On direct systems of groups

by
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1. *Introduction*

The notion of a direct limit of a direct system generalizes that of a direct limit of a sequence, which in its turn may be regarded as an outgrowth of the notion of limit as it is found in mathematical analysis. We may consider sequences in various mathematical categories; thus, in the category of sets, we consider sequences

(1.1) $$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of sets X_n and functions $f_n : X_n \longrightarrow X_{n+1}$ $n \geq 0$. However we may also study, e.g., the category of topological spaces, when the X_n in (1.1) are topological spaces and the f_n are continuous functions, or the category of groups, when the X_n in (1.1) are groups and the f_n are homomorphisms. These are clearly just a few among the many possible examples available.

The association of a direct limit with the sequence (1.1) is most familiar in the case in which the functions f_n are *inclusions*. Then the direct limit is just the set-theoretic union

(1.2) $$X = \bigcup_n X_n ;$$

this set is topologized in a natural way if the X_n are topological spaces, and it is given a natural group structure if the X_n are groups. Moreover the evident inclusion $g_n : X_n \longrightarrow X$ is then a *morphism* of the appropriate category, that is, it is continuous in the topological case and homomorphic in the case that the X_n are groups.

It turns out to be possible to associate a direct limit with the sequence (1.1) even when the functions f_n are not inclusions. Again the direct limit X comes provided with functions $g_n : X_n \longrightarrow X$. Moreover, it turns out that the direct limit may be *characterized* in terms of a *universal property* enjoyed by the functions g_n relative to the sequence (1.1) – and this in any category in which we wish to consider direct limits.

Thus a general developement of the theory of direct limits would apply to any mathematical category; moreover, the duality principle would enable us to derive properties of *inverse limits* standing in one-one correspondence with the properties of direct limits adduced. In this article we do not attempt any such generality, concentrating entirely on the *category of groups*; however, we do introduce the essential generalization which consists of passing from sequences to *direct systems over arbitrary directed sets*. By presenting this generalization, and by laying emphasis on the universal property already referred to (see Theorem 2.5 and Definition 2.12), we hope to prepare the reader for a study of the abstract theory of direct and inverse limits in any category. However we must emphasize that this article is preparatory and not in any sense definitive. The theory of direct limits of groups, so far as we take it, may be applied do direct systems over index categories much more general than directed sets (see Definition 2.9); indeed we cannot even apply our results as stated to the simple situation of the *co-equalizer* of two homomorphisms

$$G_0 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} G_1 ,$$

that is, a homomorphism ϵ with domain G_1 such that $\epsilon\gamma = \epsilon\delta$ which is *universal* for this property of "equalizing" γ and δ. We have not however thought it worthwhile in this expository article to consider a class of index categories (the so-called *quasi-filtered* categories) which includes the directed sets and to which, in fact, all our results would apply, at some cost in simplicity of demonstration.* Moreover, our enunciations and demonstrations are by no means always the most sophisticated, lending themselves naturally to further generalization and abstraction; we have been concerned to make the arguments intelligible to those to whom the actual material of this article is unfamiliar.

Although this article is basically expository, we have included a new result which forms the content of the final section. This result has been included in order to demonstrate that there are questions to be asked and answered even within the very concrete and well-established parts of the theory of direct systems of groups; in fact, the theorem we prove in Section 5 plays a key role (in slightly, but very slightly, generalized form) in a study of general cohomology theories with coefficients currently being undertaken by the author.

In Section 2 we develop the basic ideas relating to direct systems of groups, emphasizing how the direct limit may be characterized by means of a universal property. In Section 3 we apply the theory to the study of tensor products of abelian groups, using direct limits to obtain a complete characterization of the so-called *flat* abelian groups (Theorem 3.14). In Section 4 we show that direct limits preserve *exactness* (of sequences of groups). It is, of course, in Sections 3 and 4 that we most conspicuously use special forms of much more general arguments; we hint that we are really exploiting the facts that the tensor product has a right adjoint (see Theorem 3.7) and that the direct limit also has a right adjoint, but we do not develop this more basic point of view any further in this article.

* A treatment is given in B. Eckmann and P. J. Hilton, Commuting limits with colimits, Jour. Alg. 11 (1969), 116-144.

2. *Sequences and direct systems of groups*

Suppose given a sequence of groups and homomorphisms

$$\text{(2.1)} \qquad G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} G_2 \longrightarrow \cdots \longrightarrow G_n \xrightarrow{\varphi_n} G_{n+1} \longrightarrow \cdots$$

It is then customary to define the *direct limit** of (2.1) as the group G constructed as follows. On the set $\bigcup_n G_n$ we set up the equivalence relation generated by the relation $g_n \equiv \varphi_n(g_n)$, $0 \le n < \infty$, $g_n \in G_n$. The set of equivalence classes is the underlying set of G and we write $g = [g_n]$ for the element of G represented by g_n. Let $g = [g_n]$, $g' = [g'_m]$ be two elements of G; we may suppose without real loss of generality that $n \ge m$ and we set $\varphi_{mn} = \varphi_{n-1} \cdots \varphi_{m+1}\varphi_m$. We then introduce a group structure into G by the rule

$$\text{(2.2)} \qquad gg' = [g_n \cdot \varphi_{mn}(g'_m)].$$

The reader may verify that this rule is independent of the choice of representatives and does indeed give to G the structure of a group; moreover the function

$$\theta_n : G_n \longrightarrow G,$$

* The terms "inductive limit", "colimit" are also used.

given by $\theta_n(g_n) = [g_n]$ is then a homomorphism such that, for all $n \geq 0$,

$$\theta_{n+1}\,\varphi_n = \theta_n. \tag{2.3}$$

We write $(G, \theta_n) = \varinjlim_n (G_n, \varphi_n)$ or, more briefly, $G = \varinjlim_n (G_n, \varphi_n)$ or, even, $G = \varinjlim_n G_n$. We may even omit the "n" under the arrow!

Example 2.4. Let G be the union of an ascending sequence of subgroups $G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_n \leq G_{n+1} \leq \ldots, G = \bigcup_n G_n$. Then G is the direct limit of the sequence of inclusions and $\theta_n : G_n \longrightarrow G$ is just the inclusion of G_n in G.

Theorem 2.5. *Let* $(G, \theta_n) = \varinjlim_n (G_n, \varphi_n)$ *and let* $\rho_n : G_n \longrightarrow X$ *be a sequence of homomorphisms such that* $\rho_{n+1}\varphi_n = \rho_n$, $n \geq 0$. *Then there exists a unique homomorphism* ρ: $G \longrightarrow X$ *such that* $\rho\theta_n = \rho_n$.

Proof. We set $\rho[g_n] = \rho_n(g_n)$. We leave to the reader the verification that ρ is well-defined and is a homomorphism; it is plain that ρ satisfies $\rho\theta_n = \rho_n$ and that this equation (for all $n \geq 0$) uniquely determines ρ.

This simple theorem lies at the heart of the modern developments in the theory of direct limits. We will immediately draw two consequences.

