X on M.

In order to make the construction of X more amenable we give the idea of it:

At first, we construct suitable vector fields X_1 and X_2 on small neighborhoods of p_1 and p_2 (say in V_1 and V_2 respectively). Then we extend X_1 and X_2 (getting X) imposing that field is transversal to the level curves of f out $V_1 \cup V_2$; other global additional imposition to X will be considered too. It is convenient to note that each X_i is a small pertubation of Grad f in V_i, $i = 1, 2$.

Assume $V_1 \cap V_2 = \phi$.

Let (x, y) be a system of coordinates around p_1 (say on V_1) satisfying $x(p_1) = y(p_1) = 0$, $f(x, y) = (x^2 + y^2)/2$.

Consider the following vector field on $V_1 : Y_1(x, y) = (x - a, y - b)$ where a, b are small positive numbers.

The critical set of (Y_1, f) is:

$$C_1 = \{(x, y) \in V_1 : (x - a/2)^2 + (y - b/2)^2 = (a/2)^2 + (b/2)^2\}.$$

If $q \in C_1$ is distinct from p_1 and $c = (a, b)$ then it is a G_{III}-singularity of (Y_1, f).

Let U_1 be a small neighborhood of c contained in V_1 and disjoint from p_1; by using a bump function and a elementary technique we get a field Z_1 on V_1 satisfying:

i) Z_i is C^r-close to Y_1,

ii) $Z_1 = Y_1$ out of U_1,

iii) c is a non-degenerate node of Z_1 such that its strong stable manifold is tangent to C_1 at c,

iv) $Z_1(q) = 0$ only if $q = c$ (see figure 7).

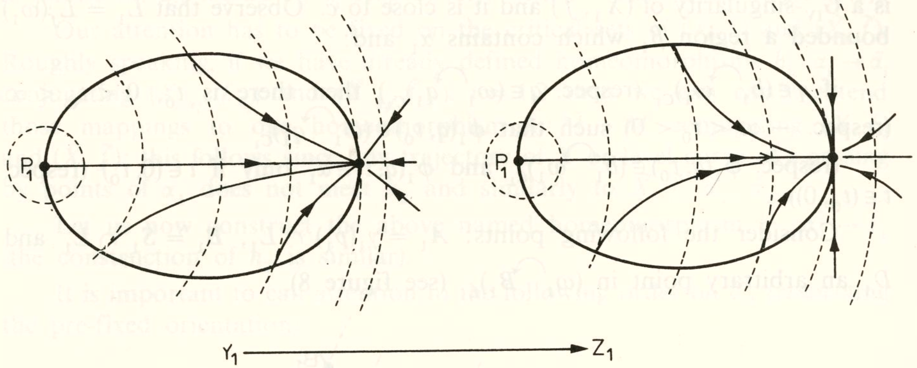

Figure 7 — The fields Y_1 and Z_1.

We again perturbe Y_1 around c(say in U_1) obtaining a new C^r-vector field X_1 on V_1, such that:

i) X_1 is C^r-close to Y_1,

ii) $X_1 = Y_1$ out of U_1,

iii) $X_1(q) = 0$ if and only if $q = q_1$ and q_1 is a G_I-singularity of (X_1, f) nearby c. As q_1 is a non-degenerate node of X_1, call by S_1 and R_1 the strong trajectories of X_1 associated to it; we know from **2** that $S_1 \cap C_1 = \phi$ and $R_1 \cap C_1 = b_1$ (we are also calling by C_1 the critical set of (X, f)). Impose that $b_1 \neq p_1$ and $b_1 \neq u_1$ where u_1 is the point of $C_1 \cap L_f(q_1)$ different from q_1.

Call by α_1(respc. β_1) the subregion of V_1 where $X(f_1)$ is negative (respec. positive).

Fix on C_1 the counter clockwise orientation and impose the following order on it:

$$p_1 < b_1 < u_1 < q_1$$

We use the notations $\phi_1(q,t) = \phi_{X_1}$ and $\gamma_{X_1}(q) = \gamma_1(q)$.

As the eigenspaces of $DX_1(q_1)$ are transversal to C_1 and $L_f(q_1)$ at q_1, we deduce that:

a) if $q \in (\overset{\frown}{q_1 b_1})_{C_1}$ then $\phi_1(q,t) \in \alpha_1$ for any $t > 0$.

b) for each $q \in (\overset{\frown}{b_1 q_1})_C$ there is a number $t_0(q) \geq 0$ such that $\phi_1(q,t) \in \beta_1$ for any $t \gtrsim t_0(q)$.

