BOL. SOC. BRAS. MAT., VOL. 9 N.º 2 (1978), 83 - 88

Estimation of the spectrum and of the covariance function of a dyadic-stationary series

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1. Introduction.

Let $\{\psi_n(x),\ n \in N,\ 0 \leq x < 1\}$ be the orthonormal system of Walsh functions in the sense of Fine (see [1] and [3]), where $N = \{0, 1, 2, \ldots\}$. Extend these functions periodically to the nonnegative real numbers. Let $\{X(t),\ t \in N\}$ be a real-valued, weakly dyadic-stationary process with

$$E[X(t)] = c_X, \tag{1.1}$$

$$B_{XX}(u) = Cov\{X(t \dotplus u), X(t)\} \tag{1.2}$$

$t, u \in N$. Here $\dotplus$ denotes addition modulo 2 defined for any two real numbers in terms of their dyadic expansions (see [5], for example).

If we assume

$$\sum_{u=0}^{\infty} |B_{XX}(u)| < \infty, \tag{1.3}$$

we then define the Walsh spectrum of $X(t)$ as

$$f_{XX}(\lambda) = \sum_{u=0}^{\infty} B_{XX}(u)\psi_u(\lambda), \qquad 0 \leq \lambda < \infty. \tag{1.4}$$

The relation (1.4) may be inverted to give

$$B_{XX}(u) = \int_0^1 \psi_u(\lambda) f_{XX}(\lambda)\, d\lambda, \tag{1.5}$$

and in particular $B_{XX}(0) = Var[X(t)] = \int_0^1 f_{XX}(\lambda)\, d\lambda$.

In [4] we discussed the problem of estimating $f_{XX}(\lambda)$ through the Walsh periodogram and the smoothed periodogram. Here we consider another class of estimates and derive some of their properties. The smoothed periodogram weights all periodogram ordinates equally. The estimates considered here allows differential weighting. We also consider the problem of estimating the dyadic auto-covariance function $B_{XX}(u)$.

Recebido em abril de 1978

For details on Walsh functions, dyadic-stationary processes and related topics see [3], [6] and [7]. We shall consider the discrete case here. The continuous case is handled similarly.

In the next section we introduce the finite Walsh transform and the periodogram, and we state for reference the discrete versions of the results in [4]. We add a result (Theorem 2.3) which will be used latter.

2. The finite Walsh transform and the periodogram.

We have values $X(0), X(1), \dots, X(T-1)$ from the series $\{X(t), t \in N\}$ and we consider the finite Walsh transform

$$d_X^{(T)}(\lambda) = \sum_{t=0}^{T-1} X(t)\psi_t(\lambda), \qquad 0 \le \lambda < \infty, \tag{2.1}$$

where we suppose that $T = 2^n$, which is suitable for computational purposes, in order to use a fast-Walsh-transform algorithm and also for theoretical reasons, since simple results are available when T is a power of 2. See (2.5) below, for example. If $T \neq 2^n$, consider S as the next power of 2 and complete with zeros.

It is known (see [3]) that $d_X^{(T)}(\lambda)$ is asymptotically normal with mean zero and variance $Tf_{XX}(\lambda)$, under certain regularity conditions. This suggests that $f_{XX}(\lambda)$ may be estimated by the (Walsh) periodogram

$$I_{XX}^{(T)}(\lambda) = T^{-1}[d_X^{(T)}(\lambda)]^2 = T^{-1}\left[\sum_{t=0}^{T-1} X(t)\psi_t(\lambda)\right]^2, \qquad 0 \le \lambda < \infty. \tag{2.2}$$

The following theorems are known (cf. [4]).

Theorem 2.1. *If*

$$\sum_{u=0}^{\infty} u|B_{XX}(u)| < \infty \tag{2.3}$$

then

$$E[I_{XX}^{(T)}(\lambda)] = f_{XX}(\lambda) + 0(T^{-1}). \tag{2.4}$$

That is, (2.2) is an asymptotically unbiased estimate of $f_{XX}(\lambda)$.

Let

$$D_T(\alpha) = D_{2^n}(\alpha) = \sum_{t=0}^{2^n-1} \psi_t(\alpha) = \begin{cases} 2^n, \ \alpha \in [0, 2^{-n}) \ (\text{mod } 1) \\ 0, \ \text{otherwise} \end{cases} \tag{2.5}$$

be the Dirichlet Kernel.