Corollary 2.6. *Suppose given a commutative diagram*

$$\begin{array}{ccccccccc}
G_0 & \xrightarrow{\varphi_0} & G_1 & \xrightarrow{\varphi_1} & \cdots & \longrightarrow & G_n & \xrightarrow{\varphi_n} & G_{n+1} & \longrightarrow & \cdots \\
\downarrow \gamma_0 & & \downarrow \gamma_1 & & & & \downarrow \gamma_n & & \downarrow \gamma_{n+1} & & \\
H_0 & \xrightarrow{\psi_0} & H_1 & \xrightarrow{\psi_1} & \cdots & \longrightarrow & H_n & \xrightarrow{\psi_n} & H_{n+1} & \longrightarrow & \cdots
\end{array}$$

and let $(G, \theta_n) = \varinjlim (G_n, \varphi_n)$, $(H, \eta_n) = \varinjlim (H_n, \psi_n)$. *Then there exists a unique homomorphism* γ: $G \longrightarrow H$ *such that* $\gamma\theta_n = \eta_n\gamma_n$, $n \geq 0$.

Proof. Let $\rho_n = \eta_n\gamma_n : G_n \longrightarrow H$. Then $\rho_{n+1}\varphi_n = \eta_{n+1}\gamma_{n+1}\varphi_n = \eta_{n+1}\psi_n\gamma_n = \eta_n\gamma_n = \rho_n$. Apply Theorem 2.5.

The reader familiar with the language of category theory will recognize that Corollary 2.6 is the essential step in establishing the fact that $\varinjlim$ is a *functor* from the category of sequences of groups to the category of groups; the remaining steps are, indeed, very easy and we will omit them.

Corollary 2.7. *Suppose, in the notation of Theorem* 2.5, *that the homomorphisms* ρ_n *have following two properties*:

(a) *Every* $x \in X$ *is in the image of* ρ_n *for some* n;

(b) *if* $\rho_n(g_n) = \mathrm{e}$, *then** $\varphi_{nk}(g_n) = \mathrm{e}$ *for some* $k \geq n$.

Then ρ: $G \longrightarrow X$ *is an isomorphism; that is,* (X, ρ_n) *is then canonically isomorphic to* $\varinjlim (G_n, \varphi_n)$.

Proof. Since $\rho[g_n] = \rho_n(g_n)$, condition (a) asserts that ρ is onto X. Likewise condition (b) asserts that ρ is one-one. For if $\rho[g_n] = \mathrm{e}$, then $\rho_n(g_n) = \mathrm{e}$, so $\varphi_{nk}(g_n) = \mathrm{e}$ and $[g_n] = [\varphi_{nk}(g_n)] = \mathrm{e} \in G$.

The force of this corollary is that it gives us a criterion for when (X, ρ_n) essentially *is* the direct limit of (G_n, φ_u). We will illustrate this by an example.

Example 2.8. Consider the sequence

$$\mathbb{Z} \xrightarrow{\varphi_1} \mathbb{Z} \xrightarrow{\varphi_2} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z} \longrightarrow \cdots,$$

where φ_{n-1} is simply multiplication by n. We claim that the direct limit of this sequence is just $\mathbb{Q}$, the group of rationals. For define ρ_n : $\mathbb{Z} \to \mathbb{Q}$ by $\rho_n(1) = 1/n!$. Then $\rho_n\varphi_{n-1} = \rho_{n-1}$, $n \leq 2$. Property (a) of Corollary 2.7 holds since $\rho_q(p(q-1)!) = p/q$; and property (b) holds trivially since each ρ_n is one-one. Notice that this example illustrates one fact typical of direct limits: *if each* φ_n *is one-one and* $\varinjlim (G_n, \varphi_n) = (G, \theta_n)$, *then each* θ_n *is one-one* (the converse follows from (2.3)).

We now proceed to generalize what has gone before, replacing the notion of a sequence by that of a *direct system*, over a *directed set*.

Definition 2.9. A *directed set* A is a (partially) ordered set with the property: for any $\alpha, \alpha' \in A$, there exists $\alpha'' \in A$ with $\alpha \leq \alpha''$, $\alpha' \leq \alpha''$.

Definition 2.10. A *direct system* of groups over the directed set A consists of

(i) A collection of groups $\{G_\alpha\}$ indexed by A, and

(ii) to each pair α_1, α_2 in A with $\alpha_1 \leq \alpha_2$ a homomorphism $\varphi_{\alpha_1\alpha_2} : G_{\alpha_1} \to G_{\alpha_2}$ such that

$$\varphi_{\alpha\alpha} = 1\colon G_\alpha \longrightarrow G_\alpha \tag{2.11}$$

and $\varphi_{\alpha_1\alpha_3} = \varphi_{\alpha_2\alpha_3}\varphi_{\alpha_1\alpha_2}$ whenever $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

* We use "e" for the neutral element of any group. Also recall that $\varphi_{nk} = \varphi_{k-1}\cdots\varphi_{n+1}\varphi_n : G_n \longrightarrow G_k$, $k \geq n$.

Notice that the set of natural numbers is directed by size and a direct system of groups over the directed set of natural numbers is just a sequence of groups and homomorphisms.

We may now generalize the entire discussion from sequences to direct systems. However, we prefer to introduce a change of outlook since it is Theorem 2.5 that constitutes the definitive property of the direct limit. With this new point of view the notion of direct limit plays its right role as a concept of universal algebra, and the construction of G given at the start of this section constitutes an existence proof for the direct limit.

Definition 2.12. Let $\{G_\alpha; \varphi_{\alpha_1 \alpha_2}\}$ be a direct system of groups over the directed set A and let $\{G; \theta_\alpha\}$ consist of a group G and, for each $\alpha \in A$, a homomorphism $\theta_\alpha : G_\alpha \longrightarrow G$ such that, for all pairs α_1, α_2 in A with $\alpha_1 \leq \alpha_2$

$$\theta_{\alpha_2} \varphi_{\alpha_1} \varphi_{\alpha_1 \alpha_2} = \theta_{\alpha_1}. \tag{2.13}$$

Suppose further that $\{G; \theta_\alpha\}$ has the following universal property: given $\{X; \rho_\alpha\}$ where, for each $\alpha \in A$, $\rho_\alpha : G_\alpha \longrightarrow X$ is a homomorphism such that

$$\rho_{\alpha_2} \varphi_{\alpha_1 \alpha_2} = \rho_{\alpha_1} \qquad \text{whenever } \alpha_1 \leq \alpha_2,$$

then there exists a unique homomorphism ρ: $G \longrightarrow X$ such that $\rho\theta_\alpha = \rho_\alpha$, $\alpha \in A$. Then $\{G; \theta_\alpha\}$ is called the *direct limit* (*inductive limit, colimit*) of the system $\{G_\alpha; \varphi_{\alpha_1 \alpha_2}\}$.

Theorem 2.14. *The direct limit, if it exists, is unique up to canoncal isomorphism.*

Proof. Suppose that $\{G'; \theta'_\alpha\}$ also has the property ascribed to $\{G; \theta_\alpha\}$ in Definition 2.12. Taking $\{G'; \theta'_\alpha\}$ for $\{X; \rho_\alpha\}$ in that definition – legitimate since $\theta'_\alpha \varphi_{\alpha_1 \alpha_2} = \theta'_{\alpha_1}$ if $\alpha_1 \leq \alpha_2$ – we find a unique ω: $G \longrightarrow G'$ such that $\omega\theta_\alpha = \theta'_\alpha$, $\alpha \in A$. We will show that ω is an isomorphism, thus proving the theorem.