From [9] X_1 can be obtained such that:

iv) there is a point $\omega_1 \in (\overset{\frown}{\mu_1 q_1})_{C_1}$ such that $t_0(\omega_1) = 0$; furthermore ω_1 is a G_{IV}-singularity of (X_1, f) and it is close to c. Observe that $L_1 = L_f(\omega_1)$ bounded a region β_1 which contains α_1 and:

if $q \in (\overset{\frown}{b_1 \omega_1})_{C_1}$ (respec. $q \in (\overset{\frown}{\omega_1 q_1})_{C_1}$) then there is t_0, $0 < t_0 < \infty$ (respec. $-\infty < t_0 < 0$) such that $\phi_1(q, t_0) \in (\overset{\frown}{\omega_1 q_1})_{C_1}$

(respec. $\phi_1(q,t_0) \in (\overset{\frown}{b_1 \omega_1})_{C_1}$ and $\phi_1(q,t) \in \alpha_1$ only if $t \in (0, t_0)$ (respec. $t \in (t_0, 0)$).

Consider the following points: $A_1 = \gamma_1(p_1) \cap L_1$, $B_1 = S_1 \cap L_1$ and D_1 an arbitrary point in $(\overset{\frown}{\omega_1 B_1})_{L_1}$ (see figure 8).

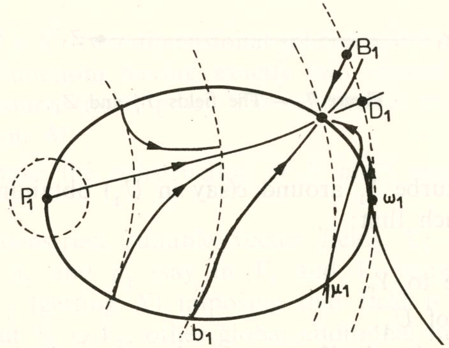

Figure 8 — The vector field X_1

By a simmetric procedure we find a vector field X_2 on V_2 and obviously, the following similar objects associated to it: L_2, C_2, p_2, q_2, w_2, B_2, D_2. Choose an arbitrary point $E_2 \in (\overset{\frown}{D_2 B_2})_{L_2}$.

Define a C^r-vector field X on S^2 satisfying:

a) $X_{|V_1} = X_1$,

b) $X_{|V_2} = X_2$,

c) $X_{M-(V_1 \cup V_2)}$ is transversal to L_f,

d) Each one of the following pairs of points (of M) belongs to the same trajectory of X: (A_1, E_2), (B_1, D_2), (D_1, B_2). This assertion is possible since they preserve the same order in C_1 and C_2.

The condition *d)* implies that there is no connection between C_1 and C_2 by the trajectories of X or level curves of f.

Proposition (3.2). *The field-function (X, f) above constructed is structurally stable in W.*

Proof. According to (3.1) it seems clear that for each (X, f) close enough to $(\tilde{X}, \tilde{f})$ in W we can find the similar objects $\tilde{p}_i$, $\tilde{V}_i$, $\tilde{C}_i$, $\tilde{q}_i$, $\tilde{w}_i$, $\tilde{u}_i$, $\tilde{b}_i$, $\tilde{\alpha}_i$ to p_i, V_i, C_i, q_i, w_i, u_i, b_i, α_i respectively, with $i = 1,2$. (see example (3.1)).

Our attention has to be fixed on the critical sets of (X, f) and $(\tilde{X}, \tilde{f})$. Roughly speaking, if we have already defined homeomorphisms $h_i : \alpha_i \to \tilde{\alpha}_i$ conjugating $(X_{|\alpha_i}, f_{\alpha_i})$ and $(\tilde{X}_{\tilde{\alpha}_i}, \tilde{f}_{\tilde{\alpha}_i})$, $i = 1,2$ then we can easily extend those mappings to one homeomorphism $h : M \to M$ conjugating (X, f) and $(\tilde{X}, \tilde{f})$; this follows since any trajectory of X or level curve of f passing by points of α_1 does not meet α_2 and similarly to $\tilde{X}$, $\tilde{f}$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$.

Let us now construct the above named homeomorphism $h_1 : \alpha_1 \to \tilde{\alpha}_1$ (the construction of h_2 is similar).

