The Early Development of Algebraic Topology

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Introduction

This is not the first time, in recent years, that I find myself engaged in writing about "early developments." The motive power has come from the same direction – from my friend Mauricio Peixoto. At first it seemed like a good idea but the more I plunged into it the less it seemed like an ideal occupation for me. For one thing I did not feel particularly possessed with the necessary historical sense. Besides when and where did early algebraic topology start? I decided that the starting points could be placed with Euler and the characteristic, perhaps also with Möbius and his strip, more truly with Riemann and his surface. As far as the end point for me, the mid thirties when the algebro-topological population began to augment most appreciably, and I began to feel more and more swamped by brilliant new knowledge. It was altogether indicated, at least for myself, as an apropriate "last station" and so I more or less decided. Even so I had to leave out many first rate contributions, mainly because they concerned me but little, and it was too late to learn!

The lack of historical sense has been the cause of my mentioning but few names, and even fewer exact dates. For these I must refer to the extensive bibliographies in two of my productions:

Topology, Am. Math. Soc. Colloquium publications vol. 12, 1930 New York reproduced by Chelsea, New York. Hereafter referred to as LT.

Algebraic Topology, same Coll. Publ., vol. 27, 1942 mostly complementary to the preceding. Hereafter referred to as LAT.

I. EARLY HISTORY

The beginnings of algebraic topology share this with the beginnings of any important chapter of mathematics that its roots are more or less obscure. Those of algebraic topology are found mostly in geometry and did not contain the promise of a major field. Since my proposed excursion has nothing archeological and hardly any historical aspect, I will concentrate on the following major points. First Euler's characteristic, then the Möbius strip and its significance for orientability. I will conclude (for special reasons) with a section on knots.

1. Euler's characteristic.

This is certainly one of the earliest manifestations of algebraic topology. Let a convex polyhedron Π in a 3-space have F faces, E edges and V vertices. Euler's formula asserts that always

$$F - E + V = 2. \tag{1.1}$$

The expression at the left is the *characteristic* $\chi(\Pi)$ of Π.

Let 0 be an interior point of the polyhedron and S a sphere of center 0. Project Π onto S from 0. This results in a partition of the sphere into F polygonal regions, with E sides and V vertices and (1.1) still holds. It is interesting however to observe that it is known to hold for any partition of a sphere into a finite number of polygonal regions. In other words it represents actually a property of the sphere S itself: topological property. In fact it holds as well for example for an ellipsoid, or for any "like" figure.

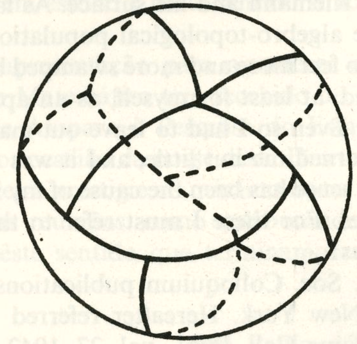

Fig.

In order to calculate this fixed value of $\chi(\Pi)$ one may therefore take any simple decomposition, for example: a great circle made into a polygon with one vertex and 1 arc plus the two hemispheres. Thus there are one vertex, one edge and two faces, so that $\chi(\Pi) = 2$. Euler's proof was à la old geometry, but his proof is easily topologized, as done much later by Poincaré (1895).

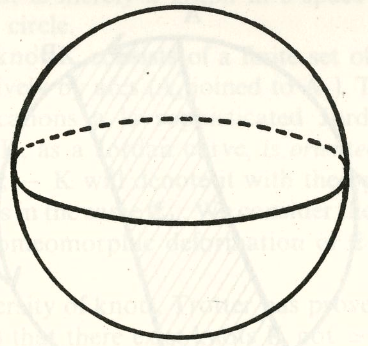

Fig. 2

2. The Möbius Strip (1850)

Let ABCD be a plane rectangle. Match AD with BC so that A coincides with C and B with D. One sees then readily that one cannot match the orientation of

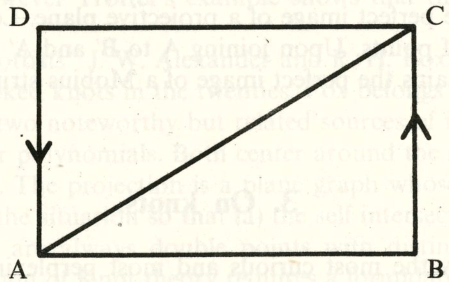

Fig 3

AD with that of BC so that any side common to two triangles is oppositely oriented to both. Intuitively one finds that a small oriented circuit on the strip may be so displaced as to return to its original position with reversed orientation. Poincaré described this as the return upside down of a fly crawling on the strip.

A smooth surface is *orientable* when it contains no part like a Möbius strip, and it is non-orientable when it does contain a part like this strip. Thus a 2-sphere is orientable, but the projective plane is not. The second statement is not quite obvious, but is easily proved along the following lines. An open line L in an ordinary plane is orientable (evident) and remains so when its two end points are made to coincide turning the line into a circle C. Take now an origin 0 in a plane and a circular region of center 0 bounded by a circle D (fig. 4). Let any diameter have the end points AA'

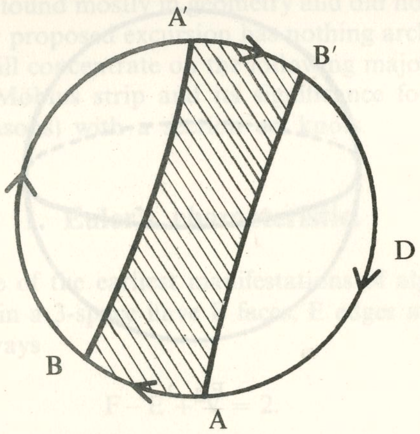

Fig. 4

on the circle D. The open interval (A, A′) is the perfect image of the line L. One closes it by bringing the two points A and A′ into coincidence. The operation on the circle D has for effect to bring all diametral pairs of points into coincidences and then one has the perfect image of a projective plane. Let (A, A′) and (B, B′) be two terminal pairs of points. Upon joining A to B′ and A′ to B one finds that the projective plane contains the perfect image of a Mobius strip and so it is nonorientable.

3. On knots.

This is assuredly the most curious and most perplexing chapter of algebraic topology. One may also say of it that while it has borrowed enormously from the rest of algebraic topology it has returned very scant interest on this "borrowed" capital. It is however full of problems *sui generis* with some of the simplest, in formulation, as yet unsolved. In this respect it resembles considerably number theory.

Our main reason for placing "something about knots" in this early location, is the impossibility to give more than a faint notion of this topic in a reasonable space. We shall therefore merely indicate a very few salient points and refer for knots to the highly interesting and thorough exposition given in the recent monograph *Introduction to Knot theory* by R. H. Crowell and R. H. Fox, Ginn & Co., Boston, 1963. (Hereafter referred to as CF.) This book contains also a good guide to the literature, an extensive bibliography and a wealth of figures. For the few points to be discussed here, no better source of references can be found. It is not possible to touch knot theory at any point without utilizing many advanced topological concepts. For most of these brief indications will be found in Chs. III and IV.

(3.1) *Definition.* A knot is merely a graph in 3-space $\mathfrak{E}_3$ which as a point-set is the homeomorph of a circle.

As a graph then the knot K consists of a finite set of points $A_1, A_2, \ldots, A_n$, which are joined consecutively by arcs (A_n joined to A_1). The arcs may be assumed differentiable (the complications à la sophisticated Jordan curves are avoided).

We will assume that K, as a Jordan curve, *is oriented.* K designates the knot with a definite orientation; $-$ K will denote it with the opposite orientation.

Let K, K_1 be two knots in the same $\mathfrak{E}_3$. We consider them as *equivalent:* $K \simeq K_1$, whenever there exists a homeomorphic deformation of $\mathfrak{E}_3$ into itself under which K_1 goes into K.

To illustrate the perversity of knots, Trotter has proved recently (by an infinity of rather simple examples) that there exist knots K not $\simeq -K$ (thus solving a long outstanding problem).

A *knot invariant* is a knot character which is the same for all equivalent knots. The central problem of knot theory is to find a collection of knot invariants which guarantee that if they are the same for two knots K, K_1 then $K \simeq K_1$ (K and K_1 are assumed inbedded in the same $\mathfrak{E}_3$). Although this central problem has been attacked, in our century at least, by many very eminent mathematicians, it is doubtful if we are nearer to a solution than a century ago.

The most important invariant of a knot K is the group of paths $\Pi(\mathfrak{E}_3 - K)$ of its complement. However Trotter's example shows that this is not a "decisive" invariant.

The following "knottists" J. W. Alexander and R. H. Fox, will be mainly mentioned. Alexander attacked knots in the twenties, Fox belongs to the forties to date. We owe to Alexander two noteworthy but related sources of invariants: Alexander matrices and Alexander polynomials. Both center around the concept of projection of a knot onto a plane. The projection is a plane graph whose sides may intersect, but one may organize the situation so that (a) the self intersections are never nodes of the graph; (b) they are always double points with distinct tangents.

The mere penetration of Knot theory requires a formidable amount of modern algebra, far more than I can go into here. A little of it is however indispensable even for a bare description of a few main concepts.

Let $G = \{g\}$ be a multiplicative group. Let J be the ring of integers. With G there is associated the goup ring JG, defined as the set of mappings $\nu : G \to J$ such that $\nu(g) = 0$ except for at most a finite set of g's $\in G$. Addition and multiplication in JG are defined by

$$(\nu_1 + \nu_2)\, g = \nu_1\, g + \nu_2\, g\,; \quad (\nu_1 \nu_2)\, g = \Sigma\, (\nu_1\, h)\, (\nu_2\, h^{-1}\, g)$$

for any ν_1, ν_2 of JG and any g of G. One may easily verify that JG is a ring under these operations; also that if n is any integer then $(n\nu)\, g = n\, (\nu g)$.

The *free calculus,* introduced by Fox, yields a most powerful technique for calculating knot invariants. The basis of this calculus is this definition of a *derivative*

D as the unique linear extension to JG of any mapping $D : G \to JG$ which satisfies for g_1, g_2 in G:

$$D(g_1 g_2) = Dg_1 + g_1 Dg_2.$$

For further information see CF Ch. VII.

The weight of algebra in Knot theory is best indicated by this: in CF out of 8 chapters 5 are on pure algebraic questions (mainly general group theory). I should like to recommend to the advanced reader and to any one interested in new and up to date problems, the *Guide to the literature* at the end of CF.

On braids

If one severs a knot at one joint one obtains a braid (Emil Artin 1925). The group of braids (Artin) is defined as:

$$\sigma_1, \sigma_2, \ldots, \sigma_n, \ \sigma_1 \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$
$$i = 1, 2, \ldots, n-1; \ \sigma_i \sigma_j = \sigma_j \sigma_i, \ (i-j) \neq 1.$$

These groups have been completely classified by Artin — they do not offer the complications of Knot groups.

II. RIEMANN AND RIEMANN SURFACES. CONSTRUCTION. NUMBER OF INTEGRALS OF FIRST KIND. THE WORK OF SCORZA.

1. Puiseux's theorem

This is really the initial place of algebraic topology. Not that Riemann himself thought of it that way, but this will cause no argument.

The problem attacked by Riemann, (around 1850) among many others, was the nature, as geometry, of a complex plane algebraic curve

$$F(x, y) = 0 \tag{1.1}$$

where F is a complex irreducible polynomial. Much was known about the geometry of real curves — since the period was post-Plücker — but "as geometry" complex curves remained obscure. Important information was contained in the

(1.2) *Theorem of Puiseux*. But even this theorem did not offer any information in the large about the curve F. However, we shall need the material provided by Puiseux.

Let $x = a$ be a value for which the roots of $F = 0$ in y remain finite. Let $y_1(x)$ be such a root. As x describes a small circle around $x = a$ in its complex plane the root $y_1(x)$ varies continuously and at all times say around $x = x_0$ on C it is holomorphic around x_0 as a function of x. Hence as x describes C once $y_1(x)$ returns to a value which is still a root of F in y but not necessarily the same root $y_1(x)$. Let it return to a different root $y_2(x)$, etc. There arises a set of say q roots $y_1(x), \ldots, y_q(x)$ which are circularly permuted as x describes C. This implies that these q roots are represented by q series in powers of $(x-a)^{1/q}$ or as

$$y(x) = b + \alpha(x-a)^{p_1/q} + \beta(x-a)^{p_2/q} + \ldots \tag{1.3}$$

The q roots of the circular system may be jointly represented by a unique series in t:

$$x = a + t^q, \qquad y = b + \alpha t^{p_1} + \beta t^{p_2} + \ldots, \qquad |t| < \rho \tag{1.4}$$

where $q, p_1, p_2, \ldots$, have no common factor.

Since the number of values $x = a$ with true circular representations is finite the points corresponding to $0 < |t| < \rho$, are *ordinary* points of the curve. That is: (a) to any such point there corresponds only one solution $y(a)$; (b) the corresponding $q = 1$.

We have assumed that a is finite. The points at infinity are taken care of by the standard transformation

$$x' = \frac{1}{x}, \ y' = \frac{y}{x}.$$

If m is the degree of F in y there are at most m such points and hence at most m series $\{x'(t), y'(t)\}$. The transformation just introduced merely means that the true space of the curve F is a projective plane. It was well known (as an analytical artifice) to the mathematicians preceding Riemann: Abel, Jacobi, Plücker, Weierstrass, and many others.

The set $\pi = \{$pair of series in t, point $t = 0$, number $\rho\}$ is called a *place* of F and $t = 0$ is the *center* of the place. The number ρ, the *convergency radius* of the series, is the *extension* of the place. It is agreed that the change in ρ, provided that it does not reach another singular place, does not affect π. (Singular place is one which is the center of several distinct places, or of a single place around which more roots than one $y_j(x)$ are permuted).

(The concepts related to places have been clearly set down by Hermann Weyl in the monograph: Die Idee der Riemannschen Fläche, Springer, Berlin (1913)). However, there are well founded reasons to believe that "places" were no strangers for Riemann. The main reason would go something like this. Let (a, b) be a center of several places $\pi_1, \ldots, \pi_s$. In his construction of the Riemann surface, Riemann always represents the π_h by s distinct points.