Now we may reverse the roles of $\{G; \theta_\alpha\}$ and $\{G'; \theta'_\alpha\}$ in the above argument. Thus we also find a unique ω': $G' \longrightarrow G$ such that $\omega' \theta'_\alpha = \theta'_\alpha$, $\alpha \in A$. But then

$$\omega' \omega\theta_\alpha = \theta_\alpha, \ \alpha \in A. \tag{2.15}$$

Reverting to Definition 2.12, the uniqueness part of the statement means that if $\rho\theta_\alpha = \rho'\theta_\alpha$ for all $\alpha \in A$, then $\rho = \rho'$; but (2.15) tells us that $\omega' \omega\theta_\alpha = 1\theta_\alpha$ for all $\alpha \in A$, so we conclude that $\omega' \omega = 1$. Again

reversing the roles of $\{G; \theta_\alpha\}$ and $\{G'; \theta'_\alpha\}$ we infer that $\omega\omega' = 1$, so that ω is indeed an isomorphism (with inverse ω').

This theorem establishes the essential uniqueness of the direct limit; it does not, of course, prove that the direct limit exists. We now imitate the construction given at the beginning of the section to establish existence. Given the system $\{G_\alpha, \varphi_{\alpha_1, \alpha_2}\}$ we form the set $\bigcup_{\alpha \in A} G_\alpha$ and, on this set, set up the equivalence relation generated by the relation

$$g_{\alpha_1} = \varphi_{\alpha_1 \alpha_2}(g_{\alpha_1}), \ \alpha_1 \leq \alpha_2, \ g_{\alpha_1} \in G_{\alpha_1}. \tag{2.16}$$

If we wish to describe this equivalence relation explicitly, we may do so as follows:

$$g_\alpha \equiv g_{\alpha'} \ \text{ if, for some } \alpha'' \text{ such that } \alpha \leq \alpha''; \ \alpha' \leq \alpha'', \quad \text{we have } \varphi_{\alpha\alpha''}(g_\alpha) = \varphi_{\alpha'\alpha''}(g_{\alpha'}). \tag{2.17}$$

We write $[g_\alpha]$ for the equivalence class containing g_α. We then introduce a group structure into the set G of equivalence classes by the rule

$$[g_\alpha][g'_{\alpha'}] = [\varphi_{\alpha\alpha''}(g_\alpha) \cdot \varphi_{\alpha'\alpha''}(g'_{\alpha'})] \tag{2.18}$$

where α'' is any element of A such that $\alpha'' \geq \alpha$, $\alpha'' \geq \alpha'$.

To show that (2.18) gives a well-defined binary operation in G, we first show independence of the choice of α''. If also $\alpha_1 \geq \alpha$, $\alpha_1 \geq \alpha'$, choose $\alpha''' \geq \alpha''$, $\alpha''' \geq \alpha_1$. Then

$$\begin{aligned} \varphi_{\alpha\alpha''}(g_\alpha) \cdot \varphi_{\alpha'\alpha''}(g'_{\alpha'}) &\equiv \varphi_{\alpha''\alpha'''}(\varphi_{\alpha\alpha''}(g_\alpha) \cdot \varphi_{\alpha'\alpha''}(g'_{\alpha'})) \\ &= \varphi_{\alpha\alpha'''}(g_\alpha) \cdot \varphi_{\alpha'\alpha'''}(g'_{\alpha'}) \end{aligned}$$

and, similarly,

$$\varphi_{\alpha\alpha_1}(g_\alpha) \cdot \varphi_{\alpha'\alpha_1}(g'_{\alpha'}) \equiv \varphi_{\alpha\alpha'''}(g_\alpha) \cdot \varphi_{\alpha'\alpha'''}(g'_{\alpha'}).$$

We next show that the right-hand side of (2.18) is independent of the choices of $g_\alpha, g'_{\alpha'}$ within their equivalence classes; it is plainly sufficient to give the argument proving independence of the choice of g_α. Suppose then that $\bar{\alpha} \geq \alpha$ and that $\bar{\alpha}'' \geq \bar{\alpha}$, $\bar{\alpha}'' \geq \alpha'$. But then $\bar{\alpha}'' \geq \alpha$, $\bar{\alpha}'' \geq \alpha'$, so that, by what we have already proved

$$\varphi_{\alpha\alpha''}(g_\alpha) \cdot \varphi_{\alpha'\alpha''}(g'_{\alpha'}) \equiv \varphi_{\alpha\bar{\alpha}''}(g_\alpha) \cdot \varphi_{\alpha'\bar{\alpha}''}(g'_{\alpha'}) = \varphi_{\bar{\alpha}\bar{\alpha}''}(g_{\bar{\alpha}}) \varphi_{\alpha'\bar{\alpha}''}(g'_{\alpha'}),$$

where $g_{\bar{\alpha}} = \varphi_{\alpha\bar{\alpha}}(g_\alpha)$. Since the equivalence relation is generated by (2.16), this establishes our claim. We now leave to the reader the proof that the operation (2.18) yields a group structure in G. Moreover there is a homomorphism $\theta_\alpha : G_\alpha \longrightarrow G$ given by $\theta_\alpha(g_\alpha) = [g_\alpha]$ and plainly

$$\theta_{\alpha_2}\varphi_{\alpha_1\alpha_2} = \theta_{\alpha_1} \text{ if } \alpha_1 \leq \alpha_2 .$$

The reader may now readily supply the proofs of the following results, generalizing Theorem 2.5 and Corollaries 2.6 and 2.7.

Theorem 2.19. *The pair* $\{G;\theta_\alpha\}$ *constructed above is the direct limit of* $\{G_\alpha;\varphi_{\alpha_1\alpha_2}\}$ *in the sense of Definition* 2.12.

Theorem 2.20. *Let* $\{G_\alpha;\varphi_{\alpha_1\alpha_2}\}$ and $\{H_\alpha;\psi_{\alpha_1\alpha_2}\}$ *be two direct systems of groups over the directed set A, and let* $\gamma_\alpha : G_\alpha \longrightarrow H_\alpha$, $\alpha \in A$, *be a collection of homomorphisms such that, whenever* $\alpha_1 \leq \alpha_2$, *the diagram*

$$\begin{array}{ccc} G_{\alpha_1} & \xrightarrow{\varphi_{\alpha_1\alpha_2}} & G_{\alpha_2} \\ \downarrow\gamma_{\alpha_1} & & \downarrow\gamma_{\alpha_2} \\ H_{\alpha_1} & \xrightarrow[\psi_{\alpha_1\alpha_2}]{} & H_{\alpha_2} \end{array}$$

commutes. Let $\varinjlim\{G_\alpha;\varphi_{\alpha_1\alpha_2}\} = \{G;\theta_\alpha\}$, $\varinjlim\{H_\alpha;\psi_{\alpha_1\alpha_2}\} = \{H;\eta_\alpha\}$. *Then there exists a unique homomorphism* $\gamma: G \longrightarrow H$ *such that* $\gamma\theta_\alpha = \eta_\alpha\gamma_\alpha$, $\alpha \in A$.
(We call the collection (γ_α) a *morphism of direct systems.*)

Theorem 2.21. *Let* $\{G_\alpha;\varphi_{\alpha_1\alpha_2}\}$ *be a direct system of groups over A and let* $\{X;\rho_\alpha\}$ *be a pair, where* $\rho_\alpha : G_\alpha \longrightarrow X$ *satisfies* $\rho_{\alpha_2}\varphi_{\alpha_1\alpha_2} = \rho_{\alpha_1}$ *if* $\alpha_1 \geq \alpha_2$. *Suppose the homomorphisms* ρ_α *hare the following two properties:*

(a) *Every* $x \in X$ *is in the image of* ρ_α *for some* α;

(b) *if* $\rho_\alpha(g_\alpha) = e$, *then there exists* $\alpha' \geq \alpha$ *such that* $\varphi_{\alpha\alpha'}(g_\alpha) = e$.
Then $\{X;\rho_\alpha\} = \varinjlim\{G_\alpha;\varphi_{\alpha_1\alpha_2}\}$.
(Of course, the direct limit *does* have properties (a) and (b).)