Theorem 2.2. *Let $I_{XX}^{(T)}(\lambda)$ be given by (2.2) and (2.3) be satisfied. Then*

$$Cov\{I_{XX}^{(T)}(\lambda), I_{XX}^{(T)}(\mu)\} = 2\left[\frac{D_T(\lambda \dotplus \mu)}{T}\right]^2 f_{XX}^2(\lambda) + 0(T^{-1}). \tag{2.6}$$

In particular if $\lambda = \mu$, the theorem gives

$$Var[I_{XX}^{(T)}(\lambda)] = 2f_{XX}^2(\lambda) + 0(T^{-1}), \tag{2.7}$$

which shows that the periodogram is not a consistent estimate of the spectrum. A way to improve the stability of (2.2) is to consider the smoothed periodogram (cf. [4]). Let Z be a standard normal variable.

Corollary 1. *Under the conditions of the theorem, $I_{XX}^{(T)}(\lambda)$ is an asymptotically $f_{XX}(\lambda)Z^2$ variable. Moreover, $\{I_{XX}^{(T)}(\lambda_j)\}$ are asymptotically independent.*

The following result is useful.

Theorem 2.3. *If $E[X(t)] \doteq c_x$ and (1.3) holds, then*

$$E[I_{XX}^{(T)}(\lambda)] = T^{-1}\int_0^1 [D_T(\lambda \dotplus \alpha)]^2 f_{XX}(\alpha)d\alpha + T^{-1}[D_T(\lambda)]^2 \cdot c_X^2 . \tag{2.8}$$

Proof. We have

$$E[I_{XX}^{(T)}(\lambda)] = T^{-1}\sum_{u=0}^{T-1}\sum_{v=0}^{T-1} \psi_{u \dotplus v}(\lambda) B_{XX}(u \dotplus v) + T^{-1}[D^T(\lambda)]^2 c_X^2. \tag{2.9}$$

Using (1.5), we find that (2.9) is equal to

$$T^{-1}\int_0^1\left[\sum_{u=0}^{T-1} \psi_u(\lambda \dotplus \alpha)\right]^2 f_{XX}(\alpha)\, d\alpha + T^{-1}[D_T(\lambda)]^2 \cdot c_X^2$$

and (2.8) follows from (2.5).

In particular, if $\lambda \notin [0, 2^{-n})$ (mod 1) or if $c_{\hat{X}} = 0$, (2.8) shows that $E[I_{XX}^{(T)}(\lambda)] \to f_{XX}(\lambda)$ as $T \to \infty$.

3. A class of estimates.

First, let the Walsh functions be extended to the whole real line in the usual manner (see [2]). Let W_j be weights such that $\sum_{j=-m}^{m} w_j = 1$, and let $s(T)$ be an integer such that $s(T)/T \sim \lambda$, as $T \to \infty$.

We define the estimate

$$f_{XX}^{(T)}(\lambda) = \sum_{j=-m}^{m} W_j I_{XX}^{(T)}\left[\frac{s(T) \dotplus j}{T}\right] \tag{3.1}$$

With

$$A_T(\lambda) = \sum_{j=-m}^{m} W_j T^{-1}[D_T(\lambda)]^2, \tag{3.2}$$

we have the following result.

Theorem 3.1. *Under the conditions of Theorems 2.1 and 2.3,*

$$E[f_{XX}^{(T)}(\lambda)]' = \int_0^1 A_T\left[\frac{s(T)}{T} \dotplus \alpha\right] f_{XX}(\alpha)d\alpha, \tag{3.3}$$

and

$$E[f_{XX}^{(T)}(\lambda) = f_{XX}(\lambda) + 0(T^{-1}). \tag{3.4}$$

Proof. This is an immediate consequence of (3.1), (3.2) and Theorems 2.3 and 2.1.

From Theorem 2.2 we have

Theorem 3.2. *Under the conditions of Theorem 2.2, the variance of $f_{XX}^{(T)}(\lambda)$ is*

$$Var[f_{XX}^{(T)}(\lambda)] = 2f_{XX}^2(\lambda) \sum_{j=-m}^{m} W_j^2 + 0(T^{-1}). \tag{3.5}$$

The asymptotic distribution of $f_{XX}^{(T)}(\lambda))$ is obtained directly from Corollary 1.