2. Construction of the Riemann surface.

Suppose that m is the degree of F in y. Take the sphere S of the variable x and mark on it two diametral points A and B so disposed that no great circle through them contains more than one of the *critical* points $a_1, a_2, \ldots, a_n$ which are centers of singular places. Mark on S arcs of great circles Aa_h and cut S along these arcs. Choose now one sphere S_h for each root $y_h(x)$ of F. Mark on S_h the cuts Aa_h which do permute $y_h(x)$. The complement Ω_h of the cuts on S_h is a 2-cell and the value $y_h(\xi)$ at $\xi \in \Omega_h$ is uniquely determined by the value $y_h(\xi(B))$. We look now at the 2-cell Ω_h and at its boundary the polygon π_h. Let all the π_h be positively oriented, that is let the Ω_h all be oriented in the same way. It follows at once that if A'_{hj} and A''_{hj} are the two sides of a cut A_{hj} then in that cut their orientations are opposite. Suppose then that A_{hj} is a cut permuting y_k with y_j. Then there correspond to it two cuts A'_{hk} in Ω_k and A''_{hj} in Ω_j and those two are oppositely oriented. Hence if we bring them back into coincidence, and similarly for all permuted pairs y_r, y_s the result is a closed surface $\Phi(F)$: the Riemann surface of the curve F. The construction has obtained these fundamental consequences:

(a) Φ (F) is covered by a finite collection of 2-cells $E_1, \ldots, E_m$, one for each root $y_j(x)$ of $F = 0$.

(b) If the polygons E_c, E_d have a common side then they are oppositely oriented relative to it.

(c) (less evident) Each point P of Φ (F) has a neighborhood in the surface which is a union of closed polygonal regions each making up a 2-cell.

(d) The surface Φ (F) is connected. This is a ready consequence of the irreducibility of the polynomial F. For if Φ (F) is not connected the roots $y_1(x), \ldots, y_m(x)$ may be divided into at least two collections say $y_1, \ldots, y_r$ and $y_{r+1}, \ldots, y_m$ whose elements are not permuted under the variation of x. Hence the symmetric functions of the y_h, $h \leqq r$ are meromorphic in x and so satisfy a relation $F_1(x, y) = 0$, where F_1 is like F, but of smaller degree in y, and hence it is a proper factor of F. Since this contradicts the irreducibility of F, the surface Φ (F) is connected;

(e) Property (b) implies that Φ (F) is orientable.

Conclusion. The preceding properties imply that Φ (F) is an orientable compact two dimensional manifold (in the sense of modern topology).

We notice also:

(f) Under an appropriate definition of place-continuity the collection of places $\{\pi\}$ is turned into a surface homeomorphic with Φ (F).

The statement just made implies the following important result:

Theorem. The Riemann surface is a birational invariant.

For the places have birational character and hence this holds also for their surface.

Characteristic. A particular case of a very general property (Euler-Poincaré characteristics) asserts the following property: Let the polygons of the decomposition of Φ (F) consist of α_2 polygons, with α_1 sides and α_0 vertices then *whatever this decomposition* we have the relation

$$X(\Phi) = \alpha_0 - \alpha_1 + \alpha_h = 2 - 2p. \tag{2.1}$$

This is a classical formula due to De Jonquière.

The number p is the well known *genus* of the curve F. It will be shown later that the *characteristic has topological character. Hence the genus* p *is a topological invariant of the Riemann surface and therefore of the curve* F.

A direct calculation of $X(\Phi)$ is of interest. Let $\beta_0, \beta_1, \beta_2$ be the analogues of the α for Φ. Evidently if α_i are the same numbers for a 2-sphere, then from Euler's result

$$\alpha_0 - \alpha_1 + \alpha_2 = 2.$$

Also $\beta_1 = m\alpha_1$, $\beta_2 = m\alpha_2$. But for each place with q permuting roots y (x) we lose $q - 1$ vertices. Hence if $N = \Sigma(q-1)$ then $\beta_0 = m\alpha_0 - N$. From this follows

$$\beta_0 - \beta_1 + \beta_2 = 2m - N = 2 - 2p$$

Hence this formula due to Riemann

$$N = 2(p + m - 1). \tag{2.2}$$

3. Topological models of a surface.

After Riemann, in the latter part of the 19th century, his surfaces, or more generally their topological type was deeply studied by a number of geometers (Klein, Clifford and others). Clifford showed that a surface of genus p was homeomorphic to a 2-sided disk with p holes. This model is identical to a sphere with p-handles (fig. 5).

From the Clifford model one may obtain with little difficulty the most significant model of all: a polygonal region with sides matched in a certain way (fig. 6). Draw on a plane a 4p-sided regular polygonal region whose boundary polygon Π is to be described so that the successive sides are labelled (with their orientations)

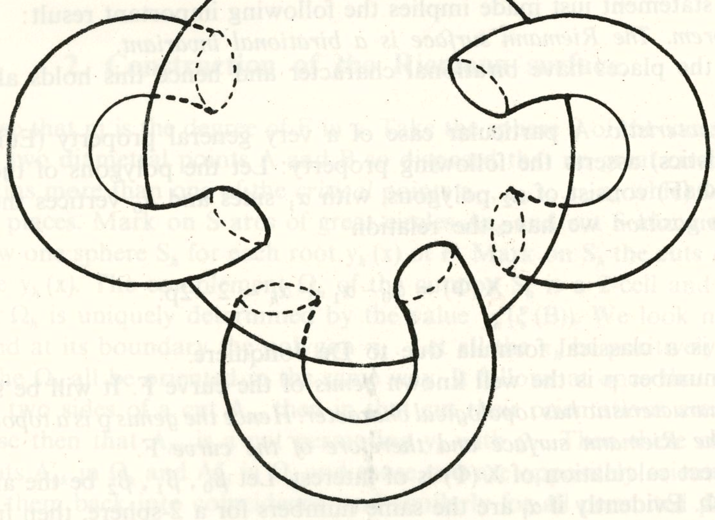

Fig. 5

$$a,\ b,\ a^{-1},\ b^{-1},\ c,\ d,\ c^{-1},\ d^{-1},\ \ldots,\ e^{-1}$$

(4p-sides). Let the 2-cell bounded by Π be Ω.

The labels are such that for instance d^{-1} means d described in the opposite direction. Let now all the 4p vertices be brought into coincidence, and match for instance d with d^{-1} so that d^{-1} is merely d described in the opposite way.

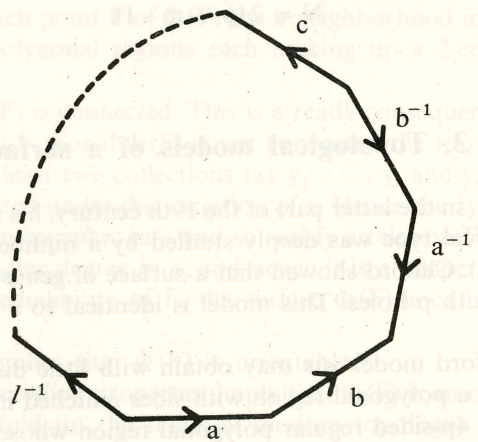

Fig. 6

From the new polygonal boundary, still called Π, say to the Clifford model is but a step and conversely. Hence the new model is a general model for a surface of genus p.

4. Analytical application.

One might get the impression from what precedes that the Riemann surface is a pure gometrical instrument without further ado. This would be entirely misleading. For Riemann, like all his mathematical contemporaries was strongly under the influence of the theories created and developed by Cauchy. His surfaces show this plainly: it is at least through analysis that he obtained some of his most beautiful results. However in expounding them I shall not endeavor to follow in Riemann's footsteps and shall not hesitate to utilize later results especially if they come under "early algebraic topology".

Consider then a function f(z) on the Riemann surface $\Phi(F)$ which is uniquely defined on $\Phi(F)$ or perhaps on a region $\mathscr{R} \subset \Phi(F)$. We assume this property: If P is a point of $\mathscr{R}$ there is a place π of center P and parameter t ($|t| < \rho$) whose points are all in $\mathscr{R}$. On Π the function f(t) is holomorphic in t. One defines f as holomorphic in $\mathscr{R}$ whenever it is holomorphic throughout $\mathscr{R}$.

5. Extended Cauchy theorem.

Let $\mathscr{R}$ be a 2-cell with boundary Π. Let f(z) *be holomorphic at all points of* $\bar{\mathscr{R}}$ ($\mathscr{R}$ *plus* Π). *Then*

$$\int_{\Pi} f(z)\,dz = 0.$$

Let L be a line dividing $\mathscr{R}$ into two similar regions $\mathscr{R}_1$ and $\mathscr{R}_2$ with boundaries Π_1 and Π_2. Then

$$\int_{\Pi} = \int_{\Pi_1} + \int_{\Pi_2}.$$

It is sufficient therefore to prove the theorem for Π_1 and Π_2.

Consider the original decomposition of $\Phi(F)$ into polygons (one for each sheet of the surface). Let A be a vertex of one of the polygons. The set U consisting of all the open polygons and edges together with A, with vertex A — called the *star* of A, written St A, is an open set of $\Phi(F)$. In fact it is a place of center A and say parameter t (t = 0 at A). The collection $\mathscr{U} = \{U\}$ of all St A, is a *finite open covering* of $\Phi(F)$. We recall that such a covering has a Lebesgue number $\lambda(\mathscr{U}) > 0$ with the property that if a set H on $\Phi(F)$ is of diameter $< \lambda(\mathscr{U})$ then H is contained in some set U.

Now upon carrying the subdivision process far enough we shall obtain sets $\mathscr{R}_0$ all of diameter $< \lambda$, hence each contained, with its boundary Π_0 in a set U_0 of $\mathscr{U}$. Let t be the uniformizing parameter of U_0. In U_0 the function f is holomorphic in t. Hence by Cauchy's theorem

$$\int_{\Pi_0} f(t) = 0$$

and this implies the theorem.

In the preceding proof it has been implicitly shown that an integral

$$\int_{z_0}^{z} f(z)\,dz$$

of a holomorphic function f(z) at all points of a path μ in a region of holomorphism of the function is well defined.

Let Q(z) be a rational function on the surface $\Phi(F)$: a function represented at all points of the curve by a *rational* function

$$S(x, y) = \frac{A(x, y)}{B(x, y)},$$

where locally the function is always represented by a convergent power series $t^k(\alpha + \beta t + \ldots)$, k a positive integer. The integral along *any* path of $\Phi(F)$

$$u = \int S(x, y)\,dx$$

is then uniquely defined and represents a holomorphic function on the entire Riemann surface. Such an integral is said to be *of the first kind.* We refer to it briefly as (ifk).

Let γ be a closed path on $\Phi(F)$. The value

$$\int_\gamma du = \int_\gamma S(x, y)\,dx = \omega$$

is then uniquely defined and called a *period* of u.

Going back to the model of a 4p sided polygonal plane region plus its boundary aba^{-1}, $b^{-1}cd\ldots$, set

$$\int_{a_\mu} du = \omega_\mu, \quad \int_{b_\mu} du = \omega_{p+\mu}, \quad \mu \leqq p.$$

Take now two (ifk) u_1 and u_2 and define their periods as

$$\omega_{i\mu}, \ \omega_{i,\,p+\mu}, \ i = 1, 2.$$

Then we have this all important

(5.1) *Theorem of Riemann:*

$$\sum \begin{vmatrix} \omega_{1\mu}, & \omega_{1,\,p+\mu} \\ \omega_{2\mu}, & \omega_{2,\,p+\mu} \end{vmatrix} = 0.$$

The proof is very simple.

Since u_1 and u_2 are holomorphic throughout $\Phi(F)$ we have

$$\int_\Pi u_1\,du_2 = 0$$

This integral is the sum of p terms each of the same type as the sum

$$(*) \qquad \int_a + \int_b + \int_{a^{-1}} + \int_{b^{-1}}$$

The first and third term combine to

$$\int_a [u_1(P) - (u_1(P + \omega_{12})]\,du_2 = -\omega_{21}\,\omega_{12}$$

The second and fourth term combine like

$$\int_b [u_1(Q) - (u_1(Q) - \omega_{11})]\,du_2 = \omega_{11}\,\omega_{22}.$$

Hence the sum (*) is

$$\begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix}$$

The μ-th set of four terms has the sum

$$\begin{vmatrix} \omega_{1\mu} & \omega_{1,\,p+\mu} \\ \omega_{2\mu} & \omega_{2,\,p+\mu} \end{vmatrix}$$

Hence

$$\int_\Pi u_1\,du_2 = \sum_{\mu=1}^{p} \begin{vmatrix} \omega_{1\mu} & \omega_{1,\,p+\mu} \\ \omega_{2\mu} & \omega_{2,\,p+\mu} \end{vmatrix} = 0$$

This proves Riemann's equality.

Let now $u = u' + iu''$ be a non constant (ifk) and let $\omega'_\mu, \omega''_\mu; 0 < \mu \leqq 2p$ be the respective periods of the *real* integrals u', u''. By Cauchy's inequality over $\Phi(F)$ we have

$$\int_\Pi u' \, du'' > 0.$$

Hence if we reason as before we obtain Riemann's inequality:

$$\sum \begin{vmatrix} \omega'_\mu & \omega'_{p+\mu} \\ \omega''_\mu & \omega''_{p+\mu} \end{vmatrix} > 0, \ \mu \leqq p. \tag{5.2}$$

(5.3) *Consequence. There are at most* p *linearly independent* (*ifk*) *mod constants.*

For if say there were p + 1: $u_1, \ldots, u_{p+1}$ then there would exist a linear combination

$$u = \lambda_1 u_1 + \ldots + \lambda_{p+1} u_{p+1}$$

such that every $\omega_\mu = 0$, $\mu \leqq p$. That is $\omega'_\mu = \omega''_\mu = 0$, which contradicts the inequality.

(5.4) *Digression. Theorem. There are exactly* p *linearly independent* (*ifk*) *modulo constants.*

That is one may find p linearly independent $\{du_h\}$, u_h is an (ifk) but no more.

We have already seen that the number p′ in question is $\leqq$ p. There remains to prove that $p' \geqq p$.

There are two distinct approaches to this property:

(a) A proof by Riemann using highly complicated analytical properties of the well known theorem of existence of potential functions. See Hermann Weyl loc. cit.

(b) A proof of a more algebraic nature based upon a reduction of singularities theorem of much more geometric nature, due in part to Max Nöther (around 1870) which states:

(5.5) *Theorem. An irreducible plane curve* F *may always be birationally transformed into a plane curve* G *whose only singularities consist of a finite number of double points with distinct tangents.*

We have already seen that the genus p has birational character. It is also evident from the definition of the (ifk) that each of them has individual birational character. Hence the number p′ of linearly independent (ifk) mod constants has the same character. Therefore to study their linear dependence we may freely replace the curve F by the curve G. That is we may assume that F has only the singularities just ascribed to G. This is the procedure that we shall follow.