We will henceforth adopt the notational abbreviations indicated following (2.3) but now applying to our more general situation; thus we may even use the *very* abbreviated symbol $\varinjlim\{G_\alpha\} = G$.

Example 2.22. Let G be a group and let A be a set in a given one-one correspondence with the set of finitely-generated subgroups of G, thus $\alpha \leftrightarrow G_\alpha$. We declare $\alpha \leq \alpha'$ if $G_\alpha \leq G_{\alpha'}$. Then A is a directed set;

for given G_α, $G_{\alpha'}$ finitely-generated subgroups of G, then the union of a (finite) set of generators of G_α and a (finite) set of generators of $G_{\alpha'}$ is a finite set of elements of G generating a finitely-generated subgroup $G_{\alpha''}$ of G such that $G_\alpha \leq G_{\alpha''}$, $G_{\alpha'} \leq G_{\alpha''}$. If $\varphi_{\alpha\alpha'} : G_\alpha \longrightarrow G_{\alpha'}$ is the inclusion when $\alpha \leq \alpha'$, then it is easy to see that

$$G = \varinjlim\{G_\alpha;\varphi_{\alpha\alpha'}\}.$$

Briefly, every group is the direct limit of its finitely-generated subgroups. We will see in the next section how we may use this fact to transfer certain properties from finitely-generated groups to arbitrary groups.

We close this section by introducing an important notion. It is clear that the direct limit of a sequence is unaffected if we throw out some of the groups G_n, provided only that we retain an infinite subsequence*. The notion of a cofinal subset of a direct set provides us with the appropriate generalization.

Definition 2.23. A subset B of a directed set A is said to be *cofinal* in A if, given any $\alpha \in A$, there exists $\beta \in B$ with $\alpha \leq \beta$.

Notice that a cofinal subset is certainly itself directed; for if β', $\beta'' \in B$, there exists $\alpha \in A$ with $\beta' \leq \alpha$, $\beta'' \leq \alpha$ and then, B being cofinal, there exists β with $\alpha \leq \beta$, so that $\beta' \leq \beta$, $\beta'' \leq \beta$. The converse is plainly false – a directed subset of a direct set may well fail to be cofinal (it is sufficient to take B to be a singleton where A is the set of natural numbers).

Given a direct system $\{G_\alpha;\varphi_{\alpha\alpha'}\}$ of groups directed by A, and given B cofinal in A, then we plainly have the direct system $\{G_\beta;\varphi_{\beta\beta'}\}$ directed by B, obtained simply by restriction of the indices.

Theorem 2.24. *If* $\varinjlim\{G_\alpha;\varphi_{\alpha\alpha'}\} = \{G;\theta_\alpha\}$, *then* $\varinjlim\{G_\beta;\varphi_{\beta\beta'}\}$ $= \{G;\theta_\beta\}$.

Proof. We apply Theorem 2.21; thus we must show (a) every $g \in G$ is in the image of θ_β for some $\beta \in B$, and (b) if $\theta_\beta(g_\beta) = e$, then there exists $\beta' \geq \beta$ such that $\varphi_{\beta\beta'}(g_\beta) = e$.

Now since $\{G;\theta_\alpha\} = \varinjlim\{G_\alpha;\varphi_{\alpha\alpha'}\}$, each $g \in G$ is expressible as $g = \theta_\alpha(g_\alpha)$ for some $\alpha \in A$, $g_\alpha \in G_\alpha$. Choose $\beta \geq \alpha$; then $\theta_\beta\varphi_{\alpha\beta} = \theta_\alpha$ so $g = \theta_\beta(\varphi_{\alpha\beta}(g_\alpha))$, proving (a). As to (b), if $\theta_\beta(g_\beta) = e$, then there exists

* This corresponds to the familiar fact that if a sequence (of real numbers, say) converges to x, then every infinite subsequence also converges to x.

$\alpha' \geq \beta$ such that $\varphi_{\beta\alpha'}(g_\beta) = e$. Choose $\beta' \geq \alpha'$; then $\varphi_{\beta\beta'} = \varphi_{\alpha'\beta'}\varphi_{\beta\alpha'}$, so $\varphi_{\beta\beta'}(g_\beta) = e$.

A very special example of a cofinal subset is furnished by the case in which A possesses an upper bound, that is, an element $\bar{\alpha}$ such that $\alpha \leq \bar{\alpha}$ for all $\alpha \in A$. If B is the singleton $(\bar{\alpha})$, then B is cofinal in A. Thus, in this case, $\underset{\longrightarrow}{\lim}\{G_\alpha;\varphi_{\alpha\alpha'}\} = G_{\bar{\alpha}}$.

3. *Direct limits and tensor products*

The notion of the *tensor product* of two abelian groups is crucial in homological algebra. Here we will show how the direct limit *interacts* with tensor products; in particular, Theorem 3.10 below enables us to obtain a completely general theorem from a rather obvious result on free abelian groups (Proposition 3.12). Since we will only be concerned in this section with *abelian* groups, we will write the group operation additively.

Definition 3.1. Given two abelian groups G and H, their *tensor product* $G \otimes H$ is the abelian group generated by the symbols $g \otimes h$, subject to the relations

$$(3.2)\qquad \begin{aligned}(g_1 + g_2) \otimes h &= g_1 \otimes h + g_2 \otimes h, \quad g_1, g_2 \in G,\ h \in H,\\ g \otimes (h_1 + h_2) &= g \otimes h_1 + g \otimes h_2, \quad g \in G,\ h_1, h_2 \in H.\end{aligned}$$

Proposition 3.3. *In* $G \otimes H$, *we have*

$$n(g \otimes h) = ng \otimes h = g \otimes nh, \text{ for any } n \in \mathbb{Z}.$$

Proof. That $g \otimes 0 = 0 \otimes h = 0$ follows trivially from (3.2). Again, an easy induction on n, based on (3.2), establishes the proposition if n is positive. Finally, (3.2) establishes that

$$-(g \otimes h) = (-g) \otimes h = g \otimes (-h)$$

and hence the proposition holds for all integers n.

Example 3.4. Let $G = \mathbb{Z}$. Then a generator of $\mathbb{Z} \otimes H$ has the form $n \otimes h = 1 \otimes nh$. It follows now from (3.2) that every element of $\mathbb{Z} \otimes H$ is expressible as $1 \otimes h$, $h \in H$, and we thus obtain a natural isomorphism $\mathbb{Z} \otimes H \simeq H$.