Theorem 3.3. *The estimate (3.1) is an asymptotically $f_{XX}(\lambda) \cdot \sum_{j=-m}^{m} W_j \cdot Z_j^2$ variable. The variables Z_j^2 (chi-squared variables with 1 degree of freedom) are independent.*

It may be difficult to use this approximating distribution in practice. A standard procedure is to approximate the distribution of such a variate by a multiple $\theta\chi_\nu^2$ of a chi-squared variable, whose mean and degrees of freedom are determined by equating first-and second-order moments. It follows that

$$\nu = 1 \Big/ \sum_{j=-m}^{m} \cdot W_j^2 \text{ and } \theta = 1/\nu.$$

4. Estimation of the dyadic auto-covariance function.

Let $\{X(t),\ t \in N\}$ be a dyadic-stationary series with mean 0 and dyadic auto-covariance funtion given by (1.2). As an estimate of $B_{XX}(u)$ we consider

$$B_{XX}^{(T)}(u) = T^{-1} \sum_{0 \le t, t \dotplus u \le T-1} X(t)X(t \dotplus u), \quad u \in N. \tag{4.1}$$

We consider the integral

$$\int_0^1 I_{XX}^{(T)}(\alpha)\psi_u(\alpha)\,d\alpha = T^{-1} \int_0^1 [d_X^{(T)}(\alpha)]^2 \psi_u(\alpha)\,d\alpha =$$

$$= T^{-1} \int_0^1 \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} X(t)X(s)\psi_t(\alpha)\psi_s(\alpha)\,d\alpha. \tag{4.2}$$

But $\int_0^1 \psi_u(\alpha)d\alpha = 0$, unless $u = 0$, hence (4.2) is equal to

$$T^{-1} \sum_{0 \le t, t \dotplus u \le T-1} X(t)X(t \dotplus u).$$

Hence,

$$B_{XX}^{(T)}(u) = \int_0^1 I_{XX}^{(T)}(\alpha)\psi_u(\alpha)\,d\alpha. \tag{4.3}$$

Theorem 4.1. *If (2.3) is satisfied, we have*

$$E[B_{XX}^{(T)}(u)] = B_{XX}(u) + 0(T^{-1}), \tag{4.4}$$

$$Cov\{B_{XX}^{(T)}(u_1),\ B_{XX}^{(T)}(u_2)\} = 2T^{-1} \int_0^1 \psi_{u_1 \dotplus u_2}(\alpha) f_{XX}^2(\alpha)d\alpha + 0(T^{-1}). \tag{4.5}$$

Proof. (4.4) is a consequence of (2.4). Now

$$Cov\{B_{XX}^{(T)}(u_1),\ B_{XX}^{(T)}(u_2)\} = \int_0^1 \int_0^1 \psi_{u_1}(\alpha)\psi_{u_2}(\beta) Cov\{I_{XX}^{(T)}(\alpha),\ I_{XX}^{(T)}(\beta)\}\,d\alpha\,d\beta.$$

By Theorem 2.2, the right-hand side becomes

$$2T^{-2}\int_0^1 \psi_{u_1}(\alpha)\, f_{XX}^2(\alpha)\,d\alpha[\psi_{u_2}(\alpha)\int_0^1 \psi_{u_2}(\alpha \dot{+} \beta)D_T^2(\alpha \dot{+} \beta)d\beta] + 0(T^{-1}) =$$

$$= 2T^{-2}\int_0^1 \psi_{u_1 \dot{+} u_2}(\alpha)\, f_{XX}^2(\alpha)d\alpha \int_0^1 \psi_{u_2}(\beta)D_T^2(\beta)d\beta + 0(T^{-1}).$$

Here, we have used the invariance relation $\int_0^1 f(\alpha \dot{+} \beta)\,d\beta = \int_0^1 f(\beta)d\beta.$

By (2.5) we have finally that the covariance in question is

$$2T^{-2}\int_0^1 \psi_{u_1 \dot{+} u_2}(\alpha)\cdot f_{XX}^2(\alpha)d\alpha\left[T^2\int_0^{T^{-1}} \psi_{u_2}(\beta)d\beta\right] + 0(T^{-1})$$

and (4.5) is proved.

In particular, (4.5) gives

(4.6) $$Var[B_{XX}^{(T)}(u)] = 2T^{-1}\int_0^1 f_{XX}^2(\alpha)d\alpha + 0(T^{-1}).$$

Acknowledgment — The author wishes to express his appreciation of the referee's comments on some aspects of the paper,

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