Consider the most general curve of degree m − 3 passing through all the double points: *adjoint* of degree m − 3, also called *canonical* curve.

The curves of degree m − 3 have $\frac{(m-1)(m-2)}{2}$ arbitrary coefficients. Those passing through the δ double points satisfy that many linear equations. Hence they have at least

$$\frac{(m-1)(m-2)}{2} - \delta$$

arbitrary coefficients. Now from an earlier Riemann formula

$$N = 2(p + m - 1)$$

since N is the class of F and it has only double points with distinct tangents

$$N = m(m-1) - 2\delta. \tag{5.6}$$

Thus

$$p = \frac{(m-1)(m-2)}{2} - \delta.$$

This expression is actually the classical definition of the genus by Plücker.

Let us ask now for the *dimension* μ of the system of adjoints of degree m − 3. Since they are merely the curves of degree m − 3 through the double points

$$\mu \geqq \frac{(m-1)(m-2)}{2} - \delta = p.$$

Now given such an adjoint Q_{m-3} we may write the integral

$$\int \frac{Q_{m-3}(x, y)\, dx}{F'_y}$$

and we prove easily that it is an (ifk). It follows that the

$$\mu = p' \geqq p.$$

But we have already proved that $\mu \leqq p$. Hence $\mu = p$. This proves the theorem.

Let then $u_1, \ldots, u_p$ be a system of p linearly independent (ifk). Form their period matrix

$$\Omega = [\omega_{j\nu}], \ j = 1, 2, \ldots, p, \ \nu = 1, 2, \ldots, p$$

Let $\eta = \lambda_1 u_1 + \ldots + \lambda_p u_p$, η_ν, $\eta_{p+\nu}$, $\nu \leqq p$. Owing to Riemann's inequality the η cannot all be zero, whatever the choice of the λ's. Hence

$$[\omega_{j\mu}],\ j,\ \mu \leqq p$$

is of rank p. We may therefore apply a linear transformation such that this matrix becomes a unit matrix. That is

$$[\omega_{j\mu}] = [1,\ [\tau_{j\mu}]],\ \mu \leqq p.$$

The corresponding (ifk), written usually v_j are the *normal* (ifk).

Let

$$\tau = \tau' + i\tau''.$$

From Riemann's equality and inequality we infer at once that

(a) the matrix τ is symmetrical;

(b) it is the matrix of a positive definite quadratic form

$$\sum \tau_{jk} x_j x_k.$$

6. Scorza's theory of Riemann matrices (1915).

The preceding results have been strongly generalized and at Scorza's hand given rise to a very interesting new theory. We will say a few words about it.

The basic scheme of Scorza was *not* to take special bases for the cycles and the (ifk). We take then p linearly independent (ifk) and 2p independent one-cycles $\gamma_1, \ldots, \gamma_{2p}$ and write down their period matrix as a $p \times 2p$ matrix Ω_1. We then define

$$\Omega = \begin{bmatrix} \Omega_1 \\ \bar{\Omega}_1 \end{bmatrix}.$$

A more or less simple calculation shows then that the Riemann equality and inequality combined are equivalent to the existence of a unimodular skew symmetric matrix $C(|C| = 1)$ such that $i^{2p}\Omega' C\Omega = M$ is of the form

$$i^{2p}\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

where A is a $p \times p$ matrix, $A^* = (\bar{A})'$, $|A| \neq 0$, so that M is a hermitian positive definite matrix.

So far we only have a "clever" reformulation of Riemann. Scorza's departure is this:

(2.17) *Definition.* A Riemann matrix is a $p \times 2p$ matrix of type $\begin{bmatrix} \Omega_1 \\ \bar{\Omega}_1 \end{bmatrix}$ such

that there exists a skew-symmetric *rational* matrix C such that

$$\Omega' C\Omega = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

No condition is placed on A. Whenever

$$i^{2p}\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} = M$$

Ω is said to be a *principal* matrix.

Given a Riemann matrix Ω there may be many matrices C which merely satisfy the definition (no hermitian matrix condition imposed). The number k of linearly independent matrices C is the *singularity index* of Ω (Scorza had 1 + k where we have k, but the latter yields much simpler formulas).

Still another index h: *multiplication* index was introduced by Scorza, when the only condition imposed on C is that C need not be skew symmetric. Both indices have highly important applications in the theory of algebraic varieties.

III. HENRI POINCARÉ AND ALGEBRAIC TOPOLOGY

1. Poincaré: the founder of algebraic topology.

Presentday topology consists of two distinct parts: point set topology and algebraic topology. The first has mainly been the prerogative of Poland plus a strong American component: the school of R. L. Moore (of Austin, Texas). At all events, I shall only deal with algebraic topology.

The enormous impetus given by Poincaré to our field deserves to call him its founder. His contribution is contained in his paper Analysis Situs (1895) together with its five complements (till 1909), two on applications to algebraic surfaces.

Incidentally Poincaré did not say "topology" but "Analysis Situs," a beautiful but awkward term at best. Since the midtwenties "topology" has been generally adopted (much earlier I believe in Germany).

My purpose in this chapter is to develop Poincaré's basic concepts, but as seen by a modern: with algebra, especially group theory, in evidence. No doubt Poincaré himself, had he lived long enough, would have adopted this mode of exposition. Where simpler proofs than his have appeared, I do not hesitate to outline them. It must be said that simplifications have largely been due to the injection in topology of group theory by Emmy Noether (through Alexandroff).

A few post Poincaré relevant contributions will be indicated by* .

In conformity with modern usage I generally omit the term "dimensional" and say: n-space, n-manifold, etc. for n-dimensional space, manifold, etc.

2. Manifolds in the sense of Poincaré.

The whole of Poincaré's first Analysis Situs paper is devoted to manifolds. However, as is often the case with him, he is never too precise about what meaning he attaches to the term (in French, variété). I have therefore endeavored to extract a more precise meaning from his description.

Let $\mathfrak{E}_r$ denote a real Euclidean r-space referred to coordinates $x = (x_1, x_2, \ldots, x_r)$. By an *absolute* n-manifold $M_n \subset \mathfrak{E}_r$, $(n < r)$ I shall understand a compact, connected subset of $\mathfrak{E}_r$ without boundary, represented by the equations

$$f_p(x) = 0 \tag{2.1}$$

where the $f_p(x)$ are of class C^k, $k \geqq 2$, and Jacobian rank $r-n$ in some bounded set $\supset M_n$. It is then known that any point ξ of M_n has in M_n a neighborhood $U(\xi)$ which is an n-cell differentiably parametrized by n local coordinates $u_1, \ldots, u_n$ with the condition that if two such neighborhoods say $U(\zeta)$, $U'(\zeta)$ overlap at ζ with respective parameters u_k, u'_k then each set is differentiable in the other with a Jacobian say $J = \dfrac{\partial(u)}{\partial(u')} \neq 0$ and continous at ζ.

Notice that compactness of M_n implies that it has a finite open convering $\{U(\zeta)\}$. If the Jacobians J have a fixed sign over M_n then M_n is *orientable*, otherwise *non-orientable.*

One may equally define M_n directly as possessing a finite open covering by parametric n-cells $\{U\}$ with the above overlapping property. This is the modern definition of "differentiable manifold." However, while Poincaré indicates its equivalence with the definition by the system (2.1), it is the latter upon which he always falls back.

I called "absolute" the manifolds just defined. This mention, however, will usually be omitted.

Suppose that M_n is orientable so that the Jacobians have a fixed sign. We may then orient M_n by choosing a given order of the parameters u_k in some $U(\zeta)$ and use that ordering, modulo an even permutation, as determining the Jacobian signs and hence the orientation of M_n. One refers then to $U(\zeta)$ as *indicatrix* of M_n.

Some examples of absolute M_n. In $\mathfrak{E}_3$, a sphere, a torus, in $\mathfrak{E}_4$ a Riemann surface are examples of orientable M_n. On the other hand a projective plane is a nonorientable M_2.

(2.2) *Relative or open manifolds.* In an M_n let M_p be a connected and compact subset contained in an open subset W of M_n. Thus W is a neighborhood of M_p in M_n. Set $V = U \cap M_p$. The collection $\{V\}$ is a finite open covering of M_p and $M_p \subset W$. The set $\overline{M}_p - M_p = \partial M_p$ is the *boundary* of M_p. We will assume that every point ζ of M_p has a neighborhood $V(\zeta)$ parametrized by p parameters $v_1, \ldots, v_p$ with the same overlapping property as for M_n. Orientability, indicatrices, etc. are defined as for M_n.

An additional hypothesis is

(2.3) ∂M_p *consists of a finite set of closures of manifolds* M^h_{p-1}.

(2.4) Let $\zeta \in M^h_{p-1}$ and let $v_1, \ldots, v_{p-1}$ be local parameters for ζ on M^h_{p-1}. Since ζ is a point of a parametric p-cell of M_p, whose intersection with M^h_{p-1} contains a small parametric (p − 1)-cell X_{p-1}, one may choose the parameters x_j of the latter so that together with one local parameter v of M_p at ζ, they make up a set of p local parameters of M_p at ζ. We shall use this property in a moment.

We refer to M_p as an *open* p-manifold.

Example. Let S_3 be a sphere in $\mathfrak{E}_4$. A solid cube in S_3 is an open M_3. Here ∂M_3 consists of the surface of the cube. The faces of the cube are manifolds $M_2 \subset \partial M_3$. Together with the edges and vertices of the cube they make up ∂M_3.

3. Boundary relations. Homologies.

The situation remaining the same write for the present M_{p-1} for M^h_{p-1}. Let $\varepsilon_p(v, x_1, \ldots, x_{p-1})$ and $\varepsilon_{p-1}(x_1, \ldots, x_{p-1})$, ($\varepsilon_p$ and $\varepsilon_{p-1} = \pm 1$) represent indicatrices for M_p and M_{p-1}. The product $[M_p : M_{p-1}] = \varepsilon_p \varepsilon_{p-1} = \pm 1$ is the *incidence number* of M_p and M_{p-1}.

More generally if M^j_p and M^h_{p-1} are all oriented p− and (p − 1)-manifolds in M_n then one defines the incidence number $[M^j_p : M^h_{p-1}]$ as 0 or ± 1 : 0 when M^h_{p-1} is not in ∂M^j_p, and ± 1 according to the preceding rule when M^h_{p-1} is in ∂M^j_p.

Call for the present (temporarily) *p-chain of* M_n a finite expression

$$c_p = \Sigma\, m_j M^j_p.$$

(The felicitous term "chain" is due to Alexander.) I define a chain-boundary ∂c_p under the rule

(a) $\partial M^j_p = \Sigma\, m_j [M^j_p : M^h_{p-1}] M^h_{p-1}$; $\partial M_0 = 0$ (M_0 is a point);

(b) $\partial c_p = \Sigma\, m_j \partial M^j_p$;

(c) if in the last sum M^j_{p-1} occurs with a total coefficient μ_h we define

$$\partial c_p = \Sigma \mu_h M^h_{p-1}.$$

Following Poincaré, if one is not interested in the special ∂c_p at the right then one expresses it by a *homology*

$$\Sigma\, \mu_h M^h_{p-1} \sim 0.$$

Such homologies do combine like linear equations. We also note:

Definition. A chain c_p such that $\partial c_p = 0$ is called a *p-cycle.*

One proves that

(3.1) ∂M_p *is a* $(p-1)$*-cycle; hence every* ∂c_p *is a* $(p-1)$*-cycle: boundary cycle.*

In operator symbolism

$$\partial\partial = 0. \tag{3.2}$$

A set of p-cycles $\gamma_p^1, \ldots, \gamma_p^s$ is *independent* whenever they satisfy no homology. The maximum number of independent p-cycles is the p-th *Betti number* $R_p^d(M_n)$.

Remarks. I. R_p^d has no topological pretension since it depends strictly upon the differential structure of M_h. No such distinction was ever made by Poincaré.

II. The notation [:] is taken from Tucker's thesis (Princeton, 1931) and will be widely utilized later.

III. The notion of *cobordism*, developed by Thom, and in full vogue nowadays finds its origin in the ideas of Poincaré.

IV. Poincaré said "one or two sided (unilatère or bilatère)" where one says today "nonorientable or orientable," suggested by Alexander. His just criticism of Poincaré's terminology was that it referred really to a relationship with the ambient space, whereas orientability or nonorientability characterize an intrinsic property of the space (of the manifold M_n).

4. Complex analytic manifolds.

These are the M_{2n} whose 2n-cells are "complex analytic", that is parametrized by n complex variables $\{x_h \mid 1 \leqq h \leqq n\}$ with the condition that if U (x) and U (y) are two of the 2n-cells overlapping at the point ζ then near ζ the complex variables y are holomorphic functions of the x.

Let $x_h = x_h' + ix_h''$. (The x', x'' are real.) Agree to orient U (x) by naming the parameters in the order $(x_1', x_1'', \ldots, x_n'')$. Then the Jacobians $\frac{\partial(x_1', \ldots, x_n'')}{\partial(y_1', \ldots, y_n'')}$ are all *positive*. Hence *analytic manifolds are all orientable, and this in a unique manner.*

Example. A nonsingular algebraic variety is always an orientable M_{2n}.

In M_{2n} the analytic manifolds M_{2p} are likewise oriented by the scheme just given. However, the arbitrary differentiable submanifolds have perfectly arbitrary orientations.

5. Intersection of orientable manifolds.

Let M_p and M_{n-p} be orientable submanifolds of an orientable M_n. Let ξ be

a common *isolated* intersection of the two with parameters $\{x_h \mid 1 \leqq h \leqq p\}$ and $\{x_j' \mid 1 \leqq j \leqq n-p\}$, and such that $\{x_h ; x_j'\}$ is a set of parameters for M_n at ξ.

Suppose now that $\varepsilon\{x_h\}$, $\varepsilon'\{x_j'\}$ and $\varepsilon_0\{x_h ; x_j'\}$ all in their proper natural order, with $\varepsilon_0, \varepsilon, \varepsilon' = \pm 1$, are indicatrices of M_n, M_p, M_{n-p}. Then we assign to ξ the coefficient $\varepsilon_0 \varepsilon\varepsilon' = \pm 1$ to be counted as *algebraic intersection* of M_p, M_{n-p} in M_n. Let ξ be described as a *simple intersection* of M_p and M_{n-p}.