Example 3.5. Let G be a finite group and $H = \mathbb{Q}$. Then a generator of $G \otimes \mathbb{Q}$ has the form $g \otimes \lambda$, $g \in G$, $\lambda \in \mathbb{Q}$. Let $ng = 0$, $n > 0$. Then $g \otimes \lambda = n(g \otimes \lambda/n) = ng \otimes \lambda/n = 0 \otimes \lambda/n = 0$. Thus $G \otimes \mathbb{Q} = 0$.

We do not propose to go into a detailed discussion of all the properties of the tensor product. The theorem below is, however, crucial in any significant application of tensor products. We first note that, given abelian groups M, N, then the set of homomorphisms from M to N, written $\mathrm{Hom}(M, N)$, is an abelian group under the binary operation given by

$$(3.6)\qquad (\varphi + \psi)(x) = \varphi(x) + \psi(x),\ x \in M,\ \varphi, \psi\colon M \longrightarrow N.$$

Theorem 3.7. *Given abelian groups G,H,K, there is a natural isomorphism*

$$\eta\colon \mathrm{Hom}(G \otimes H, K) \simeq \mathrm{Hom}(G, \mathrm{Hom}(H, K)).$$

Proof. Given $\varphi\colon G \otimes H \longrightarrow K$, define $\varphi' = \eta(\varphi)\colon G \longrightarrow \mathrm{Hom}(H, K)$ by $\varphi'(g)(h) = \varphi(g \otimes h)$, $g \in G$, $h \in H$. The second relation in (3.2) shows that $\varphi'(g)$ is a homomorphism from H to K and the first relation in (3.2) shows that φ' is a homomorphism from G to $\mathrm{Hom}(H, K)$. That η is a homomorphism follows immediately from (3.6). Plainly we may go in the other direction; that is, given $\psi\colon G \longrightarrow \mathrm{Hom}(H, K)$, we may define $\psi' = \eta'(\psi)\colon G \otimes H \longrightarrow K$ by $\psi'(g \otimes h) = \psi(g)(h)$. Then η' is inverse to η and η is an isomorphism.

The reader familiar with the language of category theory will know what to understand by the word "natural" in the statement of this theorem. We will be content to explain what it means to say that η is *natural with respect to* G.

We first observe that, given homomorphisms $\varphi\colon G_1 \longrightarrow G_2$, $\psi\colon H_1 \longrightarrow H_2$, there is a well-defined homomorphism

$$\varphi \otimes \psi\colon G_1 \otimes H_1 \longrightarrow G_2 \otimes H_2,$$

given by

$$(\varphi \otimes \psi)(g \otimes h) = \varphi(g) \otimes \psi(h),\ g \in G_1,\ h \in H_1.$$

Let us write $\bar{\varphi}$ for $\varphi \otimes 1\colon G_1 \otimes H \longrightarrow G_2 \otimes H$. Then plainly $\bar{\varphi}'\bar{\varphi} = \overline{\varphi'\varphi}$ for $\varphi'\colon G_2 \to G_3$. We also notice that φ induces

$$\varphi^*\colon \mathrm{Hom}(G_2, X) \longrightarrow \mathrm{Hom}(G_1, X),$$

for any abelian group X, where φ^* is given by

$$\varphi^*(\theta) = \theta\varphi, \quad \theta\colon G_2 \longrightarrow X,$$

and again it is plain that $\varphi^*\varphi'^* = (\varphi'\varphi)^*$.

With these preliminaries we may describe the naturality of η in Theorem 3.7 with respect to G in the following way. Given $\varphi: G_1 \longrightarrow G_2$ we have $\bar{\varphi}: G_1 \otimes H \longrightarrow G_2 \otimes H$ and hence

$$(3.8)\qquad \begin{aligned} \bar{\varphi}^*&: \operatorname{Hom}(G_2 \otimes H, K) \longrightarrow \operatorname{Hom}(G_1 \otimes H, K) \\ \varphi^*&: \operatorname{Hom}(G_2, \operatorname{Hom}(H,K)) \longrightarrow \operatorname{Hom}(G_1, \operatorname{Hom}(H,K)). \end{aligned}$$

Then the diagram

$$(3.9)\qquad \begin{array}{ccc} \operatorname{Hom}(G_2 \otimes H, K) & \overset{\eta_2}{\simeq} & \operatorname{Hom}(G_2, \operatorname{Hom}(H,K)) \\ \Big\downarrow \bar{\varphi}^* & & \Big\downarrow \varphi^* \\ \operatorname{Hom}(G_1 \otimes H, K) & \underset{}{\overset{\eta_1}{\simeq}} & \operatorname{Hom}(G_1, \operatorname{Hom}(H,K)) \end{array}$$

commutes; the proof is an easy and immediate consequence of the definition of η.

We are now ready to demonstrate the relation of direct sums to tensor products. Our main theorem is the following.

Theorem 3.10. *Let* $\varinjlim \{G_\alpha\,;\,\varphi_{\alpha\alpha'}\} = \{G;\,\varphi_\alpha\}$. *Then*

$$\varinjlim \{G_\alpha \otimes H;\, \bar{\varphi}_{\alpha\alpha'}\} = \{G \otimes H;\, \bar{\theta}_\alpha\}.$$

In other words, direct limits commute with tensor products.

Proof. Let us first recast Definition 2.12 in slightly different form. Then to say that $\varinjlim \{G_\alpha\,;\,\varphi_{\alpha\alpha'}\} = \{G;\,\theta_\alpha\}$ is to say, first, that $\varphi^*_{\alpha\alpha'}(\theta_{\alpha'}) = \theta_\alpha$, for all $\alpha \le \alpha'$ and, second, that if, for any X and any $\rho_\alpha \in \operatorname{Hom}(G_\alpha, X)$, $\alpha \in A$, we have $\varphi^*_{\alpha\alpha'}(\rho_{\alpha'}) = \rho_\alpha$, then there exists a unique $\rho \in \operatorname{Hom}(G, X)$ such that $\theta^*_\alpha(\rho) = \rho_\alpha$.

It is obvious that $\bar{\varphi}^*_{\alpha\alpha'}(\bar{\theta}_{\alpha'}) = \bar{\theta}_\alpha$ if $\varphi^*_{\alpha\alpha'}(\theta_{\alpha'}) = \theta_\alpha$, so we take an arbitrary abelian group* X and homomorphisms $\rho_\alpha \in \operatorname{Hom}(G_\alpha \otimes H, X)$ such that $\bar{\varphi}^*_{\alpha\alpha'}(\rho_{\alpha'}) = \rho_\alpha$. Let $\tilde{\rho}_\alpha = \eta(\rho_\alpha) \in \operatorname{Hom}(G_\alpha, \operatorname{Hom}(H, X))$. In view of (3.9) we infer that $\varphi^*_{\alpha\alpha'}(\tilde{\rho}_{\alpha'}) = \tilde{\rho}_\alpha$, so that there exists a unique $\tilde{\rho} \in \operatorname{Hom}(G, \operatorname{Hom}(H, X))$ such that $\theta^*_\alpha(\tilde{\rho}) = \tilde{\rho}_\alpha$. Define $\rho \in \operatorname{Hom}(G \otimes H, X)$ by $\eta(\rho) = \tilde{\rho}$. Then since η is an isomorphism, we may again invoke (3.9) to conclude that ρ is the unique homomorphism from $G \otimes H$ to X such that $\bar{\theta}^*_\alpha(\rho) = \rho_\alpha$. This proves the theorem.

