On Hyperbolic Linear $\mathbf{R}^h \times \mathbf{Z}^1$ actions

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Preliminaries: A linear action of a Lie group G on a vector space V is a continuos homomorphism $\rho : G \to \mathrm{Aut}(V)$ where Aut (V) denotes the group of automorphisms of V with the usual topology. The isotropy group of a vector $v \in V$ is defined as $G_v(\rho) = \{g \in G;\ \rho(g)(v) = v\}$. The orbit of v by ρ is the set $\mathfrak{O}_v(\rho) = \{\rho(g)(v);\ g \in G\}$. The compact-open topology is given to the space of linear actions. In what follows $G = R^k \times Z^1$. The space of linear actions will then be denoted by $\mathfrak{A}_v(k, 1)$.

The Splitting Process. Let ρ be as above. Given $v \in V$ such that $G_v(\rho) \neq 0$, a linear action $\varphi : G_v(\rho) \to \mathrm{Aut}(T_v V)$ is defined by $\varphi(g) = d\rho(g)(V)$. Since G is abelian, two points in the same orbit by ρ have the same isotropy groups. So any vector in $T_v V$ lying on $T_v\mathfrak{O}_v(\rho)$ (the tangent space to $\mathfrak{O}_v(\rho)$ at v) is a fixed point of the action φ. We assumme there exists $g \in G$ such that $\rho(g)$ has only different eigenvalues. Then V splits as a direct sum of φ-invariant subspaces $T_v\mathfrak{O}_v(\rho)$ and $N_v(\rho)$, where $N_v(\rho) \approx T_v V / T_v\mathfrak{O}_v(\rho)$. Therefore φ induces a linear action $\chi_\rho : G_v(\rho) \to \mathrm{Aut}(N_v(\rho))$ defined by $\chi_v(g)(\pi u) = \pi\varphi(g)(u)$ where $\pi : T_v V \to N_v(\rho)$ is the quotient map.

Definition: Let ρ be as in the splitting process. Suppose G is isomorphic to either Z or R. Then we say that ρ is hyperbolic if for any $v \in V$, $v \neq 0$, $\overline{\mathfrak{O}_v(\rho)}$ is not compact.

Suppose now that G is isomorphic to $R \times Z$ then we call ρ hyperbolic if .he following conditions hold:

(i) For any $g \in G$, $\rho(g)$ has only real eigenvalues

(ii) For any $v \in V$, $v \neq 0$, $\overline{\mathfrak{O}_v(\rho)}$ is not compact

Moreover, for any non zero $v \in V$ for which $G_v(\rho) \neq 0$ the induced action χ_v is hyperbolic.

Let now $G \sim R^k \times Z^l$; $k \geqq 2$, $l = 0,1$. Proceeding by induction on k, suppose we have already, defined hyperbolic linear actions for $s > k$, $l = 0,1$. We then call ρ hyperbolic if for any $v \in V$, $v \neq 0$, we have:

(j) $\underline{G_v(\rho)}$ is not isomorphic to $R^s \times Z^j$, $j \geqq 2$

(jj) $\overline{\mathfrak{O}_v(\rho)}$ is not compact

Moreover, for any non zero $v \in V$ for which $G_v(\rho) \neq 0$, the induced linear action χ_v is hyperbolic.

Definition: A linear action $\rho \in \mathfrak{A}_v(k, 1)$ is called locally structurally stable in $\mathfrak{A}_v(k, 1)$ if for any $\varepsilon > 0$ there exists a neighborhood $\mathscr{U}$ of ρ in $\mathfrak{A}_v(k, 1)$ such that for any $\psi \in \mathscr{U}$ there exist a neighborhood W of zero in V and a homeomorphism $h: W \to V$ satisfying:

(1) h is $\mathscr{E}$-close to the identity map

(2) h maps orbits of ρ onto orbits of ψ

(3) For any $v \in V$, $G_v(\rho)$ is isomorphic to $G_{h(v)}(\psi)$

Theorem A. Let $\rho \in \mathfrak{A}_v(k, 1)$ satisfying any of the following items:

(a) There exits $v \in V$ $v \neq 0$ such that $\overline{\mathfrak{O}_v(\rho)}$ is compact

(b) $k = l = 1$ and there exists $g \in R \times Z$ such that $\rho(g)$ has non real eigenvalues.

(c) $l \geqq 2$ and $\dim V \geqq k + 2$.

Then ρ is not locally structurally stable in $\mathfrak{A}_v(k, 1)$.

Theorem B. Any hyperbolic linear action $\rho \in \mathfrak{A}_v(k, 1)$ is locally structurally stable in $\mathfrak{A}_v(k, 1)$

Theorem C: The set of hyperbolic linear actions of $R^k \times Z^l$ on a vector space V is open and dense in $\mathfrak{A}_v(k, 1)$.