Let M_p, M_{n-p} have only isolated intersections $\xi_1, \ldots, \xi_s$ all simple with coefficient ε_h for ξ_h. By the *intersection number*, (M_p, M_{n-p}) is meant the sum

$$(M_p, M_{n-p}) = \Sigma\, \varepsilon_h. \tag{5.1}$$

Note that

$$(M_{n-p}, M_p) = (-1)^{p(n+1)}(M_p, M_{n-p}) \tag{5.2}$$

By approximations one may extend the meaning of (M_p, M_{n-p}) when ∂M_p and ∂M_{n-p} are disjoint. By a far from simple argument Poincaré proved:

(5.3) *Theorem.* N.a.s.c. *in order that* $M_p[M_{n-p}] \sim 0$ *is that*

(5.4) $(M_p, M_{n-p}) = 0$ for every $M_{n-p}[M_p]$.

(5.5) *Remark.* All the preceding results were obtained by Poincaré in his first paper Analysis Situs (§ 9). However, he had recourse to his first definition of a manifold together with a very subtle analytical argument.

The treatment which I have given is essentially parallel to that of chain intersections in a manifold, of LT., Ch. 4.

6. Duality in manifolds.

Let now $\{M_p^h \mid 1 \leqq h \leqq R_p^d\}$ and $\{M_{n-p}^j \mid 1 \leqq j \leqq R_{n-p}^d\}$ be maximal independent sets relative to $\sim$ of M_p's and M_{n-p}'s of M_n *Let* ρ *be the rank of the intersection matrix* $[(M_p^h, M_{n-p}^j)]$.

Applying (5.3) we find at once that $R_p^d = \rho = R_{n-p}^d$. This is the

(5.6) *Duality theorem of Poincaré. The Betti numbers* $R_p^d(M_n)$ *and* $R_{n-p}^d(M_n)$ *for an orientable* M_n *are equal.*

As we shall see later (Ch. V) I have greatly generalized this fundamental property.

7. Group of paths.

Let X be an arcwise connected metric space and let A be a given point of X. Let l be the directed segment $\alpha \leqq x \leqq \beta$, $\alpha < \beta$, and let ϕ map $l \to X$ so that $\phi(\alpha) = \phi(\beta) = A$. The image $\lambda = \phi(l)$ is a loop from A to A. Take the collection $\Lambda = \{\lambda\}$ with the following conventions: (a) if λ is homotopic to A in X. write $\lambda = 1$; (b) λ described in the opposite sense is written λ^{-1}; (c) if ψ maps l in a second loop λ'

then λ followed by λ' is a loop written $\lambda' \lambda$. Under these conventions Λ is a group g (A). If B is a second point of X and μ = BA a directed arc from B to A the operations of g (B) may be represented by $\{\mu^{-1} \lambda\mu\}$ where λ is any operation of g. Hence the groups g (A) and g (B) are *similar*. Upon identifying the operations λ and $\mu^{-1} \lambda\mu$, for all points $B \in X$, there results an abstract group π (X), the *Poincaré group*, or *group of paths* of X. It is generally non-commutative. *It is also* (obviously) a *topological invariant of the space* X. In the ulterior investigations of Poincaré this group plays a very important role. For a reason to appear in a moment its general designation is (X), and it is also called first homotopy group of the space X.

8*. Homotopy groups and homotopy type of Hurewicz.

The group π_1 has been generalized (around 1935) in a very fortunate way. Let X, A be as before and let S_n be an n-sphere (generally $n > 1$) on which a certain point P is designated as fixed. Let ψ be a map $S_n \rightarrow X$ such that $\psi P = A$. The collection of the maps $(\psi S_n, A, P)$ may be made into a group, more or less as done by Poincaré for π_1. The only point, not obvious, is the mode of combination of these operations. Let me merely say that ψ_1 and ψ_2 are combined *additively*, as the combination is commutative, except for the Poincaré group π_1. The new groups are freed from dependence upon A and P and called *n-th homotopy groups of* X, written π_n (X). This explains the π_1 designation for the Poincaré group.

Hurewicz groups have occupied a central position in modern algebraic topology. Although they are commutative, they do not have the rather simple properties of homology groups. This has greatly enhanced their importance.

Homotopy type. This is another noteworthy concept introduced by Hurewicz. Two topological spaces X, Y are of the same homotopy type whenever there exist mappings $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ such that $\psi\phi$ is homotopic to the identity as a mapping $X \rightarrow X$ and $\phi\psi$ is homotopic to the identity as a mapping $Y \rightarrow Y$. This is not quite homeomorphism, but the closest approach to it and assuredly much more elastic. This is why it has been in much favor among modern topologists.

9. Examples.

In Analysis Situs Poincaré constructed 8 examples of 3-spaces by matching appropriate faces of a cube (first four examples) or of a regular vectahedron. His purpose was to obtain explicit 3-manifolds whose Betti numbers and groups π_1 could be computed. The second example is to be rejected as not corresponding to an M_3.

Of particular interest is his 5th example for this reason. Poincaré desired to settle the question whether Betti numbers alone were sufficient to characterize an M_n, $n > 2$. The examples in question enabled him to answer in the negative. For

he obtained a whole family of 3-manifolds with the same Betti numbers but different groups π_1 and hence topologically distinct. In fact a careful study of these manifolds have produced $R_0 = R_3 = 1$, $R_1 = R_2 = \{1, 2, 3\}$ and yet there are an infinity of distinct groups π_1 (see Analysis Situs, p. 83).

10. Complexes.

Soon after Poincaré's first Analysis Situs paper the Danish mathematician Heegard criticized his approach, more particularly for having missed torsion. In the Introduction to his first Complement Poincaré answered in part Heegard, but perhaps did not realize that his general "homology" description failed to cover a variety of cases. It was also clear to him that his general method was far from suitable for deriving for example his very general formula for the characteristic.

The upshot was that he introduced an entirely new approach to algebriac topology: the concept of *complex* and the highly elastic algebra going so naturally with it.

While Poincaré's complexes were formally only applied by him to manifolds, they have a far broader range. Moreover his complexes were made up of quite general cells. It has been found more and more expedient to base everything on simplicial complexes, and their easy proofs.

(10.1) *Simplexes.* Take (n + 1) linearly independent points (= vectors) in $\mathfrak{E}_{n+p}$, $p \geqq n$, say $A_0, A_1, \ldots, A_n$. The set of points,

$$A = k_0 A_0 + k_1 A_1 + \ldots + k_n A_n\,;\; 0 < k_h < 1,\; \Sigma k_h = 1,$$

constitutes an n-simplex σ_n. It is, and will always be, assumed oriented, by the order of naming the A_h, modulo an even permutation.

By replacing $n-p$ of the "< 1" by "= 0" one obtains a p-face σ_p of σ_n with a suitable orientation. Given σ_{n-1} let $\varepsilon_n, \varepsilon_{n-1} = \pm 1$ be such that $\varepsilon_n \{A_{i_0} \ldots A_{i_n}\} = \sigma_n$, and $\varepsilon_{n-1} \{A_{i_0} \ldots A_{i_{n-1}}\} = \sigma_{n-1}$. Then $\varepsilon_n \varepsilon_{n-1} = \pm 1 = [\sigma_n : \sigma_{n-1}]$ is the *incidence number* of the two simplexes.

(10.2) *Simplicial complex:* $K = \{\sigma\}$ is a finite collection of disjoint simplexes such that if $\sigma \in K$ then every face of $\sigma \in K$.

For any σ_p and σ_{p-1} of K there is an incidence number $[\sigma_p : \sigma_{p-1}] = 0$ or ± 1, 0 when σ_{p-1} is not a face of σ_p, and ± 1 according to the above rule when σ_{p-1} is a face of σ_p.

I will now follow the modern treatment, rather than the very details contained in the second and third Complements. Let α_p denote the number of p-simplexes of K.

A p-chain is a linear integral expression

$$c_p = \Sigma\, m_h \sigma_p^h,\; 1 \leqq h \leqq \alpha_p\,.$$

One defines a boundary (p – 1)-chain of σ_p as

$$\partial\sigma_p = \Sigma\,[\sigma_p : \sigma^h_{p-1}]\,\sigma^h_{p-1}$$

and the boundary of c_p by linear extension as

$$\partial c_p = \Sigma\, m_h\, \partial\sigma^h_p .$$

It is then easily shown that

$$\partial\partial = 0. \tag{10.3}$$

A c_p with $\partial c_p = 0$ is a *p-cycle*. Hence:

(10.4) *Every* ∂c_p *is a* (p – 1)*-cycle called a bounding cycle.*

(10.5) *Evidently the collections* C_p, Z_p, B_p *of chains, cycles, bounding cycles are additive groups. Moreover*

$$C_p \supset Z_p \supset B_p$$

where each term is a subgroup of its predecessor.

From this follows that $H_p = Z_p/B_p$ *is likewise an additive group: integral* p-th *homology group of* K.

From a fundamental result of Frobenius, rediscovered by Poincaré (2nd Complement) we have:

(10.6) *Theorem. The group* H_p *has the following structures:*

$$H_p \simeq I_1 \oplus I_2 \oplus \ldots \oplus I_{R_p} \oplus T_p$$

where the I_h *are infinite cyclic and* T_p *is finite. More precisely*

$$T_p \simeq \Theta_1 \oplus \ldots \oplus \Theta_r$$

where the Θ_h *are finite cyclic. If* t^h_p *is the order of* Θ_h *then* t^h_p *divides* t^{h+1}_p.

The t^h_p are the *torsion* coefficients of Poincaré and R_p is the p*-th Betti number* of K.

By a fairly simple calculation one obtains the relation

$$R_p = \alpha_p - r_{p+1} - r_p \tag{10.6}$$

where r_h is the rank of the *incidence matrix*

$$\eta_p = [\sigma^j_p : \sigma^k_{p-1}] .$$

Hence

(10.7) *Theorem of Poincaré. The characteristic* $\chi(K) = \Sigma(-1)^p \alpha_p$ *satisfies the relation*

$$\chi(K) = \Sigma(-1)^p R_p .$$

(10.8) *Barycentric subdivision. A subdivision* $K_1 = \{\zeta\}$ *of* K, *with simplexes* ζ, is defined by the condition that every $\sigma \in K$ is a union of ζ' s. The barycentric type is particularly simple.

Let n = dim K.

Let the *derived* of K to be defined, be denoted by K'. If n = 0 (a finite set of points) let $K' \equiv K$. If K has ν simplexes suppose that K' has been defined for $\nu - 1$.

Let σ be an n – simplex of K and let $K_1 = K - \sigma$. Thus K'_1 is known. Call P the barycenter of σ. Join P by arcs to all the points of $(\partial\sigma)'$. Replacing σ by the resulting new simplexes, including P yields σ'. The orientations are defined by the condition that in $\bar{\sigma}$ they are determined so that

$$\partial(\sigma)' = (\partial\sigma)' .$$

Once K' is defined, one determines the derived sequence $K', K'' \ldots K^{(n)} \ldots$, by the condition $K^{(n+1)} = K^{(n)'}$. One proves then easily

$$\mathit{Mesh}K^{(n)} \to 0 \text{ with } 1/n. \tag{10.9}$$

This is the most important property of $\{K^{(n)}\}$.

(10.10) *Special case of manifolds.* When K is an M_n one may construct a dual complex K^*_n which has the same Betti numbers and torsion coefficients as K_n itself but with complementary dimensions.

The construction of K^*_n is simple enough. Let K'_n be the first derived of K_n and let $\{\zeta\}$ be its simplexes. Given $\sigma_p \in K_n$ the simplexes ζ with a single vertex (centroid) of σ_p and all others exterior to σ_p make up an (n – p)-cell σ^*_{n-p} and $K^*_n = \{\sigma^*_{n-p}\}$.

The relation between K_n and K^*_n leads to Poincaré's famous duality relations

(a) *for Betti numbers*

$$R_p(M_n) = R_{n-p}(M_n) \tag{10.11}$$

(b) *for torsion numbers*

$$t^h_p = t^h_{n-p-1}$$

[For details see LT Ch. 1].

(10.13) *Remark.* We recall again the origin of "homology." When two chains c_p, c'_p differed by a boundary ∂c^{p+1}, Poincaré wrote $c_p \sim c'_p$ or $c_p - c'_p \sim 0$. These relations, called homologies combined like linear relations. In other words they form groups: homology groups.

(10.14)* *Various types of coefficients.* While Poincaré only dealt with integral chains, cycles, etc., wide extensions were soon made to other types. I just mention: mod 2, Tietze; mod m (m prime) Alexander; rational coefficients, Lefschetz; (these are the same as Poincaré's: $\sim$ with division allowed: my later homologies $\approx$); any number system (real or complex) which is a field, Pontrjagin. These last led

Pontrjagin to his famous duality: simultaneous in the complex and the coefficients.
(10.15)* *Wide extensions to infinite complexes* will be found in LAT.

11. Subdivision invariance.

In the first Complement Poincaré dealt at length with subdivision and barycentric subdivision of a complex and proved that under them his Betti numbers, characteristic relation, torsion numbers, and for manifolds the manifold property and duality relations were subdivision invariant. He seems never to have attacked topological invariance.

Problems posed by Poincaré will be discussed at the end of Ch. 4.

IV. ALGEBRAIC TOPOLOGY AFTER POINCARÉ

1. A touch of topological history.

After 1904 Poincaré turned his attention to some arduous problems suggested by his previous work. He attacked applications to algebraic geometry (see my note "A page of mathematical autobiography") and to dynamics, more particularly to the famous theorem of Poincaré-Birkhoff discussed below.

Three important events mark the period before 1910 and immediately after: (a) the introduction by Tietze (1909) of chain coefficients mod 2, the first departure beyond Poincaré; (b) the advent on the scene of the powerful figure of L. E. J. Brouwer the advocate par excellence of strict rigor. Curiously in his early years the Poincaré concepts played little role in his work; (c) the definition by Lebesgue of dimension for compact metric spaces. Finally the most salient features of the period before 1923 (I omit my own work on algebraic geometry) are the appearance of Oswald Veblen and J. W. Alexander at Princeton.

Beyond 1923 we find my extensive work on coincidences and fixed points together with their extensive and necessary ramifications; the related work of Hopf (Berlin); various contributions by Alexander notably on knots already mentioned; the contributions of Morse on critical points and applications to the calculus of variations; the research of Alexandroff (Moscow) on compact and dimension theory. This will take us more or less to 1930: roughly my intended terminal point.