---

*It is easy to see that, for a direct system of abelian groups, it is sufficient to take X abelian in Definition 2.12.

We will give an application of this theorem. It is fairly easy to see that

$$(3.11)\qquad (G_1 \oplus G_2) \otimes H = G_1 \otimes H \oplus G_2 \otimes H.$$

From this observation and Example 3.4 it follows that if G is a free abelian group of rank n (where, in fact, n may be any cardinal), then $G \otimes H$ is isomorphic to the direct sum of n copies of H. Now let $\iota : H_1 \longrightarrow H_2$ be an inclusion – or, more generally, any monomorphism. We then have

$$1 \otimes \iota : G \otimes H_1 \longrightarrow G \otimes H_2$$

and the remarks above immediately lead to the

Proposition 3.12. *If* $\iota : H_1 \longrightarrow H_2$ *is a monomorphism and* G *is free abelian, then* $1 \otimes \iota : G \otimes H_1 \longrightarrow G \otimes H_2$ *is also a monomorphism.*

For $1 \otimes \iota$ can be regarded as the direct sum of n copies of the monomorphism ι. However, $1 \otimes \iota$ need not be a monomorphism for every G. Take, for example, $G = \mathbb{Z}_2$ and ι the embedding of the even integers $2\mathbb{Z}$ in $\mathbb{Z}$. Then $G \otimes 2\mathbb{Z} \simeq G = \mathbb{Z}_2$, but, if g generates G, then

$$(1 \otimes \iota)(g \otimes 2) = g \otimes 2 = 2g \otimes 1 = 0 \otimes 1 = 0,$$

so that $1 \otimes \iota$ is the zero homomorphism. Thus it is important to find out for just what abelian groups G is it true that $1 \otimes \iota$ is always a monomorphism. We use Theorem 3.10 to prove

Proposition 3.13. *If* $\iota : H_1 \longrightarrow H_2$ *is a monomorphism and* G *is a direct limit of free abelian groups, then* $1 \otimes \iota : G \otimes H_1 \longrightarrow G \otimes H_2$ *is a monomorphism.*

Proof. Let $\{G; \theta_\alpha\} = \varinjlim \{G_\alpha\,;\,\varphi_{\alpha\alpha'}\}$ with each G_α free abelian. By Theorem 3.10

$$\{G \otimes H_i\,;\, \theta_\alpha \otimes 1\} = \varinjlim \{G_\alpha \otimes H_i\,;\, \varphi_{\alpha\alpha'} \otimes 1\}, \qquad i = 1,2.$$

Let $x \in G \otimes H_1$ and let $(1 \otimes \iota)(x) = 0$. Then $x = (\theta_\alpha \otimes 1)(x_\alpha)$ for some index α, $x_\alpha \in G_\alpha \otimes H_1$, so that $(\theta_\alpha \otimes 1)(1 \otimes \iota)(x_\alpha) = 0$. It thus follows (see the parenthetic remark following Theorem 2.21) that, for some $\alpha' \ge \alpha$, $(\varphi_{\alpha\alpha'} \otimes 1)(1 \otimes \iota)(x_\alpha) = 0$ or

$$(1 \otimes \iota)(\varphi_{\alpha\alpha'} \otimes 1)(x_\alpha) = 0.$$

But $G_{\alpha'}$ is free so that, by Proposition 3.12, $1 \otimes \iota : G_{\alpha'} \otimes H_1 \longrightarrow G_{\alpha'} \otimes H_2$

is a monomorphism. Thus $(\varphi_{\alpha\alpha'} \otimes 1)(x_\alpha) = 0$. But $\theta_\alpha \otimes 1 = (\theta_{\alpha'} \otimes 1)(\varphi_{\alpha\alpha'} \otimes 1)$, so

$$x = (\theta_\alpha \otimes 1)(x_\alpha) = (\theta_{\alpha'} \otimes 1)(\varphi_{\alpha\alpha'} \otimes 1)(x_\alpha) = 0,$$

and the proposition is proved.

It is now easy to prove the following general theorem.

Theorem 3.14. *The abelian group G has the property that* $1 \otimes \iota : G \otimes H_1 \longrightarrow G \otimes H_2$ *is a monomorphism for all monomorphisms* $\iota : H_1 \longrightarrow H_2$ *if and only if it is torsion free.*

Proof. An abelian group is the direct limit of its finitely-generated subgroups. Now a finitely-generated torsion free abelian group is free abelian, so that Proposition 3.13 implies that $1 \otimes \iota$ is a monomorphism. Conversely, suppose G is not torsion free and let $g \in G$ be an element of finite order $n > 1$. Let $\iota : \mathbb{Z} \to \mathbb{Q}$ be the embedding. Now $G \otimes \mathbb{Z} \simeq G$ under the isomorphism $g \otimes 1 \leftrightarrow g$. Thus $g \otimes 1 \neq 0$ in $G \otimes \mathbb{Z}$. But in $G \otimes \mathbb{Q}$,

$$g \otimes 1 = ng \otimes 1/n = 0 \otimes 1/n = 0.$$

Thus $1 \otimes \iota$ is not a monomorphism.

Abelian groups enjoying the property, above, that tensor products preserve monomorphisms, are also called *flat*. This concept, applied to the more general algebraic structures called *modules* (over some ring Λ), plays a crucial role in modern algebraic geometry.

4. *Direct limits and exactness*

One of the most important concepts of modern algebra is that of an *exact sequence*. We say that a sequence of groups and homomorphisms

$$(4.1) \qquad \ldots \to G_{n-1} \xrightarrow{\xi_{n-1}} G_n \xrightarrow{\xi_n} G_{n+1} \to \ldots, -\infty < n < \infty$$

is *exact at* G_n if

$$(4.2) \qquad \text{Image } \xi_{n-1} = \text{Kernel } \xi_n .$$

It is a very important feature of direct limits (and distinguishes them from inverse limits) that they *preserve exactness*. Precisely, this means the following. Let A be a directed set and suppose that, for each $\alpha \in A$, we have a sequence of groups*

$$(4.3) \qquad \ldots \longrightarrow G^\alpha_{n-1} \xrightarrow{\xi^\alpha_{n-1}} G^\alpha_n \xrightarrow{\xi^\alpha_n} G^\alpha_{n+1} \longrightarrow \ldots ;$$

suppose further that, for each n, we have a direct system $\{G^\alpha_n ; \varphi^{\alpha\alpha'}_n\}$ over A such that each collection (ξ^α_n), $\alpha \in A$, is a morphism of direct systems (see Theorem 2.20). This means that, for all n and all $\alpha \le \alpha'$, we have a commutative diagram

$$(4.4) \qquad \begin{array}{ccc} G^\alpha_n & \xrightarrow{\xi^\alpha_n} & G^\alpha_{n+1} \\ \downarrow \varphi^{\alpha\alpha'}_n & & \downarrow \varphi^{\alpha\alpha'}_{n+1} \\ G^{\alpha'}_n & \xrightarrow{\xi^{\alpha'}_n} & G^{\alpha'}_{n+1} \end{array}$$

Let $\varinjlim \{G^\alpha_n ; \varphi^{\alpha\alpha'}_n\} = \{G_n ; \theta^\alpha_n\}$. Then, by Theorem 2.20, we have homomorphisms $\xi_n : G_n \longrightarrow G_{n+1}$ such that

$$(4.5) \qquad \xi_n \theta^\alpha_n = \theta^\alpha_{n+1} \xi^\alpha_n , \quad \alpha \in A, \quad -\infty < n < \infty .$$

We assert

Theorem 4.6. *If each sequence* (4.3) *is exact at* G^α_n, *then the limit sequence*

$$\ldots \longrightarrow G_{n-1} \xrightarrow{\xi_{n-1}} G_n \xrightarrow{\xi_n} G_{n+1} \longrightarrow \ldots$$

is exact at G_n.