It is important to call attention to the following order on C_1 (respecting the pre-fixed orientation):

$$p_1 < b_1 < u_1 < w_1 < q_1 < p_1 \quad \text{(and similarly on } \tilde{C}_1\text{)}.$$

The curve $L_f(b_1)$ (respc. $L_{\tilde{f}}(\tilde{b}_1)$) meets C_1(respec. $\tilde{C}_1$) at $v_1 \in (\overset{\frown}{q_1 p_1})_{C_1}$ (respec. $\tilde{v}_1 \in (\overset{\frown}{\tilde{q}_1 \tilde{p}_1})_{\tilde{C}_1}$).

We recall that (X, f) induces a homeomorphism $\Theta = [\overset{\frown}{u_1 w_1}]_{C_1} \to$ $\to [\overset{\frown}{u_1 w_1}]_{C_1}$ as follows: for each $q \in [\overset{\frown}{u_1 w_1}]_{C_1}$, $L_f(q)$ meets $[\overset{\frown}{w_1 q_1}]_{C_1}$ at R_1 and $\gamma_X(k_1)$ meets $[\overset{\frown}{u_1 w_1}]_{C_1}$ in $\Theta_1(q)$; we notice that $\Theta(w_1) = w_1$ and Θ contracts to w_1.

Define an arbitrary homeomorphism $h_1 : [\overset{\frown}{p_1 q_1}]_{C_1} \to [\overset{\frown}{\tilde{q}_1 \tilde{p}_1}]_{\tilde{C}_1}$ satisfying $h_1(p_1) = \tilde{p}_1$, $h_1(q_1) = \tilde{q}_1$ and $h_1(v_1) = \tilde{v}_1$.

We extend the last homeomorphism to $h_1 : C_1 \to \tilde{C}_1$ as follows:

i) if $q \in (\overset{\frown}{p_1 u_1})_{C_1}$ then $L_f(q)$ meets $(\overset{\frown}{q_1 p_1})_{C_1}$ in q_0; we do $h_1(f) = \tilde{q}$, $\tilde{q}$ being the point where $L_{\tilde{f}}(h_1(q_0))$ meets $(\overset{\frown}{\tilde{p}_1 \tilde{u}_1})_{\tilde{C}_1}$; this implies, in particular, that $h_1(b_1) = \tilde{b}_1$ and $h_1(u_1) = \tilde{u}_1$.

ii) using the above defined mapping Θ we immediately construct h_1 on $(u_1 \overset{\frown}{\ } q_1)_{C_1}$.

Finally we apply a direct process to get the required homeomorphism $h_1 : \alpha_1 \to \tilde{\alpha}_1$.

This completes the proof.

4. Necessary condition for stability.

In this section we establish a necessary condition for structural stability in W. The result of this section will not be used in the sequel. The proofs here, will be omitted since either they come up from known techniques and results (see [4], [5], [8]) or they are trivial.

We begin by considering the following subsets of $C(X, f)$ for $(X, f) \in W$:

$C_1 = \{p \in C(X, f) : p$ is a G_I-singularity of $(X, f)\}$
$C_2 = \{p \in C(X, f) : p$ is a G_{II}-singularity of $(X, f)\}$
$C_3 = \{p \in C(X, f) : p$ is a G_{IV}-singularity of $(X, f)\}$
$C_4 = \{p \in C(X, f)$: there is a saddle separatrix of X tangent to L_f at $p\}$.
$C_5 = \{p \in C(X, f)$: there is a strong trajectory of X tangent to L_f at $p\}$.
$C_6 = \{p \in C(X, f)$: there is a closed trajectory of X tangent to L_f at $p\}$.
$C_7 = \{p \in C(X, f)$: p is a regular point of f, $X(p) \neq 0$, $f(p)$ is a critical value of $f\}$.

Call by C^* the union of C_i, $i = 1, \dots, 7$.

Proposition (4.1). *If (X, f) is structurally stable in W, then:*

(1) *X is structurally stable in X^r,*
(2) *f is a Morse Function,*
(3) $(X, f) \in \Sigma^l$,
(4) *$C_i \cap C_j = \phi$ for $i \neq j$,*
(5) *Each trajectory of X meets C^* at most at one point,*
(6) *Each level curve of f meets C^* at most at one point,*
(7) *No saddle separatrix of X is a strong trajectory of X,*
(8) *Let S_1 be a strong trajectory of X associated to p_1 and S_2 be a strong trajectory of X associated to p_2. If $p_1 \neq p_2$ then $S_1 \neq S_2$,* .
(9) *Each trajectory of X is tangent to one level curve of f at most at one point.*

5. Pair of real diffeomorphisms.

The result of this section will be used in the sequel.

Let $J = [0, 1]$ be the closed interval contained in the reals with extremes 0 and 1.