2. The Poincaré-Birkhoff theorem

This is the last partly topological question that occupied Poincaré. In a long mémoire (Circolo di Palermo) he stated the theorem, exposed his unsuccessful endeavors to prove it and motivated his publication with the expressed hope that perhaps a younger man would be more successful. This hope was fulfilled with the solution of young Birkhoff which appeared in the Transactions (1912) soon after Poincaré's death! In a sense this marked the entrance of the U. S. into the new world of topology.

The problem consists in this: – Let T be a topological mapping, area preserving, of a plane closed annular ring between two circles sending the two into opposite directions. To prove that T has at least one fixed point. Birkhoff's solution (1912 Transactions of the Am. Math. Soc.) is not only brilliant but very short. It marks the beginning of his extensive work on celestial mechanics: his later research. Birkhoff not only proved the theorem but completed it by showing that if T is not area preserving then either there is a fixed point or else some Jordan curve in the ring surrounding the inner circle is mapped by T into its interior or else into its exterior. This is a strictly topological property – which is not the case for the theorem itself.

The initial theorem has many applications to dynamics, notably in the study of the various periodic solutions near one such solution.

3. Henri Lebesgue and his definition of dimension

Let X be a compact metric space and let $F = \{F_k\}$ be a finite closed covering of X. The *order* of F, written ω (F), is the least number of sets F_σ minus unity which have a common point. Lebesgue defines the dimension of X, written dim X, as the least order of F of mesh $< \varepsilon$ as $\varepsilon \to 0$, and this for all possible F. This is the first appearance of the concept of "order of a covering", found so useful later. This dimension was identified later with the Menger-Urysohn classic by Brouwer.

4. The early work of L. E. J. Brouwer

This work was done around 1910. One of his early contributions was a rather short proof of the Jordan curve theorem (the second accurate proof; the first was given several years earlier by Veblen). He also gave a proof of the invariance of regionality. That is if Ω_m, Ω_n are two Euclidean regions, with $m \neq n$, then they could not be homeomorphic. In more modern language assuming that Ω_m and Ω_n could correspond under an homeomorphism T_m of Ω_m and Ω_n, their local Betti numbers (defined later) would have to be equal. But those R_h of Ω_m are zero for $0 < h < m$, with $R_m = 1$; similarly for $\Omega_n : R_h = 0$ for $0 < h < n$ and $R_n = 1$, which contradicts $m \neq n$.

The more striking result of Brouwer coming a lot closer to our topic is this:

– Let M_n, M'_n be two absolute orientable, manifolds. Let T be a mapping $M_n \to M'_n$. Assuming the two manifolds simplicial a suitable subdivision of M'_n has its n-simplexes covered the same algebraic number μ of times by images of those of M_n and μ is a topological invariant of the triple (M_n, M'_n, T). In terms of more modern topology the result is readily obtained. For if γ_n, γ'_n are the fundamental n-cycles of the manifolds: $T\gamma_n \sim \mu\gamma'_n$ in M'_n and μ is known to be a topological invariant. It is called degree of the mapping.

Noteworthy corollaries for mappings of spheres were obtained by Brouwer. Many other striking topological results are due to Brouwer but we cannot deal with them here.

5. Oswald Veblen as topologist.

He really began his work in the early part of the century. He was as much a rigorist as Brouwer, but operated first out of Chicago with E. H. Moore as mentor (under whom he took his doctorate). Moore was likewise given to full rigor, but less exclusively than the early Veblen. At any rate Veblen, perhaps under Moore's influence, or under the appearance of David Hilbert's (Gottingen): Uber die Grundlagen der Geometrie, was early launched into geometry. For some years he studied polyhedra – source of his proof (first correct) of the Jordan curve theorem. He then launched into his major work: Projective geometry: 2 volumes, close to 1000 pages, first volume coauthored by J. W. Young. The second volume already shows leanings towards topology. This occupied him till about 1913. As a professor at Princeton he was fortunate to have as a disciple J. W. Alexander the outstanding topologist. Their collaboration led to a significant but short paper in the Annals of Mathematics of 1913 in which their aim – fully accomplished – was to present Poincaré's main ideas in Analysis Situs and complements, in strict rigorous manner. This led to Veblen's monograph "Analysis Situs" (Colloquium Lectures 1921; lectures given in 1916), which had the same objective as the short note, but with far more details. Noteworthy in it is a proof of the invariance of the homology groups for an n-complex (first for a 3-manifold is due to Alexander).

6. J. W. Alexander as topologist

As a mathematician and above all as topologist Alexander was distinguished by exceptional originality. At first he was attracted by the many questions left pending by Poincaré. Thus in 1915 when he was still a graduate student he gave the first proof of the topological invariance of the Betti numbers of a 3-manifold M_3. During a one-year visit to Italy he also showed that the algebraic invariant of Zeuthen-Segre had really topological character (proof by extension of the Riemann surface concept).

The invariance proof of the numbers R_3 introduced the first ideas of the future classical deformation theorem.

Thereupon World War I produced a 3-years interruption. In the early postwar period Alexander produced several noteworthy results. I mention particularly:

(a) Poincaré had already produced two orientable M_3's with equal Betti numbers but with different group of paths hence topologically distinct. Therefore homology groups were insufficient to characterize 3-manifolds. Alexander went further and produced a very simple example of two topologically distinct M_3's but with the same group of paths and therefore equal Betti numbers (and no torsion). Hence homology and group of paths identity were insufficient to distinguish two M_3's. The example is simple enough. Two solid tori (in $\mathfrak{E}_3$) could be identified at their bounding surfaces so that the characters just mentioned be the same. However this could be done so that one obtains two topologically distinct M_3's (so-called lense spaces.)

(b) The generalized Jordan curve theorem. One may presume that a 2-sphere S_2 in $\mathfrak{E}_3$ has for (bounded) complement a 3-cell. Alexander gave an example where the complement has an infinite group of paths. On the other hand he proved also that if S_2 is analytical then the complement was effectively a 3-cell.

(c) A remarkable result of Alexander was his famous duality theorem (1922), the first beyond Poincaré. Given a complex immersed in an n-sphere S_n the Betti numbers satisfy:

$$R_p(S_n - K) = R_{n-p-1}(K) + \delta_{p0} - \delta_{p,\,n-1},$$

where the δ's are Kronecker indices: $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. As an application

$$R_0(S_n - K) = R_{n-1}(K) + 1$$

which expresses the number of components of $S_n - K$ (the number R_0) in terms of the Betti number $R_{n-1}(K)$.

Actually Alexander's result holds for any, say compact subset of S_n. It is also a special case of my, more general duality theorem proved several years later.

(d) From 1926 on Alexander dealt at considerable length with an improved organization of complexes and in particular obtained a new proof of the topological invariance of homology groups of a complex. Given two homeomorphic simplicial complexes K, K_1 he interprojected their derived sequences (using his deformation theorem). He then showed that the limit of the corresponding homology groups of the sequence is merely the corresponding ones of K, K_1 so that the two are the same. (It is actually not necessary to pass to the limit – one may show that

$$H(K) = H^{(p)}(K) = H^{(q)}(K_1) = H(K_1)$$

where H stands for homology groups of same type: equal dimension and same coefficient system.

(e) *Singular Theory.* This scheme is actually implicit in part in Alexander's first topological invariance proof of a Betti number (for an M_3). I have since organized it into a highly elastic theory, which, together with a deformation theorem of Alexander (see LT) is applicable to a large number of topological invariance properties. I will say a few words about this singular theory.

Let X be an arcwise connected metric space and let ϕ map a rectilinear closed p simplex σ_p into X. The pair (ϕ, σ_p) is, by definition a *singular* p-*simplex* in X. One agrees however that if τ_p is another rectilinear p-simplex and f is a rectilinear homeomorphism $\tau_p \to \sigma_p$ then $(\phi f, \sigma_p) \equiv (\phi, \sigma_p)$.

Orientation of (ϕ, σ_p) is copied from that of σ_p. Hence if $\sigma_{p-1} \in \partial \sigma_p$ one defines $(\phi, \sigma_{p-1}) \in \partial (\phi, \sigma_p)$ with the same incidence number. Hence if

$$c_p = \Sigma\, m_h (\phi_h, \sigma_p^h)$$

then

$$\partial c_p = \Sigma\, m_h\, \partial (\phi_h, \sigma_p^h).$$

The definitions of singular cycles, bounding cycles, homology groups is then automatic.

I merely metion that one may prove:

(6.1) *Theorem. The collection of singular p-cycles, is isomorphic with the special subcollection (identity, cycles of K).*

Corollary. Since the singular cycle collection has obvious topological character this holds also for the homology groups of K.

For Alexander's central contributions to knot theory see Ch. I.

7. Marston Morse: Critical point theory.

In the twenties and later Morse initiated his classical work based on the study of critical points of functions and applications, most particularly to the calculus of variations. The results for the period in question are developed in his Colloquium Lectures, vol. 18, 1934. The particular point of interest for us is Ch. 6. This volume contains also an extensive bibliography.

The results of Morse are far more general than what we describe, but it seems preferable for the short space at our disposal to lean more to clarity than to generality.

Let then R be a real (closed) bounded region on an analytical manifold referred to Euclidean coordinates $x_1, \ldots, x_n$. Consider also on R another analytic function $g(x) = g(x_1, \ldots, x_n)$, likewise analytic and such that on $R: a \leqq g \leqq b$, $a < b$. The *critical points* in R of g are its extreme points and the points where

$$\frac{\partial g}{\partial x_i} = 0, \; 1 \leqq i \leqq n. \tag{7.1}$$

Assume that they are all isolated. Moreover grant that at the critical points all the hessians

$$\left| \frac{\partial^2 f}{\partial x_h\, \partial x_k} \right| \neq 0. \tag{7.2}$$

These are all simplifying assumptions, which Morse has abandoned.

Let $b_1, b_2, \ldots, (b_i = g(a_i))$ be the successive critical values. The problem dealt with by Morse is to find the variation of the Betti numbers of the region $b_1 \leqq \leqq g < b_h$ with increasing h as g crosses b_h. This has been determined in terms of certain integers, the *type numbers* t_j which are defined in the following manner.

Corresponding to the critical point b_k the hessian $H(b_k)$ determines a non-degenerate quadratic form

$$\phi(x) \equiv \Sigma\, h_{jk}\, x_j\, x_k . \tag{7.3}$$

This quadratic form reduced to normal form has say m negative roots. The number t_h is the total number of critical points where (7.3) has m negative signs in its canonical form. As Morse showed the Betti numbers R_h of the region and the type numbers satisfy

$$t_0 \geqq R_0$$

$$t_0 - t_1 \leqq R_0 - R_1$$

$$\sum_{h=1}^{k} (-1)^h t_h \geqq (-1)^k \Sigma (-1)^h R_h$$

$$\sum_{h=1}^{n} (-1)^h t_h = \Sigma (-1)^h R_h . \tag{7.4}$$

The last relation for n = 2 is due to Poincaré.

As given by Morse these relations were proved by him only for coefficients mod 2, but the proofs for integral coefficients or coefficients in a field is the same.

In his book Morse deals directly with the most general case but the proof for the simpler case is found in his paper.

In the same book Morse treats a great many applications, which cannot be discussed here as they usually involve a large amount of analytical technique, especially of the Calculus of Variations type. I merely mention by way of example: (a) information about the *number* of normals to a variety V in Euclidean space from a point of the space; (b) the number of chords to V normal at both end points; (c) information about closed geodesics on V.

In all this research Morse rarely imposes analyticity and freely accepts C^1 or C^2 classes of functions. This of course adds considerably to the difficulties.

8. The work of A. W. Tucker

In his thesis (Princeton 1932) Tucker algebraized the Poincaré scheme to the last degree, yet preserving a strong contact with algebraic topology. Briefly speaking, he considered a complex as a finite collection of unspecified elements and assigned "dimensions" from o to p. Let σ_q^h, $1 \leqq h \leqq \alpha_q$ be the q-elements. There were introduced incidence numbers $[\sigma^{q+1} : \sigma^q]$ under the sole condition that there takes place the general matrix relation

$$\left[[\sigma_{q+1}^h : \sigma_q^j] \right] \cdot \left[[\sigma_q^k : \sigma_{q-1}^l] \right] = 0.$$

One may define chains, their boundary relations, cycles and homology groups in the standard way. The boundary relation is given by

$$\partial \sigma_q^h = \Sigma \, [\sigma_q^h : \sigma_{q-1}^j] \, \sigma_{q-1}^j$$

and for a chain c_q by standard linear extension. This leads to the usual functional relation $\partial\partial = 0$. Betti and torsion numbers arise in the usual way. Briefly then the whole theory of complexes follows. The same holds for *manifolds* and their duality provided one specifies that every St σ has the homology groups of a point.

What attracted me most to Tucker's work is an extremely simple derivation of my fixed point formula (see the next chapter). Tucker's attack was not to be excelled for *single-valued* transformations. It did not seem to go over to multiple valued transformations. Here my early intersection method had the best of it.

9. The work of Walter Mayer.

This author went to the extreme of abstraction. His first contribution was simply to take a finite sequence of additive groups: chain groups $G_0, G_1, \ldots, G_p$ with homomorphism $\tau_q : G_{q+1} \to G_q$ (boundary relations) satisfying $\tau_{q+1}\tau_q = 0$. One may then define the boundary subgroups $\tau_q G_{q+1} \subset G_q$, cycle subgroups, homology groups. I will not enter into a description save to say that Mayer's scheme has had quite a vogue later.

Another contribution of Mayer was most curious. Having defined the boundary operators — call them just τ for simplicity — he had the interesting idea of subjecting them to a relation $\tau^3 = 0$. It was proved later by Spanier (Michigan Thesis) that the resulting scheme was reducible to the standard type.

10. Some open problems left by Poincaré.

(10.1) One of the problems that evidently occupied Poincaré was to what

extent the integral Betti numbers plus the group of paths π sufficed to characterize a manifold (let alone a complex). He actually showed by an example that a sphere S_3 and an M_3 with the same homology groups but with different π could be distinct. Furthermore, we have already observed that Alexander showed by an example that two M_3 with the same homology theory and same π need not be homeomorphic.

(10.2) There the question has rested, except that nowadays one expects much more, namely identity of all homotopy groups in addition to the identity of the homology groups. In fact whenever a new topological character is discovered one asks if it suffices to distinguish two given complexes. No such character has been discovered at the present time.

(10.3) Let our discourse be limited to compact differentiable manifolds. An absolute M_n is *differentiable* whenever it admits a finite open covering $\mathscr{U} = \{U_h\}$ such that: (a) each U_h is parametrizable by variables x_{hj}, $1 \leqq j \leqq n$, (b) whenever U_h and U_j overlap say at a point P then about P the x_i^h are differentiable functions of the x_l^j with a nowhere zero jacobian.