Proof. Let $g \in G_{n-1}$. Then $g = \theta^\alpha_{n-1}(g^\alpha)$ for some $\alpha \in A$, $g^\alpha \in G^\alpha_{n-1}$. Thus

$$\begin{aligned} \xi_n \xi_{n-1}(g) &= \xi_n \xi_{n-1} \theta^\alpha_{n-1}(g^\alpha) \\ &= \theta^\alpha_{n+1} \xi^\alpha_n \xi^\alpha_{n-1}(g^\alpha), \quad \text{by (4.5)} \\ &= e, \text{ since the } \alpha\text{-sequence is exact at } G^\alpha_n . \end{aligned}$$

Conversely, let $g \in G_n$ with $\xi_n(g) = e$. Then $g = \theta^\alpha_n(g^\alpha)$ for some $\alpha \in A$, $g^\alpha \in G^\alpha_n$. Then

$$e = \xi_n(g) = \xi_n \theta^\alpha_n(g^\alpha) = \theta^\alpha_{n+1} \xi^\alpha_n(g^\alpha) \quad \text{by (4.5)}.$$

* We write α *superscript* in this section for notational convenience.

It then follows (compare the proof of Proposition 3.13) that, for some $\alpha' \geq \alpha$, $\varphi_{n+1}^{\alpha\alpha'} \xi_n^\alpha (g^\alpha) = e$ or, in view of (4.4),

$$\xi_n^{\alpha'} \varphi_n^{\alpha\alpha'} (g^\alpha) = e.$$

Since the α'-sequence is exact at $G_n^{\alpha'}$, we conclude that $\varphi_n^{\alpha\alpha'} (g^\alpha) = \xi_{n-1}^{\alpha'} (g^{\alpha'})$ for some $g^{\alpha'} \in G_{n-1}^{\alpha'}$, so that

$$g = \theta_n^\alpha (g^\alpha) = \theta_n^{\alpha'} \varphi_n^{\alpha\alpha'} (g^\alpha) = \theta_n^{\alpha'} \xi_{n-1}^{\alpha'} (g^{\alpha'}) = \xi_{n-1} \theta_{n-1}^{\alpha'} (g^{\alpha'}),$$

by (4.5), and the theorem is proved.

This theorem has important applications; e.g., to Čech cohomology theory.

5. *A new result*

In this final section we consider a possible converse to Theorem 2.20. In that theorem we saw how a morphism of direct systems of groups induces a homomorphism of their direct limits. We may therefore ask whether every homomorphism of the direct limits is induced in this way. Precisely if $\{G_\alpha\} \longrightarrow G$, $\{H_\alpha\} \longrightarrow H$ and $\gamma: G \longrightarrow H$, does there exist a morphism (γ_α) where $\gamma_\alpha : G_\alpha \longrightarrow H_\alpha$ such that (γ_α) induces γ?

One quickly convinces oneself that the answer is negative. For example, we have the constant sequence $\{\mathbb{Q}\}$ tending to $\mathbb{Q}$ and the sequence of Example 2.8 tending to $\mathbb{Q}$. Now the only homomorphism $\mathbb{Q} \longrightarrow \mathbb{Z}$ is the zero homomorphism, so that only the zero homomorphism $\mathbb{Q} \longrightarrow \mathbb{Q}$ is induced by a morphism of the constant sequence $\{\mathbb{Q}\}$ into the sequence of Example 2.8.

However this negative answer does not destroy the interest of the question, it merely forces us to modify it. We consider throughout a fixed but arbitrary directed set and prove the following.

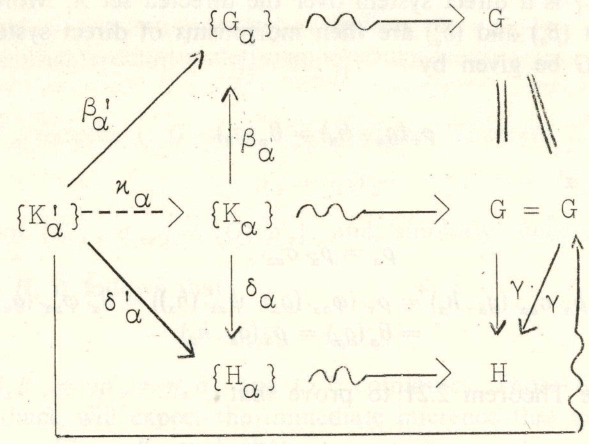
Theorem 5.1. *Let* $\varinjlim \{G_\alpha\} = G$, $\varinjlim \{H_\alpha\} = H$ *and let* $\gamma: G \longrightarrow H$. *Then there exists a direct system* $\{K_\alpha\}$ *such that* $\varinjlim \{K_\alpha\} = G$ *and morphisms* (β_α): $\{K_\alpha\} \longrightarrow \{G_\alpha\}$, (δ_α): $\{K_\alpha\} \longrightarrow \{H_\alpha\}$ such that

(i) (β_α) *induces* 1; $G \longrightarrow G$ *and* (δ_α) *induces* γ: $G \longrightarrow H$; *and*

(ii) *given* $\varinjlim \{K'_\alpha\} = G$ *and* (β'_α): $\{K'_\alpha\} \longrightarrow \{G_\alpha\}$, (δ'_α): $\{K'_\alpha\} \longrightarrow \{H_\alpha\}$ *such that* (β'_α) *induces* 1: $G \longrightarrow G$, (δ'_α) *induces* γ: $G \longrightarrow H$, *there exists*

a unique morphism (κ_α) : $\{K'_\alpha\} \to \{K_\alpha\}$ *such that* (κ_α) induces $1 : G \to G$ *and* $\beta_\alpha \kappa_\alpha = \beta'_\alpha$, $\delta_\alpha \kappa_\alpha = \delta'_\alpha$ *for all* $\alpha \in A$.