We denote by D^r the set of pairs $\phi = (\phi_0, \phi_1)$ such that:

a) $\phi_i : J \to J$ is a C^r-diffeomorphism (not necessarily onto) $i = 0, 1$,
b) $\phi_0(0) = 0$ and $\phi_1(1) = 1$; furthermore 0 (respec. 1) is the unique fixed point ϕ_0(respec. ϕ_1),
c) ϕ_0(respec. ϕ_1) contracts to 0 (respec. 1).

We topologize D^r by the C^r-topology.

Definition (5.1). Two pairs $\phi = (\phi_0, \phi_1)$, $\tilde{\phi} = (\tilde{\phi}_0, \tilde{\phi}_1)$ in D^r are equivalent (denoted $\phi \sim \tilde{\phi}$) if there exists a homeomorphism h: $J \to J$ such that $\phi_0 \circ h = h \circ \tilde{\phi}_0$ and $\phi_1 \circ h = h \circ \tilde{\phi}_1$.

By the relation $\sim$ the structural stability in D^r can easily be stablished.

Proposition (5.2). *If $\phi_1(0) < \phi_0(1)$ then $\phi = (\phi_0, \phi_1)$ is not structurally stable in D^r.*

Before the proof of the Proposition (5.2) be given, we need some preliminaries:

Let $\phi = (\phi_0, \phi_1) \in D^r$ be given such that $\phi_1(0) = a_1 < \phi_0(1) = b_1$. Denote by ψ_0, ψ_1 the inverse diffeomorphisms ϕ_0, ϕ_1 respectively obviously ϕ_0 (respec. ϕ_1) is an expansion in 0 (respec. 1).

Define a function $\alpha : J \to J$ by

$$\alpha(x) = \begin{cases} \psi_0(x) & \text{for} \quad x \le a_1 \\ \psi_1(x) & \text{for} \quad x > a_1 \end{cases}.$$

It is clear that α is a piecewise C^r-diffeomorphism, a_1 is the unique discontinuity point of α and $\alpha^n = \alpha \circ \dots\dots\dots \circ \alpha$ has 2^{n-1} discontinuity points.

Associated with each $\phi = (\phi_0, \phi_1) \in D^r$ satisfying $\phi_1(0) < \phi_0(1)$, there exists the countable set $S(\phi)$ of points $p \in J$ such that α^n is discontinuous at p for some $n > 0$.

Lemma (5.3). *$S(\phi)$ is dense in J.*

Proof. Suppose that there exists an open interval contained in $J - S(\phi)$: call by J_0 the biggest such interval. Note that:

$$0 \notin S(\phi), \quad 1 \notin S(\phi), \quad a_1 \in S \quad \text{and} \quad J_0 \subsetneq \phi_0(J_0) \cup \phi_1(J_0)$$

Moreover, $\phi_0(J_0) \cup \phi_1(J_0)$ still is an open interval contained in $J - S$; but this is a contradiction since J_0 has been supposed to be the biggest open interval in $J - S$. This proves the lemma.

Proof of (5.2). If $\tilde{\phi} = (\tilde{\phi}_0, \tilde{\phi}_1)$ is a small perturbation of $\phi = (\phi_0, \phi_1)$ in D^r we have $\tilde{a}_1 = \tilde{\phi}_1(0) < \tilde{\phi}_0(1) = \tilde{b}_1$ and consequently there exists the corresponding set $S(\tilde{\phi})$.

For any conjugacy (if there exists) $h : J \to J$ between ϕ and $\tilde{\phi}$, necessarily it satisfies $h(a_1) = \tilde{a}_1$, $h(b_1) = \tilde{b}_1$ and $h(S(\phi)) = S(\tilde{\phi})$.

Now, pick a sequence $\phi_n(\phi_{0,n}, \phi_{1,n}) \in D^r$ converging to $\phi = (\phi_0, \phi_1)$ in D^r satisfying:

$$\phi_{0,n}(x) = \begin{cases} \phi_0(x) & \text{if } \ x \le \psi_0(a_1) \\ \phi_0(x) + \dfrac{1}{n} & \text{if } \ \psi_0(a_1) + \dfrac{1}{n} \le x \end{cases}$$

$\phi_{1,n}(x) = \phi_1(x)$ (see figure 9).