Now the question arises for a given M_n, with one system of differentials, is it unique? This has been answered in the negative, in 1956, by John Milnor, by exhibiting a 7-sphere S_7 with two *distinct* (unrelated) differential systems. This has been extended by Stallings and Smale up to S_5. Several authors have even computed for some of these spheres the exact number of "disjoint" differential systems.

(10.4) One may also raise this question: given a polyhedron Π with *two* covering complexes K, K_1 (say simplicial), do they possess subdivision K^*, K_1^* with the same (algebro-geometric) structure? In 1960 Milnor gave an example which showed that in general this did not hold.

V. ABSOLUTE AND RELATIVE COMPLEXES AND MANIFOLDS.

DUALITY, COINCIDENCES AND FIXED POINTS.

1. General program

The goal of this chapter is the treatment of coincidences and fixed points of transformations. The importance of this question not only for topology but for analysis is well known. When I first attacked this problem (1923) very little had been done on it. There was mainly the classic of Brouwer and a noteworthy result of Alexander on fixed points for 2-manifolds (surfaces). My earlier major interest had attracted my attention to the question of coincidences dealt with by algebraic geometers solely for algebraic transformations between algebraic curves. There were in existence two highly interesting contributions: an analytical one by Adolph Hurwitz (around 1890) and a more geometric one by Francisco Severi (1902), even with a certain topological flavor where products of curves were considered. In this general research the main concern was the evaluation of numbers of coincidences.

Now moved by the elementary geometry of Rolle's theorem it occurred to me that the existence of fixed points in a mapping f of an n-manifold M_n into itself could be dealt with as follows. Take a copy M'_n of M_n and in the product 2n-manifold $M_n \times M'_n$ represent f by an n-cycle γ_n, the identity by another δ_n. The *intersection* number $\varphi(f) = (\gamma_n, \delta_n)$ would give the number of *signed* fixed points, and $\varphi \neq 0$ would guarantee the existence of a fixed point. More generally if M'_n is just another manifold and $f: M_n \to M'_n$, $g: M'_n \to M_n$, the same reasoning relative to their cycles in $M_n \times M'_n$ would lead to a number $\psi(f, g)$ which if $\neq 0$ would guarantee the presence of coincidences (a novel point of view in topology).

What was required then was first the creation of a theory of cycle intersections in a manifold. Next Brouwer's result showed the value of extending the investigation to *open* manifolds again a theory to be created.

All this creation was essentially terminated about 1928, when a result of Hopf (communicated by writing) extended my results on mappings (single-valued) to complexes. I succeeded in deriving them by a *new* duality in complexes. This appeared in LT (1930). Within 2 years (Princeton 1932 thesis) A. W. Tucker had indicated a major shortcut to *fixed* elements in *algebraic* transformations of complexes. By 1935 I extended his method to strict *mappings* (single valued transformations) of complexes. Here more theory was needed and was provided.

The object of this chapter is now clear: (a) to provide this novel theoretical method; (b) to apply it to coincidence and fixed points of transformations of complexes and manifolds (absolute or relative). In the process I found (1969) a novel approach to complex duality (cotheory) related, but more complete than my partial treatment of LT.

While I accepted throughout familiarity with standard finite complex theory, I could not feel so certain about manifolds and their intersection theory. This has moved me to present this theory as a short resumé in § 2.

2. Compact Manifolds (without boundary).

Let $K = \{\sigma\}$ be a finite simplicial complex and let K' be its derived. Take any $\sigma_p \in K$ and let ζ_{n-p} be the (point set) collection of all the simplexes of K' with a vertex at the barycenter $\hat{\sigma}_p$ of σ_p but, except for $\hat{\sigma}_p$ exterior to σ_p. By definition K is an n-*manifold* M_n, whenever: (a) dim K = n; (b) K is cell-wise connected; (c) every $\bar{\zeta}_{n-p}$ is a closed (n - p)-cell.

Notice the following properties: (d) *every* σ_{n-1} *is the face of exactly two* σ_n;

(e) K' *is also the (formal) derived of the collection* $K^* = \{\zeta\}$. K^* *is the "dual complex" of* K; (f) *when it is possible to orient every* σ_n *so that the two with common face* σ_{n-1} *are oppositely oriented to it then* M_n *is "orientable", otherwise it is "nonorientable".* (g) *the incidence relations in* K, K^* *may be so disposed that*

$$[\sigma_p : \sigma_{p-1}] = (-1)^{p+1} [\zeta_{n-p+1} : \zeta_{n-p}]. \tag{2.1}$$

Hence the related incidence matrices (η_p *and* η_{n-p}) *satisfy*

$$\eta_p = (-1)^{p+1} \eta'_{n-p}. \tag{2.2}$$

This leads to Poincaré's duality relations:

$$\begin{aligned} R_p(K) &= R_{n-p}(K^*) \\ \theta^i_p(K) &= \theta^i_{n-p-1}(K^*) \end{aligned} \tag{2.3}$$

where the θ' *s are torsion coefficients.* (h) *The homology groups of* K *and* K^* *are topologically invariant. Hence* (2.3) *implies*

$$\begin{aligned} R_p(M_n) &= R_{n-p}(M_n) \\ \theta^i_p(M_n) &= \theta^i_{n-p-1}(M_n) \end{aligned} \tag{2.4}$$

Notice this special property: —

(2.4a) *When* M_n *is orientable the sum of the* σ_n *properly oriented is an* n-*cycle* γ_n. *All the integral* n-*cycles, or cycles any modulo except* 2 *are given by* $\lambda\gamma_n$. *For this reason* γ_n *is called "fundamental* n-*cycle". Another choice for such a cycle is* $-\gamma_n$. *One may orient* M_n *by the choice of one or the other* $\pm\gamma_n$ *as fundamental* n-*cycle. When* M_n *is nonorientable the only* n-*cycle is* $\Sigma\,\sigma_n$ *taken mod* 2.

(2.4b) *In view of* (h) *orientation or nonorientation of* M_n *is a topological property.*

I conclude with this important proposition due to van Kampen:

(2.5) *Theorem. N.a.s.c. in order that* K *define an* M_n *is that every point possess an arbitrary small neighborhood which is an* n-*cell.*

Finally the following intersection property holds:

(2.6) *Theorem. The geometric intersection of* σ_p *and* ζ_{n-q} *is either vacuous or else a* (p – q)-*cell. One may define for it a unique orientation and the cell is then represented by* $\sigma_p \zeta_{n-q}$. *This cell is actually a subcomplex of* K'. *For chains* c_p, c_{n-q}

$$c_p = \Sigma\, x_i \sigma^i_p, \quad c_{n-q} = \Sigma\, y_j \zeta^j_{n-q},$$

the intersection is defined as

$$c_p c_{n-q} = \Sigma\, x_i y_j \sigma^i_p \zeta^j_{n-q}.$$

Note also that

$$c_p c_q = (-1)^{(n-p)(n-q)} c_q c_p.$$

Special case: $q = p$. *The intersection is a zero-chain. The sum of the coefficients*

$$(c_p, c_{n-p}) = \Sigma x_i y_j$$

is the "intersection-number" of the two chains.

(2.7) *Boundary relations.* One proves

$$\partial(c_p c_q) = c_p \partial c_q + (-1)^{n-q} (\partial c_p) c_q$$

In particular for $q = n - p$ *and since the zero-chain is a cycle:*

$$(c_{p+1}, \partial c_{n-p}) + (-1)^p ((\partial c_{p+1}), c_{n-p}) = 0. \qquad (2.8)$$

From (2.7) we may infer:

(2.7a) *If say* c_p *is a cycle the intersection with* c_{n-q} *is* $\pm c_p \partial c_{n-q}$. *Similarly the other way around. When both* c_p *and* c_{n-q} *are cycles the intersection is a* $(p-q)$*-cycle. Similarly when the chains do not meet one another's boundaries.*

(2.7b) *When* $\lambda c_p = \partial c_{p+1}$ *not intersecting* ∂c_{n-p} *then* $(c_p, c_{n-p}) = 0$. *In particular this holds when* c_{n-p} *is a cycle.*

Finally we have the all important:

(2.9) *Theorem. N.a.s.c. to have* $\gamma_p [\gamma_{n-p}]$ *a zero-divisor is that the intersection number*

$$(\gamma_p, \gamma_{n-p}) = 0$$

whatever the cycle $\gamma_{n-p} [\gamma_p]$.

(2.10) *Theorem. All the elements considered and their properties as between orientable manifolds concordantly oriented are topologically invariant.*

For details and proofs regarding § 2 see LT, chapter 4.

3. Relative theory

Let L be a closed subcomplex of K. The theory mod L merely aims to "neglect" everywhere chains in L. Thus a p-cycle γ_p of K mod L or *relative* L is a chain such that $\partial \gamma_p \subset L$. If there exists a c_{p+1} such that

$$\partial c_{p+1} = \gamma_p + d_p, \; d_p \subset L$$

then we write

$$\gamma_p \sim 0 \text{ mod } L.$$

One may define this new ∂ as ∂_L, generally omitting the subscript since it is otherwise clear. In any case

$$\partial_L \partial_L = 0.$$

Hence the usual groups (mod L) C, Z, B, H may be defined and the usual homology theory mod L follows.

Take now the derived K′, L′, ... and form the succession

$$G = K - \text{St}\, L, \quad G' = K' - \text{St}\, L', \; \ldots.$$

One verifies readily that the homology groups of the $G^{(k)}$ are the same. We call them: *homology groups of* K − L, written $H_p(K - L)$. By means of singular chains one may prove:

(3.1) *The homology groups of* K mod L *and of* K − L *are topological invariants of the pair* (K, L) (*i.e. of the polyhedra* $|K|$, $|L|$).

(3.2) *Local groups.* Instead of the homology theory mod L one may well consider the *homology theory* mod (K − L) that is relative to an open subset of K, or more generally relative to an open subset of the polyhedron $|K|$. I shall not dwell excessively upon this "dual" theory, but only consider the theory modulo a point P of $|K|$. This theory was the basis of van Kampen's proof of the *topological* invariance of a manifold.

We may so construct the derived K′ that P is a vertex of K′. One may as well assume then that it is a vertex of K. Topologically a p-cycle of *P* (= mod K − *P*) is any chain c_p passing through P with $\partial c_p \subset K - P$. We write $c_p \sim 0$ mod K − *P* when $c_p = \partial(c_{p+1} + d_{p+1})$, $d_{p+1} \subset K - P$. The definition of the homology groups of *P* is clear.

Example. If K is any closed n-cell then the Betti numbers $R_p = 0$ for $p < n$, and $R_n = 1$.

In a complex one may identify the homology theory of *P* with that of St *P* and hence (via singular chains) prove it *topologically invariant.*

(3.3) *Open manifolds.* We call K − L an *open* M_n or an M_n relative L, or mod L, or even an *n-circuit* mod L (old terminology) whenever every $\sigma_p \subset K - L$ satisfies the basic n-manifold condition: if $\hat{\sigma}$ is the barycenter of σ_p then the cells of K′ with vertex $\hat{\sigma}$ but exterior to σ_p have a closure which is a closed (n − p)-cell written $\bar{\zeta}_{n-p}$. Denote again $K^* = \{\zeta_{n-p}\}$. The various properties of § 2 hold with these modifications: replace K by K mod L and K* by K − L. In theorem (2.5) replace K by K − L. In (2.7a) c_p must be a cycle mod L and c_{n-p} a cycle of K − L (= M_n mod L). In (2.7b): $\lambda c_p = \partial c_{p+1}$ mod L; c_{n-p} is an absolute cycle. Finally in (2.9) γ_p is a cycle mod L and γ_{n-p} an (absolute) cycle of K − L.

The arguments leading to (2.4) yield also the general duality

$$R_p(M_n - L) = R_{n-p}(M_n, L). \qquad (3.4)$$

(3.5) *Duality theorem of Alexander.* This is a special duality for an n-sphere S_n: the first beyond Poincaré. The exact statement is this:

(3.6) *Theorem. Let* L_0 *be a simplicial complex and let* μ *map* L_0 *into a set* L *of* S_n, $n > 0$. *Then there takes place the duality relation*

$$R_p(S_n - L) = R_{n-p-1}(L) + \partial_{p0} - \partial_{pn}.$$

(a) Let first L be a subcomplex of S_n (μ is the identity). Then (3.4) holds. From the fact that in $S_n : R_0 = R_n = 1$ (no torsion) and that $R_p = 0 : 0 < p < n$ one infers at once that

$$R_{n-p}(S_n - L) = R_{n-p-1}(L) + \delta_{p0} - \delta_{pn}, \tag{3.7a}$$

so that in this case (3.7) holds.

(b) *General case.* Take a small enough ε and a derived S'_n of S_n of mesh $< \varepsilon$. Choose then L_0 of mesh so fine that the deformation theorem applied to L yields a subcomplex L′ of S′ ε-homotopic to L. Hence the singular homology groups of L are isomorphic to the (natural) groups of L′. That is

$$R_{n-p-1}(L) = R_{n-p-1}(L'). \tag{3.8}$$

Given s independent p-cycles of $S_n - L$ we may choose (deformation theorem) as many independent cycles of $S_n - L$ which are absolute cycles and, for suitably chosen ε, farther than 2ε from L. They may also be chosen to be independent for $S'_n - L'$. Hence for these cycles and L′ formula (3.7) will be satisfied. Therefore from the result just obtained and (3.8) the proof of (3.7) is completed.

Application. For p = 0, we find $R_0 = R_{n-1}(L) + =$ the number of component regions of $S_n - L$.

This is one of the main applications of Alexander's theorem.

4. Duality in complexes (co-theory)

In 1928-30 I introduced a complex-duality but limited to cycles, which I labeled *pseudo-cycles.* I needed this theory for fixed point purposes. Some years later (1935) Alexander, Čech and Whitney introduced (independently) a more direct method labeled *co-theory* by Whitney. This term has generally been accepted and will be followed here.

I have just found (1969) a way to extend my earlier argument to a full duality and this will be described here.

Let $K = \{\sigma\}$, dim K = n, be our usual complex. One may immerse it isomorphically as a subcomplex of an absolute M_r, $r - n > 0$ and even. For example M_r might be $\partial\sigma_{r+1}$.

Denote by σ_i^p the dual of σ_p^i of K in M_r. This avoids any true reference to M_r. The collection $\{\sigma_i^p\} = K^*$ is the *dual* of K.

