A Version of Morse-Sard Theorem for Hilbert Spaces

J. SOTOMAYOR

1. *Introduction:* Let $f:\Omega \to F$ be a differentiable function defined on the open subset Ω of a Banach space E, into a Banach F. If Df_p, the derivative of f at p, is onto, p is called a *regular point of f*; otherwise it is called a *critical point* of f. $C = C_f$ will denote the set of critical points of f.

Morse-Sard Theorem [1] states that if a) E, F are finite dimensional and b) f is of class C^k, $k \geqq \{\dim F\text{-}\dim E + 1\}$ [1] then f(C) has null (Lebesgue) measure. When a) is not satisfied I. Kupka [2] has given an example where the above proposition fails. However, under convenient hypothesis – when f is a Fredholm map – S. Smale [3] has proved a version of Morse-Sard Theorem when E, F are infinite dimensional.

In this paper we restrict ourselves to the case where E is a Hilbert space and F is the real line and announce the following.

Theorem 1. Let E be a separable Hilbert space. Let Ω be an open subset of E. Let $f:\Omega \to R$ be a C^k $k \geqq 2$ function such that, for every $p \in C$, $D^2 f_p$ is a *Fredholm bilinear form* (defined below) with nullity n_p.

If $k \gg \sup \{n_p + 1, 2;\ p \in C_f\}$, then f(C) has null measure.

2: *Outline of proof* – The proof depends on the following two lemmas.

Lemma 1. Let E be a Hilbert space with inner product $<,>$. If β is a continuous symetric bilinear form on E and E_1 is the null space of E_1 i.e. $E_1 = v \in E; \beta(v, W) = 0$ for all $w \in E\}$, then $E_1 = \text{Ker } \tilde{\beta}$, where $\tilde{\beta}$ is the continuous endomorphism of E defined by $< \tilde{\beta}(v), w> = \beta(v, w)$ for all $v, w \in E$.

Also, if $E_2 = \tilde{\beta}(E)$ is closed, then E_2 is the orthogonal complement of E_1, $E = E_1 \oplus E_2$ (topological sum), and $\tilde{\beta} | E_2 : E_2 \to E_2$ is a topological isomorphism.

Definition. β is called a *Fredholm bilinear form* if $\tilde{\beta}$ is a Fredholm operator.

Note that, for Fredholm bilinear forms β, dimension of $\tilde{\beta}^{-1}(0)$, equal to nullity of β, is finite, and $\tilde{\beta}(E)$ is closed.

Lemma 2: Let f as in Theorem 1. Each $p \in C$ has a neighborhood V_p such that $C \cap V_p$ is contained in a submanifold S_p of E, of dimension n_p and class $k-1$.

The proof of Theorem 1 follows from Lemma 2, taking into account that if $S \subset \Omega$ is a submanifold of E then $C_f \cap S \subset C_{f/s}$, and applying Morse-Sard Theorem to f restricted to a countable subcovering of C by submanifolds S_p. Lemma 1 is important for the proof of Lemma 2.

3. *Remarks and a Problem* a) Theorem 1 is also valid when E is a Hilbert manifold with countable basis. b) The counter examples to Morse-Sard Theorem (in infinite dimension) known to the author-that of Kupka [2] and that shown in J. Eells [4. p. 759] – are stated for C^∞ functions which are not analytic (the reminder of their Taylor expansions are not uniformily convergent to zero). It semms reasonable to ask the following

Problem. To prove (or disprove) Morse-Sard Theorem when $f : \Omega \to R$ is analytic and Ω is open in an infinite dimensional Hilbert space.

References

1. *A. Sard.* The measure of the critical value of differentiable maps. Bull. Am. Math. Soc. 48 (1942)

2. *I. Kupka,* Counter example to Morse-Sard Theorem in the case of infinite dimensional manifolds; Proc. Am. Math. Soc. 16 (1965)

3. *S. Smale,* An infinite dimensional version of Sard's Theorem. Am. J. Math. 87 (1965).

4. *J. Eells.* A setting for global analysis, Bull. Am. Math. Soc. 72 (1966)