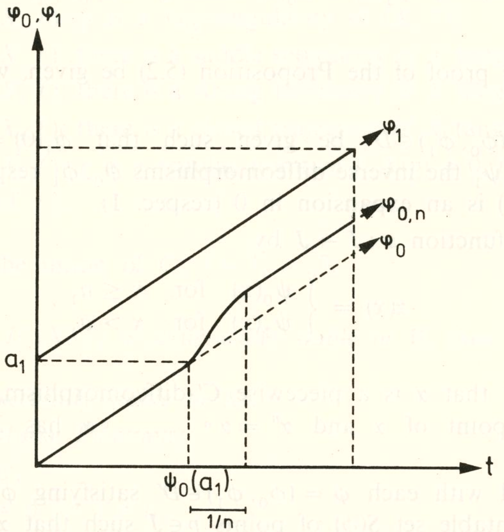

Figure 9 — The grafic of ϕ_n.

We can easily deduce that $S(\phi) = S(\phi_n)$ for every n; furthermore if there exists a conjugation h_n between ϕ and ϕ_n then it must satisfy $h_{n|S(\phi)} =$ $=$ identity and $h_n(\phi_0(1)) = \phi_{0,n}(1)$. As $1 \notin S$, these two last conditions are contradictory; this follows immediately from the density of S in J.

Remark (5.4). We can say more about the structural stability in D^r; i.e. the following result is true: "$\phi = (\phi_0, \phi_1)$ is structurally stable in D^r if and only if $\phi_1(0) > \phi_0(1)$".

Sketch of Remark (5.4) 's proof: Observe that, when $\phi_1(0) = b_1 > \phi_0(1) = a_1$ the iterates

$$\phi_0^{n_r} \circ \phi_1^{n_r - 1} \circ \dots\dots\dots\dots\dots \circ \phi_0^{n_1} \circ \phi_1(0)$$

and

$$\phi_0^{m_r} \circ \phi_1^{m_r - 1} \circ \dots\dots\dots\dots\dots \circ \phi_1^{m_1} \circ \phi_0(1),$$

for n_i, $m_j = 0, 1, \dots, n, \dots$ determine a cantor set $I \subset J$.

Let $\tilde{\phi} = (\tilde{\phi}_0, \tilde{\phi}_1)$ be a small perturbation of ϕ in D^r with $\tilde{\phi}_1(0) = \tilde{b} >$ $> \tilde{\phi}_0(1) = \tilde{a}_1$.

Define an arbitrary homeomorphism $h : [a_1, b_1] \to [\tilde{a}_1, \tilde{b}_1]$ imposing that $h(a_1) = \tilde{a}_1$ and $h(b_1) = \tilde{b}_1$.

The next step is to extend h to J in such way that a conjugacy between ϕ and $\tilde{\phi}$ in D^r is obtained. In order, associated with each $q \in J - I$ there is one unique set of non negative integers $\{n_1, \dots, n_r\}$ such that

$$(\phi_0^{-n_1} \circ \phi_1^{-n_2} \circ \dots\dots\dots\dots\dots \circ \phi_0^{-n_r})(q) = q_1 \in (a_1, b_1)$$

with $n_i > 0$ if $i = 2, \dots, n_{r-1}$.

We define: $h(q) = (\phi^{n_r} \circ \dots\dots \circ \phi^{n_1})(h(q_i))$ if $q \in J - I$ and if $q \in I$ then $h(q) = \lim\limits_{i \to \infty} h(q_i)$ where (q_i) is a sequence in $J - I$ converging to q.

Obviously h is well defined and it is a conjugation between ϕ and $\tilde{\phi}$ in D^r.

6. An example.

In this section we show the non density of Σ in W.

We begin with:

Example (6.1). Take a C^∞-Morse Function $f : M \to \mathbb{R}$ and $I = (a, b)$ an open interval in $f(M)$ disjoint from the critical values of f. Call by V a small neighborhood of some point $p \in f^{-1}(I)$.

We now define a C^r-vector field X on M by requiring:

(1) There are in V a pair γ_0, γ_1 of stable closed trajectories of X; those trajectories bound two disjoint (plane) regions in V (see figure 8).

(2) $f_0 = f_{|\gamma_0}$ (respec. $f_1 = f_{|\gamma_1}$) is a Morse Function. Call by M_0 (respec. M_1) the point of maximum of f_0 (respec. f_1) such that $f(M_0) \leq f(M_1)$. For simplicity, impose that $q \in V$ and $f(M_0) \leq f(M_1)$.

(3) The condition (2) implies that M_i, $i = 0, 1$, is a G_{III}-singularity of (X, f); we can find imbeddings $\alpha_i : (-\varepsilon, \varepsilon) \to V$ $(i = 0, 1)$, transversal to X and L_f, with $\alpha_0(0) = M_0$ and $\alpha_1(0) = M_1$. We impose that $f(M_0) \in f(L_1)$ and $f(M_1) \in f(L_0)$ where $L_i = \alpha_i(-\varepsilon, \varepsilon)$, $i = 0, 1$ are lines of contact between X and L_f.