We refer to σ_i^p as p-cocell, and p-cochains, cocycles, etc. will have their obvious meaning. Duality will be recognized for example by denoting by R^p the Betti numbers of K^*.

It is immediate that all the properties of § 2 are applicable provided ζ_{n-p} is replaced by σ^p. However one must replace (2.3) and (2.4) by

$$R_p(K) = R^{n-p}(K^*)$$

$$\theta_p^i(K) = \theta_i^{n-p-1}(K^*),$$

both relations being topologically invariant. It is also advisable to write "∂" for K^* as ∂^*, since its basic property is to *raise dimensions by one unit.* The remaining properties are either unchanged or else readily adaptable. Exception:

From the relation (with the only reference to r):

$$c_{r-p}\, c_{r-q} = c_{r-(p+q)}$$

we infer

$$c^p c^q = c^{p+q}$$

and hence for cocycles

$$\gamma^p \gamma^q = \gamma^{p+q}.$$

That is

(4.1) *The cocycles generate by intersection an intersection ring. This ring is a topological invariant of the complex* K.

(4.2) *Duals of open and closed subcomplexes.* Let L be a closed subcomplex of K and J = K − L its open complement. By dualization "open" and "closed" subcomplexes are interchanged. Thus L^* is open in K^* and J^* is closed in K^*. We may note this duality between Betti numbers (stated without proof)

$$R_p(L) = R^p(J^*). \tag{4.3}$$

5. Chain-mappings

This is a remote algebraic analogue of continuous linear transformations. Continuity will be replaced by a commutation between the boundary operator ∂ and a new linear chain-transformation.

Let $K = \{\sigma\}$, $K_1 = \{\zeta\}$ be two finite simplicial complexes. *A chain-mapping* $f : K \to K_1$ is merely a linear transformation from the group $C_p(K)$ to the group $C_p(K_1)$ (for all p) which commutes with the boundary operator ∂. That is

$$f\sigma_p^i = \Sigma\, a_p^{ij} \zeta_p^j\,;\ a_p^{ij} \text{ an integer};\ \partial f = f\partial. \tag{5.1}$$

Evidently f sends the groups C, Z, B, H of K into the same for K_1.

For later purposes it is convenient to make use here of matrix-vector notations. If $\{u_1, u_2, \ldots,\}$ is a base for example for the p-chains of K we just write it U_p. The context will usually tell what the u are.

Since $\{\sigma_p^i\}$ and $\{\zeta_p^j\}$ may be replaced by any bases for p-chains we denote them by A_p and B_p. The matrix $[a_p^{ij}]$ will be written a_p.

As we shall make an extensive use of these matrix notations we state explicitly some of their meanings:

Transpose of a matrix $C_p : C^p$.

Dual of complexes: K^*, K_1^*. Their incidence matrices are η_p for K and η^p for K^*, likewise θ_p and θ^p for K_1 and K_1^*. Boundary operators ∂ for K, K_1, and ∂^* for K^*, K_1^*. Collections of elements are A_p, A^p for K, K^* and B_p, B^p for K_1, K_1^*. Thus

$$\partial A_p = \eta_p A_{p-1}, \qquad \partial^* A^{p-1} = \eta^p A^p.$$

With these general conventions the basic chain-mapping takes the form

$$fA_p = a_p B_p. \tag{5.2}$$

By comparing ∂f and $f\partial$ we obtain the n.a.s.c. for f to be a chain-mapping $K \to K_1$ as

$$a_p \theta_p - \eta_p a_{p-1} = 0. \tag{5.3}$$

Consider now the dual $f^* : K_1^* \to K^*$ defined by

$$f^* B^p = a^p A^p. \tag{5.4}$$

The chain-mapping condition $\partial^* f^* = f^* \partial^*$ reduces to $a^{p-1} \eta^p - \theta^p a^p = 0$. This follows from (5.3) by mere dualisation. Hence if one of f, f^* is a chain-mapping so is the other.

6. Coincidences and fixed elements of chain-mappings

Let A_p, D_p, C_p be a minimum base for p-chains of K such that:
A_p = minimum base for non p-cycle chains;
D_p = minimum base for zero-divisors or ~ 0 p-cycles;
C_p = minimum base for independent p-cycles.

If p is a chain-mapping $K \to K$ then

$$\begin{aligned} fA_p &= a_p A_p + a'_p D_p + a''_p C_p, \\ fD_p &= \qquad\quad d_p D_p + d'_p C_p, \\ fC_p &= \qquad\qquad\qquad c_p C_p \end{aligned} \tag{6.1}$$

where the coefficients $a_p, \ldots,$ are integral matrices.

Set

$$\varphi(f) = \Sigma (-1)^p (\operatorname{tr} a_p + \operatorname{tr} d_p + \operatorname{tr} c_p) \tag{6.2}$$

where tr = trace.

Now if say tr $d_p = 0$ then fd_p has no fixed d_p element. Therefore

(6.3) (a) $\varphi \neq 0$ *implies that* φ *has some fixed element;* (b) *when* f *has no fixed element then* $\varphi = 0$.

From the definitions one infers that $b_p = a_{p+1}$. Hence

$$\varphi(f) = \Sigma (-1)^p \operatorname{tr} c_p. \tag{6.4}$$

Let now f be a chain-mapping $K \to K_1$ and g one $K_1 \to K$. Then gf is a chain-mapping $K \to K$. Its fixed elements correspond to coincidences of f and g, that is to pairs c_p, c_{1p} of chains c, c_1 of K, K_1 such that c_1 meets fc and c meets gc_1. The integer

$$\psi(f, g) = \Sigma (-1)^p \operatorname{tr} c_p c_{1p} \tag{6.5}$$

leads to an analogue of (6.3) for coincidences of elements.

7. Coincidences and fixed points for complexes.

Let first $|K|$ be a polyhedron and let F be a mapping $|K| \to |K|$. Suppose that F has no fixed point. Since $|K|$ is compact there is an $\varepsilon > 0$ such that if $x \in |K|$ then $d(x, Fx) > \varepsilon$ whatever x. Let $K^{(n)}$ be a derived whose mesh $< \frac{1}{2}\varepsilon$. It follows that if $\{\zeta\}$ are the simplexes of $K^{(n)}$ no ζ meets $F\zeta$. Hence the chain mapping $f : K^{(n)} \to K^{(n)}$ resulting from F has no fixed element. Thus if the c_p's refer to f

$$\varphi(f) = \Sigma (-1)^p \operatorname{tr} c_p = 0. \tag{7.1}$$

But the matrices c_p are manifestly functions of F. In fact they represent also the effect of F on a base for the p-cycles of K. Hence we may write

$$\varphi(F) = \Sigma (-1)^p \operatorname{tr} c_p = 0. \tag{7.2}$$

We may therefore state:

(7.3) *Theorem. To* $|K|$ *there corresponds an integer* $\varphi(F)$, *a function with values on any* F *such that:*
(*a*) $\varphi(F) \neq 0$ *implies the existence of a fixed point of* F;
(*b*) *the absence of a fixed point implies that* $\varphi(F) = 0$.

(7.4) *Applications.* Let F be a deformation of $|K|$ into itself. Then every c_p is the identity. Hence tr $c_p = R_p$ and

$$\varphi(F) = \Sigma (-1)^p R_p = \chi(F),$$

the characteristic of K. Thus for an n-sphere S_n

$$R_0 = R_n = 1, \qquad R_p = 0, \qquad 0 < p < n.$$

Hence $\chi = 2$ and so every deformation of S_n has a fixed point. Similarly for a closed n-cell $\bar{E}_n : \chi = 1$, and so every mapping $\bar{E}_n \to \bar{E}_n$ (deformation) has a fixed point. This is Brouwer's theorem.

On the other hand let $|K|$ be the ring of the Poincaré-Birkhoff theorem and F the homeomorphism of the theorem. Again F is a deformation but $\chi = 0$. Hence one cannot decide (by the formula) that there is a fixed point: a special method was required for the proof and it was given by Birkhoff.

Now let K, K_1 be as before with F a mapping $|K| \to |K_1|$ and G a mapping $|K_1| \to |K|$. A coincidence (x, y) is a pair of points $x \in |K|$ and $y \in |K_1|$ such that $y \in Fx$ and $x \in Gy$. We may now state

(7.5) *There is attached to every pair of mappings*

$$F : |K| \to |K_1| \text{ and } G : |K_1| \to |K|$$

an integer $\psi(F, G) = \psi(G, F)$ *such that:* (a) *if there are no coincidences* $\psi = 0$; (b) *if* $\psi \neq 0$ *there are coincidences.*

(7.6) When K and K_1 are absolute, orientable 2-manifolds one may show that F and G may both proceed in the same direction, say $|K| \to |K_1|$. (See LAT page 320.)

8. Coincidences and fixed points of an open complex

Let K, L be a simplicial complex and let L be a closed normal subcomplex of K. We are concerned with our problem for mappings of the open subcomplex K – L into itself.

Observe that K – L is not altered if we suppress in L simplexes which do not belong to the boundary $\partial(K - L)$. That is we may freely assume that $L = \partial(K - L)$. This being done let K_1 be another copy of K and let the image L_1 of L in K_1 be brought into coincidence with L, each simplex of L_1 to coincide point for point with its image in L. Let K_1 be the resulting closed simplicial complex.

There are now two possibilities: – It may be that F has fixed points on L. Then we are through. Suppose now that L is without fixed points of F. Then we extend F to a mapping $F_1 : |K_1| \to |K|$ without fixed points on L. Finally we replace F_1 by a new mapping $F_0 : K_1 \to K$ with the *same fixed points* as F, by assigning to $x_1 \in K_1 - L$ the same transform Fx as to the image x of x_1 in K – L. Thus F_0 answers the question.

Clearly the analogue for K_1 of the matrix c_p for K is merely a matrix of type

$$\begin{bmatrix} 0 & \\ & c_p \end{bmatrix}$$

where c_p is the same as c_p for K. Hence its c_p representation in the sum (7.2) is $(-1)^p$ trace c_p where c_p is the part of c_p^* corresponding to the p-cycles of (K – L), that is to the p-cycles of K mod L. In other words this time $\varphi(F_0) = \varphi(F)$, where F is the

given mapping $|K - L| \to |K - L|$. That is for our present F and fixed points the earlier result for K holds unchanged.

The method just utilized holds also for coincidences but with considerably larger complications. I will merely refer the reader to my paper: Manifolds with a boundary and their transformations, Trans. Am. Math. Soc., vol. 29, (1927), pp. 429-462. See notably page 440.

VI. VARIOUS QUESTIONS CONCERNING CERTAIN SPACES

1. General Program

In this last chapter we discuss certain problems on spaces not too remote from polyhedra. We shall first extend coincidence and fixed point results to certain moderately restricted compacta. These provide an easy access, without excessive technique, to more advanced questions.

2. Retraction

There is no reason to believe that the results of the preceding chapter have universal application even to all compacta. Some limitation will have to be imposed. One of the simplest is based upon the concept of *retraction*, due in its very general form to the Polish topologist Borsuk.

Let X be a compactum and let Y be a closed subset of X. Borsuk calls Y a *retract* of X whenever there exists a mapping $R : X \to Y$ such that $R | Y = 1$. Then R is a *retraction* $X \to Y$.

Borsuk distinguished several noteworthy special cases:

(a) Y : an *absolute* retract, that is a retract for every $X \supset Y$.

(b) Y: a *neighborhood* retract like (a) but Y merely a retract of a neighborhood N (Y) in X.

(c) Y: *absolute neighborhood retract* (= ANR) whenever it is one for a neighborhood N (Y) in every $X \supset Y$.

(d) Y: *deformation retract*, *absolute deformation retract*, etc. whenever R is always a deformation in X, N (Y), etc.

3. Coincidences and fixed points for certain compacta

Let X be a compactum deformation ANR. Let it also have a topological image in an Euclidean space $\mathfrak{E}_n$. Then we may as well assume that $X \subset \mathfrak{E}_n$.

Since X is compact it is bounded in $\mathfrak{E}_n$. We may then assume that it is in some closed square Q of $\mathfrak{E}_n$. One may cover Q with a closed n-complex K. Let N (X) be a neighborhood of X in Q deformation retracted into X. There is a derived $K^{(p)} \subset N$ for p high enough and hence K is deformation retractable into X.

The collection of all closed simplexes of $K^{(p)}$ which meet X form a closed neighborhood $\bar{N}_1$ (X) of X retractable by deformation into X. Hence (easily proved), $\bar{N}_1$ (X) has the same homology groups as X. In particular one may select for $\bar{N}_1$ a maximal set of independent q-cycles which are also such cycles for X.

Upon recalling that $X \subset \mathfrak{E}_n$ implies that dim $X \leqq n$, we may state:

(3.1) *Theorem. The coincidence and fixed point results of* (V, § 8) *are valid for a compactum of finite dimension, which is a deformation ANR.*

4. Some results of Alexandroff on compacta

Let X be an unrestricted compactum. Let $\mathfrak{U} = \{U_h\}$ be a finite open covering of X. Consider each set U_h as a point, each pair of intersecting sets U_h, U_k as an abstract segment, each triple U_h, U_k, U_l with a common intersection as a σ_2 etc. The various simplexes thus obtained give rise to an abstract simplicial complex N ($\mathfrak{U}$) called the *nerve* of $\mathfrak{U}$. In a sense the nerve is the intersection diagram of the sets of $\mathfrak{U}$.

At times it will be found necessary to carry in the nerve a type of operation natural in a geometric, but less so in an abstract complex. For this purpose it is convenient to construct a geometric isomorph of the nerve. This is done as follows: To each set U_h of $\mathfrak{U}$ let there correspond in a suitable $\mathfrak{E}_q$ a point u_h, the space $\mathfrak{E}_q$ being so chosen that the u_h are linearly independent. Then to each abstract simplex $U_h \dots U_k$ of N ($\mathfrak{U}$) one associates the geometric simplex $u_h \dots u_k$ in $\mathfrak{E}_q$. This clearly gives rise to a *geometric* isomorph N^g of N. Incidentally its metric may be assigned to N.

Recall that the *order* ω of $\mathfrak{U}$ is one less than the largest number of overlapping sets of $\mathfrak{U}$. Hence dim N (U) = ω. Let $\mathfrak{U}_\varepsilon$ denote a $\mathfrak{U}$ of mesh $< \varepsilon$. Essentially also by the result of Lebesgue-Brouwer we have:

(4.1) *Theorem. The least order* $\omega(\varepsilon)$ *of any* $\mathfrak{U}_\varepsilon \to$ dim X *as* $\varepsilon \to 0$.