We may assume further that $\rho_i'(M_i) \neq 1$ where ρ_i is the C^r-Poincaré map associated to X, γ_i, L_i and M_i, $i = 0, 1$.

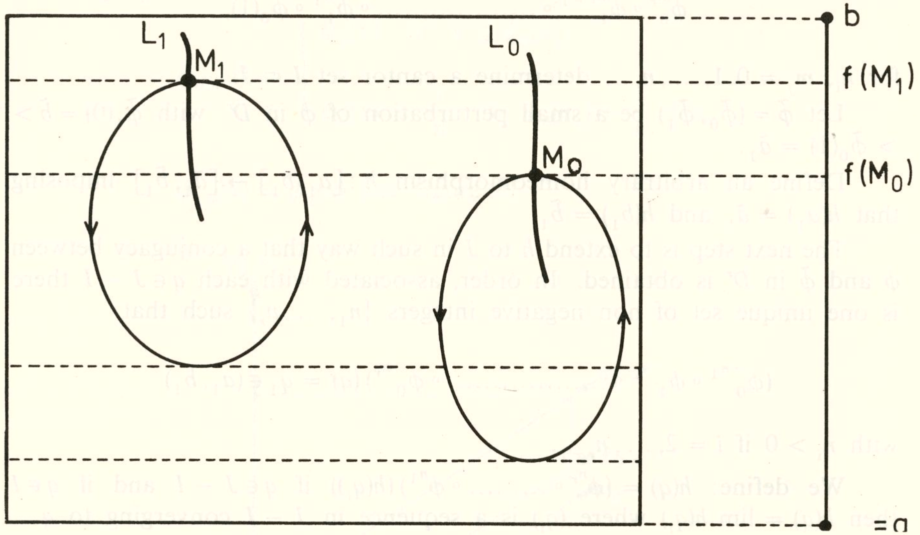

Figure 10 — The vector field X on $f^{-1}(a, b)$.

Remark (6.2). The field X above defined could rigorously be constructed by using a elementary mathematical technique.

For simplicity, consider $f(M_0) = 0$ and $f(M_1) = 1$. Then the Poincaré mappings above named, determine the element $\phi = (\phi_0, \phi_1) \in D^r$ (vide 5) given by:

$\phi_i(t) = f(\rho_i(x_1(t))$ where $x_i(t) = L_i \cap f^{-1}(t)$, $i = 0, 1$ and $t \in [0, 1]$.

Of course, if ϕ is not structurally stable in D^r then (X, f) is not structurally stable in W; this follows immediately from the definitions (1.1) and (5.1).

Under the above considerations, the following result is a corollary of Proposition (5.2):

Theorem (6.3). Σ is never dense in W.

7. The weak structural stability.

The concept of weak structural stability in W is reached from the following definition:

Definition (7.1). Two field functions (X, f) and (Y, g) (in W) are said to be weakly conjugate if there is a homeomorphism $h : M \to M$ such that:

(1) h carries level curves of f to level curves of g:

(3) for each $x \in M$, (X, f) (at x) is germ equivalent to (Y, g) at $h(x)$ (up definition (1.3)).

Definition (7.2). Let Σ^w be the set of elements $(X, f) \in W$ such that:
a) $(X, f) \in \Sigma^l$

b) if p, q are G_I, G_{II} or G_{IV}-singularities of (X, f) with $p \neq q$, then $f(p) \neq f(q)$.

It follows directly from (7.2) that "Σ^w is open and dense in W".

Lemma (7.3). *If* $(X_0, f_0) \in \Sigma^w$ *then* (X_0, f_0) *is weakly structurally stable in* W.

Proof. Because Σ^w is open in W there is a neighborhood U of (X_0, f_0) in W such that for each $(X_1, f_1) \in U$ and $t \in I = [0, 1]$, $(X_t, f_t) = (tX_1 + (1-t)X_0,\ tf_1 + (1-t)f_0) \in U \cap \Sigma^w$.

Now consider the following subsets of $M \times I$:

$$P = \{(m, t) : X_t(f_t)(m) = 0\},$$
$$Q = \{(m, t) \in P : (m, t) \text{ is a } G_I, G_{II} \text{ or } G_{IV}\text{-singularity of } (X_t, f_t)\}$$
$$P_0 = C_0 \times I \text{ where } C_0 \text{ is the critical set of } (X_0, f_0).$$
$$Q_0 = [(C_0 \times \{0\}) \cap Q] \times I.$$

We know from definition (1.5) that P is a union of a finite number of smooth curves which are transversal to $M \times \{t\}$ for each t, Q is a union of a finite number of smooth curves which are transversal to $M \times \{t\}$ for each t.

It is important to observe that:

Remark (*). for each $t \in I$ and for each connected component S_t of the critical set (X_t, f_t), $f_t(S_t) = [a, b]$ with $a \neq b$, $f^{-1}(a) \cap S_t \in Q$ and $f^{-1}(b) \cap S_t \in Q$.

Define $F : M \times I \to R \times I$ by $F(m, t) = (f(m, t), t)$ where $f(m, t) = f_t(m)$.

It is clear that $\tilde{Q} = F(Q)$ consists of a finite number of smooth curves joining $\mathbb{R} \times \{0\}$ with $\mathbb{R} \times \{1\}$ which are transversal to $\mathbb{R} \times \{t\}$. From this, one can take a one parameter family of diffeomorphisms $h_{\mathbb{R},t}: \mathbb{R} \to \mathbb{R}$, $t \in I$, satisfying:

$h_{\mathbb{R},t}[\tilde{Q} \cap (\mathbb{R} \times 0)] = \tilde{Q} \cap (\mathbb{R} \times t)$ and such that $(u, t) \to (h_{\mathbb{R},t}(u), t)$ is a diffeomorphism.

The remark (*) and continuity arguments show that

$$h_{\mathbb{R},t}[\tilde{P} \cap (\mathbb{R} \times \{0\})] = \tilde{P} \cap (\mathbb{R} \times t) \text{ where } \tilde{P} = F(P).$$

We may of course replace $h_{\mathbb{R},t}$ by identity.

Before we construct a suitable diffeomorphism $\alpha: M \times I \to M \times I$, it is convenient to note that critical set of (X_t, f_t) is C^r-close (as submanifold of M) to the critical set of (X_0, f_0) for each $t \in I$.

We start defining α on Q:

We know that, it S is any connected component of Q and $(m, t) \in S$ then $\alpha(m, t) = (b, t)$ where $b = [f^{-1}(f_t(m), 0)] \cap S$.

Let A be any component of P; assume on $A \cap [M \times t]$, for example the counter clockwise orientation for each $t \in I$. As A contains a finite (non zero) number of components of Q, one may, by continuity, consider them oriented as follows $S_1, S_2, \dots, S_n$.

If $p = (m_1, t_1) \in A - Q$ then there exists i, such that $S_i < p < S_{i+1}$. The set $B_1[f_{t_1}^{-1}(f_{t_1}(p))]_e \cap (S_i, S_{i+1})$ consists of a finite union of points, oriented as follows

$$p_1, p_2, \dots, p_j = p, \dots, p_r.$$

We do $\alpha(p) = (q, t_1)$ where q is the j-th term of the sequence $q_1, \dots, q_k$ where

$$\bigcup_{i=1}^{r} q_r = [f_0^{-1}(f_{t_1}(p))] \cap (S_i, S_{i+1}).$$

Now, as P is a compact set in $M \times I$, we use standard techniques to get α satisfying the following conditions: i) $\alpha(m, t) = (\alpha_t(m), t)$, ii) α is C^r-close to identity, iii) $h_{|(M \times I) - U}$ = identity where U is a small neighborhood of p in $M \times I$, iv) $\frac{\partial}{\partial t}(f_t \circ \alpha^{-1})(\alpha(m, t)) = 0$ for $(m, t) \in P$.

We are still denoting by f_t the function $(f_t \circ \alpha^{-1})$.

Next we want to find a one parameter family of diffeomorphism $h_{M,t}: M \to M$, $t \in I$, such that $f_t \circ h_{M,t} = f_0$ and $(m, t) \to (h_{M,t}(m), t)$ is a diffeomorphism. We shall construct $h_{M,t}$ by integrating on $M \times I$ a vector field on the form $\frac{\partial}{\partial t} + s$ where s is a vector field on $M \times I$ such that for each $(m, t) \in M \times I$ the projection of (m, t) on I is zero and $s(q) = 0$ for $q \in P$. It is clear that s must satisfy the equation $s(f) = -\frac{\partial f}{\partial t}$. For simplicity, we use the following notation $g = -\frac{\partial f}{\partial t}$.

Let's go to construct s.