Here and later the dimension is understood in the classical Menger-Urysohn sense.

Alexandroff has proved:

(4.2) *Theorem. If* dim X = n *is finite,* X *may be arbitrarily closely approximated by an n-dimensional polyhedron.*

Spectral sequences. Let $\mathfrak{U}$ be as before and let $\mathfrak{U}_1 = \{\mathfrak{U}_{1k}\}$ be a *refinement* of $\mathfrak{U}$: every U_{1k} is contained in some U_h.

Let $\sigma_p = U_{1o} \dots U_{1p}$ and let $U_{1k} \subset U_{k'}$, where the U_k need not be distinct. Since the U_{1k} determine a σ_p the sets U_{1k} have a non-void intersection. Hence this holds likewise for the U_k of $\mathfrak{U}$. It follows that they are the vertices of a $\sigma' \in N(\mathfrak{U})$. Hence the assignment $U_{1k} \to U_{k'}$ is a simplicial mapping $\mu' : N(U_1) \to N(\mathfrak{U})$. As a consequence μ' is a chain-mapping $N(\mathfrak{U}_1) \to N(\mathfrak{U})$. Since μ' depends upon the choice of a set $U_{k'} \supset U_{1k}$ it need not be unique. Suppose that μ'' is a second chain-mapping such as μ', defined by choices $U_{k''} \supset U_{1k}$. Then the sets $\{U_{k'}, U_{k''}\}$ all contain the intersection of the sets U_{1k}. Hence they are the vertices of a simplex $\sigma \in N(\mathfrak{U})$. In N^g the pair $U_{k'}$, $U_{k''}$ may be joined by a segment. Hence μ' and μ'' are homotopic in $N^g(\mathfrak{U}_1)$. It follows that they determine the *same* homomorphism of the images under μ' and μ'' of the homology groups $N(\mathfrak{U}_1)$ into those of $N(\mathfrak{U})$.

Conclusion. If γ_p is any cycle of $N(\mathfrak{U}_1)$ its images $\mu' \gamma_p$ and $\mu'' \gamma_p$ in $N(\mathfrak{U})$ are $\sim$ in $N(\mathfrak{U})$. Choose therefore one of the two operations, say μ', and call it *projection* $\pi : N(\mathfrak{U}_1) \to N(\mathfrak{U})$. The possible cycles $\pi\gamma$ will all be $\sim$ in $N(\mathfrak{U})$.

Take now a sequence $\mathfrak{U}_1, \mathfrak{U}_2, \dots,$ of finite open coverings of X determined (not necessarily uniquely) as follows. Let mesh $\mathfrak{U}_h = \varepsilon_h$, and let λ_h be the Legesgue number of $\mathfrak{U}_h$. Choose $\mathfrak{U}_{h+1}$ so that mesh $\mathfrak{U}_{h+1} < \lambda_h$. In addition one imposes that $\varepsilon_n \to 0$ with $\frac{1}{n}$. The sequence $\{\mathfrak{U}_h\}$ is a *spectral sequence* (Alexandroff).

Remark. When dim X is finite one may choose the sequence $\{\mathfrak{U}_h\}$ so that order U_h = dim X.

Since $\mathfrak{U}_{h+1}$ refines $\mathfrak{U}_h$ there exists a projection $\pi_{k+1} : \mathfrak{U}_{h+1} \to \mathfrak{U}_h$.

Let now G designate the additive group of integers or the field mod m, m prime. It is convenient to call the integers: numbers mod 0, and thus to allow m to be a prime or zero. A sequence $\gamma_p = \{\gamma_p^h\}$, where γ_p^h is a cycle of $\mathfrak{U}_h$ mod m, defines a p-cycle of X mod m. If $\gamma'_p = \{\gamma'^h_p\}$ define $\lambda\gamma_p + \lambda'\gamma'_m$, λ and λ' numbers mod m, as $\{\lambda\gamma_p^h + \lambda'\gamma'^h_p\}$. This gives a perfectly consistent definition of a homology group H_p (X) mod m.

By passing to the dual mappings one obtains a definition of the cohomology group H^p (X), mod m. (Details in (5.4)).

5. The Čech theory

While Alexandroff's spectral sequences are not topological the more general Čech scheme not only remedies this but also produces a theory with much wider applications.

Let X be this time any topological space, i.e with a topology based on open sets. Let $\mathfrak{U} = \{\mathfrak{U}_\alpha\}$ be the collection of all the finite open coverings of X with the property that any pair, hence any finite collection of coverings of the collection have a common refinement in the collection.

Let then $\mathfrak{U}_\alpha$ be refined by $\mathfrak{U}_\beta$. One may introduce the nerves, as in Alexandroff's scheme, define the projection $\pi_\alpha^\beta : \mathfrak{U}_\beta \to \mathfrak{U}_\alpha$ and the cycle projections $\pi_\alpha^\beta \gamma_p^\beta$, etc. A cycle γ_p of X is then defined as a collection $\gamma_p = \{\gamma_p^\alpha\}$, $\gamma_p^\alpha \subset \mathfrak{U}_\alpha$, such that if $\mathfrak{U}_\beta$ refines $\mathfrak{U}_\alpha$ then $\gamma_p^\alpha \sim \pi_\alpha^\beta \gamma_p^\beta$ in $\mathfrak{U}_\alpha$. The rest is defined as with Alexandroff.

It is now evident that homology groups and related characters are topological.

A subcollection $\mathfrak{B} = \{\mathfrak{B}_\beta\}$ of $\mathfrak{U}$ and behaving like it is *cofinal* with $\mathfrak{U}$ whenever any $\mathfrak{U}_\alpha$ has a refinement $\mathfrak{B}_\beta$. One may define cycles and their groups in terms of $\mathfrak{B}$ alone. By the evident property

(5.1) *Any finite set* $\{\mathfrak{U}_{\alpha 1}, \ldots, \mathfrak{U}_{\alpha k}\}$ *has a common refinement in* $\mathfrak{U}$.

One proves readily:

(5.2) *The homology groups determined by means of* $\mathfrak{B}$ *are isomorphic with the corresponding groups determined in terms of* $\mathfrak{U}$.

(5.3) *Application.* Let $K = \{\sigma\}$ be our usual finite complex. Let $K^{(n)} = \{\zeta\}$ be its n-th derived and $\mathfrak{B}_n = \{\mathrm{St}\,\zeta\}$ the finite open covering of $|K|$ by the stars of $K^{(n)}$. Let also $\mathfrak{U}$ be the same as defined for the space X. If λ is the Lebesgue number of $\mathfrak{U}_\alpha$ one may choose n so that mesh $\mathfrak{B}_n < \lambda$. Thus $\mathfrak{B}_n$ will refine $\mathfrak{U}_\alpha$ and $\mathfrak{B} = \{\mathfrak{B}_n\}$ is cofinal in $\mathfrak{U}$. Hence it has the same homology groups as $\mathfrak{U}$, and they are topological. However these groups are precisely those of K itself. Hence the homology groups of K have topological character. This is a second proof of the property.

(5.4) *Cohomology theory.* Since the projections are chain-mappings they have chain duals. We write them $\pi_\beta^{*\alpha}$ and consider them as chain-mappings of duals written $\mathfrak{U}^\alpha$ and $\mathfrak{U}^\beta$. That is $\pi_\beta^{*\alpha} : \mathfrak{U}^\alpha \to \mathfrak{U}^\beta$. Their cycles are designated as *cocycles* and written γ_α^p. Such a cocycle defines a *p-cocycle* of X. The addition and product is automatic. The cocycle is the same as $\pi_\beta^\alpha \gamma_\alpha^p = \gamma_\beta^p$. It is also the *dual* of $\gamma_p = \{\gamma_p^\alpha\}$.

Notice that $\gamma^p \sim 0$ if any $\pi_\beta^{*\alpha} \gamma_\alpha^p \sim 0$ in $\mathfrak{U}^\beta$.

We have thus a clear cut definition of *cohomology classes* and therefore also of *cohomology groups.*

(5.5) *Relative theory.* For this theory it is convenient, for the sake of simplicity, to limit the argument to X a *compactum.*

Let then A be a closed subset of X and hence also a compactum. Under the circumstances the intersections $A \cap \mathfrak{U}_\alpha = \{A \cap \mathfrak{U}_{\alpha h}\}$ determine a finite open covering of A. Since mesh $A \cap \mathfrak{U}_\alpha \leqq$ mesh $\mathfrak{U}_\alpha$ the Lebesgue number of $A \cap \mathfrak{U}_\alpha <$ Lebesgue number $\mathfrak{U}_\alpha$.

We now refine $\mathfrak{U}_\alpha$ by $\mathfrak{U}_\beta$ by the condition Lebesgue number $A \cap \mathfrak{U}_\alpha >$ mesh $A \cap \mathfrak{U}_\beta$. Hence $A \cap \mathfrak{U}_\beta$ will refine $A \cap \mathfrak{U}_\alpha$. It follows that π_α^β is also a projection for the finite open coverings $\{A \cap \mathfrak{U}_\alpha\}$ of A. This will enable us to introduce the subcomplexes $\mathfrak{U}_\alpha - A \cap \mathfrak{U}_\alpha$ and so to define the relative cycles of X mod A as those of $\mathfrak{U}_\alpha$ mod $A \cap \mathfrak{U}_\alpha$ under the same projections π_α^β as the absolute cycles. As for the relative cocycles, we leave them as exercises.

For references to the relative theory see LAT p. 248.

6. The Vietoris theory

Although this theory is formally implied by the Čech theory, it is actually more

direct and for this reason well suited for applications. In any case we restrict its development to compacta.

Consider the points of X as vertices of all possible simplexes. This gives rise to an ever so infinite complex K. This need not confuse us for actually only finite chains and cycles will be utilized.

By an ε *p-chain* c_p^ε of K we understand a chain whose mesh $< \varepsilon$. A p-cycle γ_p of X is a sequence $\gamma_p = \{\gamma_p^n\}$ of p-cycles of K such that for any ε there is an N such that if n′, n″ > N there is an ε chain c_{p+1}^ε such that $\gamma_p^{n'} - y_p^{\prime n''} = \partial c_{p+1}$. Addition and λ multiplication are defined in the obvious way. This implies that $\gamma_p = \{\gamma_p^n\} \sim 0$ whenever under the above conditions for n > N then $\gamma_p^n = \partial c_{p+1}$. Thus all the elements for the definition of Vietoris homology groups are at hand. Observe that these groups are topological as between compacta.

7. Some special homology theories

About 1935 Alexander, Kolgomoroff and Kurosh developed rather similar theories closely related to Čech's. The principal difference was the use of finite *closed* instead of finite *open* coverings. Let me give a few details about the Alexander *grating* theory. For further information see LAT, Ch. 7.

According to Alexander an open set U of X is *regular* whenever it is the *interior* (largest open subset) of $\bar{U}$. A grating is a finite closed covering $\mathfrak{U} = \{\bar{U}_h\}$ by closures of regular open sets. Some interesting special properties are:

(a) If $\mathfrak{U} = \{U_h\}$ and $\mathfrak{U}' = \{U_h'\}$ are gratings so is $\mathfrak{U} + \mathfrak{U}' = \{U_h\} + \{U_k'\}$.

(b) Every grating $\mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2 \cap \ldots \cap \mathfrak{U}_r$ where all the $\mathfrak{U}_h$ are binary (composed of two elements).

(c) *Gratings give rise to a Čech type homology theory which is the same as the Čech theory by finite closed coverings.*

Alexander has also brought out a close parallel between these theories and Élie Cartan's theory of differential forms. De Rham has shown that the parallel becomes an identity when X is an absolute differentiable manifold (see loc. cit.).

8. Relations between algebraic topology and dimension theory

It is a worthwhile observation that the impulse to "topologize dimension" is due to Poincaré. In fact I believe that he was the one to introduce the basic and early concept that, roughly speaking, an n-dimensional space was characterized by "walls" of dimension n − 1 plus the property that o-dimensional space has no walls (walls understood in the topological sense). This general idea, carried out with due mathematical rigor, was at the root of the definition (obtained independently) by Menger and Urysohn (early twenties). Lebesgue on the other hand introduced an entirely different idea: dimension = order of a finite covering (open or closed), altogether close to (future) by arbitrarily small sets. This concept is of algebraic topology and was exploited with marked success by Paul Alexandroff and his Moscow students. A short description follows.

Calculations relating to cycles will more or less follow the Vietoris scheme but with this modification. Let $\gamma_p = \{\gamma_p^1, \gamma_p^2, \ldots\}$ denote a Vietoris sequence representing a p-cycle of X. The only question to be raised about γ_p is whether it does or does not bound. It is not planned however to combine such cycles by homology relations. Therefore it is not necessary to restrict particularly the moduli. We have then two possible cases:

(a) The modulus m is the same for all cycles. That is γ_p^h is a chain mod m. The number m may be any integer $\geqq 0$ and is not restricted to being a prime. Conventionally one agrees to consider integral cycles as *cycles mod* 0. Let Δ (m) denote the largest dimension (p for γ_p) for which there exists a nonbounding cycle γ (m). We call Δ (m): dimension of X mod m, written $\dim_m X$.

(b) The modulus of each cycle γ_p^h (of the sequence of γ_p) is a variable positive integer and we denote the corresponding Δ as Δ (v), or merely Δ dimension by $\dim_v X$. It is defined as before as the largest dimension of a nonbounding cycle of the same type.

Notice that the deviations from the Vietoris definitions (imposed by Alexandroff) are entirely logical. His main result is:

(8.1) *Theorem.* (a) $\dim_m X \leqq \dim X$; (b) $\dim_v X = \dim X$.

Following immediately the Comptes Rendus Note (190 (1930), pp. 1105-1107) in which Alexandroff described his result there appeared a note by Pontrjagin in which he directed attention to this problem: Given a second compactum Y, does

$$\dim X \times Y = \dim X + \dim Y? \qquad (8.2)$$

He outlined the following results:

(a) the relation (8.2) holds for all $\dim_m X$ when m = 0 (integral domain) or m is prime;

(b) for m composite or for the classical dimension the relation (8.2) may fail;

(c) the failures of (8.2) recur only when $\dim X \times Y \geqq 4$, more precisely when $\dim X = \dim Y = 2$ (shown by examples);

(d) for compacta in $\mathfrak{E}_n$, $n \leqq 3$, all the dimensions whatsoever are equal.