Around each point in $M \times I$, choose an open neighborhood U, as follows:

a) if $p \notin K$, choose U_p so small that $X_t(f_t)(q) \neq 0$ for every $q \in U_p$. Choose a vector field r^p on U_p such that $r^p(f) \neq 0$ on U_p and the projection to I in zero,

b) If $p = (m, t_0) \in K$ and it is a G_{I}, G_{III} or G_{IV}-singularity of (X_{t_0}, f_{t_0}) then one can choose coordinates (x_1, x_2) on a small neighborhood V of m in M and $\varepsilon > 0$ such that $f(x_1, x_2, t) = C_1(t)x_1 + C_2(t)\ x_2 + h(t)$ with $(C_1(t))^2 + ((C_2(t))^2 \neq 0$ and $-\varepsilon < t < \varepsilon$; this follows essentialy from the fact that p is a regular point of p.

c) When $p = (m, t_0) \in K$ and it is a G_{II}-singularity of (X_0, f_{t_0}) we need the following auxiliar computation:

(1) Let $x \in M$ be a G_{II}-singularity of $(X, f) \in W$. From **1** we can get coordinates (x_1, x_2) around x such that $x_1(x) = x_2(x) = 0$ and $f(x_1, x_2) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2$ with $\varepsilon_i = \pm 1$; furthermore $C(X, f)$ can be described by a C^r-function $x_2 = \alpha(x_1)$ such that $\alpha'(0) = -X^1(0)/X^2(0)$ which is different from ± 1. The function $F(x_1) = f(x_1, \alpha(x_1))$ satisfies $F'(0) = 0$ and $F''(0) \neq 0$.

(2) For the same x above we know that X is transversal to $C(X, f)$ on a neighborhood of x in M. Consider $C(X, f)$ parametrized by μ (with $\mu(x) = 0$).

For $\varepsilon > 0$ and small enough, consider the following family of functions:

$$f_\mu: (-\varepsilon, \varepsilon) \to \mathbb{R} \text{ given by } f_\mu(\tau) = f(\phi_X(\mu, \tau)).$$

From the definition of G_{II}-singularity we get $f'_\mu(0) = 0$ and $f''_\mu(0) \neq 0$. Hence τ can be choosen such that $f_\mu(\tau) = \varepsilon\tau^2$ with $\varepsilon = \pm 1$; this implies that $f(\mu, \tau) = b(\mu) + \varepsilon\tau^2$. But the above computation shows that μ can be choosen such that $f(\mu, \tau) = \eta\mu^2 + \varepsilon\tau^2$ with $\eta = \pm 1$. In fact, μ and τ are normal coordinates around x in M satisfying $(\mu, \tau) \in K$ only if $\tau = 0$.

Returning to case c), we are now able to choose a neighborhood U_p of p in $M \times I$ and coordinates (μ, τ) on $U_p \cap (M \times t_0)$ such that $f(\mu, \tau, t) = \eta\mu^2 + \varepsilon\tau^2 + h(t)$ with $\eta = \pm 1$, $\varepsilon = \pm 1$ and $(0, 0, t_0) = p$.

Let $U_1, \dots, U_m$ be a finite subcovering of $\{U_p\}_{p\varepsilon M \times I}$, corresponding to $p_1, \dots, p_m$. Let $\rho_1, \dots, \rho_m$ be a partition of unity subordinate to that covering. Choose vector fields s_i on $M \times I (1 \leq i \leq m)$ as follows:

case *a)*

$$s_i(q) = \begin{cases} g(q)\rho_i(q) r^{p_i}(q)/r^{p_i}(f)(q) & \text{on } U_i \\ 0 & \text{off } U_i \end{cases}$$

case *b)*

$$s_i = \begin{cases} \left(C_1 \dfrac{\partial}{\partial x_1} + C_2 \dfrac{\partial}{\partial x_2}\right) g\rho_i/(C_1 + C_2) & \text{on } U_i \\ 0 & \text{off } U_i \end{cases}$$

case *c)*

as $\rho_i(g_{|K}) = 0$ then $\rho_i g = \tau G(\mu, \tau, t)$ for selected function G defined on U_i

Let $s_i = {}^1/_2\, \varepsilon h \dfrac{\partial}{\partial \tau}$ on U_i and extend it to be zero off U_i.

Finally the required vector field is given by

$$S = s_1 + \dots\dots\dots + s_m.$$

It is now easy to prove the following result:

Theorem (7.4). *The set of weakly structurally stable field functions* (X, f) *in* W *coincides with* Σ^w.

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