Before proving the theorem we illustrate it by a diagram, where the wavy arrow indicates passage to the limit,



Proof of Theorem 5.1. Consider, for each α, the diagram

$$(5.2) \qquad \begin{array}{c} G_\alpha \\ \downarrow \gamma\theta_\alpha \\ H_\alpha \xrightarrow{\eta_\alpha} H \end{array}$$

Here we adopt the usual notation, $\varinjlim \{G_\alpha ; \varphi_{\alpha\alpha'}\} = \{G; \theta_\alpha\}$, $\varinjlim \{H_\alpha ; \psi_{\alpha\alpha'}\} = \{H; \eta_\alpha\}$. We now form the *pull-back* of (5.2); that is, we construct from (5.2) the commutative square

$$(5.3) \qquad \begin{array}{ccc} K_\alpha & \xrightarrow{\beta_\alpha} & G_\alpha \\ \downarrow \delta_\alpha & & \downarrow \gamma\theta_\alpha \\ H_\alpha & \xrightarrow{\eta_\alpha} & H \end{array}$$

where $K_\alpha \leq G_\alpha \times H_\alpha$ consists of pairs (g_α, h_α) with $\gamma\theta_\alpha (g_\alpha) = \eta_\alpha (h_\alpha)$ and β_α, δ_α are the restrictions to K_α of the projections $G_\alpha \times H_\alpha \longrightarrow G_\alpha$, $G_\alpha \times H_\alpha \longrightarrow H_\alpha$. We may define, for $\alpha \leq \alpha'$, a homomorphism

$$\sigma_{\alpha\alpha'} (g_\alpha, h_\alpha) = (\varphi_{\alpha\alpha'} (g_\alpha), \psi_{\alpha\alpha'} (h_\alpha)).$$

For $\gamma\theta_{\alpha'}\varphi_{\alpha\alpha'}(g_\alpha) = \gamma\theta_\alpha(g_\alpha)$, $\eta_{\alpha'}\psi_{\alpha\alpha'}(h_\alpha) = \eta_\alpha(h_\alpha)$. It is then plain that if $\alpha \leq \alpha' \leq \alpha''$, then

$$\sigma_{\alpha'\alpha''}\sigma_{\alpha\alpha'} = \sigma_{\alpha\alpha''},$$

so $\{K_\alpha\,;\,\sigma_{\alpha\alpha'}\}$ is a direct system over the directed set A. Moreover, it is plain that (β_α) and (δ_α) are then morphisms of direct systems. Let $\rho_\alpha : K_\alpha \longrightarrow G$ be given by

$$\rho_\alpha(g_\alpha,\, h_\alpha) = \theta_\alpha(g_\alpha). \tag{5.4}$$

Then if $\alpha \leq \alpha'$

$$\rho_\alpha = \rho_{\alpha'}\sigma_{\alpha\alpha'}, \tag{5.5}$$

for $\rho_{\alpha'}\sigma_{\alpha\alpha'}(g_\alpha, h_\alpha) = \rho_{\alpha'}(\varphi_{\alpha\alpha'}(g_\alpha),\ \psi_{\alpha\alpha'}(h_\alpha)) = \theta_{\alpha'}\varphi_{\alpha\alpha'}(g_\alpha)$
$= \theta_\alpha(g_\alpha) = \rho_\alpha(g_\alpha, h_\alpha)$.

We now use Theorem 2.21 to prove that

$$\varinjlim\,\{K_\alpha\,;\,\sigma_{\alpha\alpha'}\} = \{G\,;\,\rho_\alpha\}. \tag{5.6}$$

First, let $g \in G$. Then $g = \theta_\alpha(g_\alpha)$ for some $\alpha \in A$, $g_\alpha \in G_\alpha$, and $\gamma(g) = \eta_{\alpha'}(h_{\alpha'})$ for some $\alpha' \in A$, $h_{\alpha'} \in H_{\alpha'}$. Choose $\alpha'' \geq \alpha,\ \alpha'$; then, if $g_{\alpha''} = \varphi_{\alpha\alpha''}(g_\alpha)$, $h_{\alpha''} = \psi_{\alpha'\alpha''}(h_{\alpha'})$, we have

$$\gamma\theta_{\alpha''}(g_{\alpha''}) = \gamma\theta_\alpha(g_\alpha) = \gamma(g) = \eta_{\alpha'}(h_{\alpha'}) = \eta_{\alpha''}(h_{\alpha''}).$$

Thus $(g_{\alpha''}, h_{\alpha''}) \in K_{\alpha''}$, and $\rho_{\alpha''}(g_{\alpha''}, h_{\alpha''}) = \theta_{\alpha''}(g_{\alpha''}) = \theta_\alpha(g_\alpha) = g$. This proves that condition (a) in Theorem 2.21 is satisfied. To establish condition (b), let $\rho_\alpha(g_\alpha, h_\alpha) = e$. Thus $\theta_\alpha(g_\alpha) = e$ so that, for some $\alpha' \geq \alpha$, $\varphi_{\alpha\alpha'}(g_\alpha) = e$. But also $\eta_\alpha(h_\alpha) = \gamma\theta_\alpha(g_\alpha) = e$, so that, for some $\alpha'' \geq \alpha$, $\psi_{\alpha\alpha''}(h_\alpha) = e$. Choose $\alpha_1 \geq \alpha', \alpha''$. It then follows that $\varphi_{\alpha\alpha_1}(g_\alpha) = e$, $\psi_{\alpha\alpha_1}(h_\alpha) = e$, so that

$$\sigma_{\alpha\alpha_1}(g_\alpha,\, h_\alpha) = e.$$

Thus (5.6) is established. Now (5.4) asserts that $\theta_\alpha\beta_\alpha = \rho_\alpha$ so that (β_α) induces $1\colon G \longrightarrow G$; and

$$\eta_\alpha\delta_\alpha = \gamma\theta_\alpha\beta_\alpha = \gamma\rho_\alpha,$$

so that (δ_α) induces $\gamma\colon G \longrightarrow H$. Thus assertion (i) of the theorem is com-

pletely established. It remains to establish assertion (ii). Consider, for each α, the diagram

$$\begin{array}{ccc} K'_\alpha & \xrightarrow{\beta'_\alpha} & G_\alpha \\ \downarrow \delta'_\alpha & & \downarrow \gamma\theta_\alpha \\ H_\alpha & \xrightarrow{\eta_\alpha} & H \end{array} \tag{5.7}$$

Since (β'_α) induces $1\colon G \longrightarrow G$, it follows (see Theorem 2.20) that

$$\rho'_\alpha = \theta_\alpha\beta'_\alpha$$

where $\varinjlim\,\{K'_\alpha\,;\,\sigma'_{\alpha\alpha'}\} = \{G;\ \rho'_\alpha\}$; and, similarly, since (δ'_α) induces $\gamma\colon G \longrightarrow H$, it follows that

$$\gamma\rho'_\alpha = \eta_\alpha\delta'_\alpha.$$

Thus $\gamma\theta_\alpha\beta'_\alpha = \gamma\rho'_\alpha = \eta_\alpha\delta'_\alpha$ and (5.7) commutes. Those familiar with the pull-back will expect the immediate inference that there exists a unique $\kappa_\alpha\colon K'_\alpha \longrightarrow K_\alpha$ such that

$$\beta_\alpha\kappa_\alpha = \beta'_\alpha, \quad \delta_\alpha\kappa_\alpha = \delta'_\alpha\,;$$

but, in any case, this is easily deduced, with

$$\kappa_\alpha(v) = (\beta'_\alpha(v),\ \delta'_\alpha(v)), \quad v \in K'_\alpha.$$

We show that (κ_α) is a morphism of direct systems. Let $\alpha_1 \geq \alpha$; then

$$\begin{aligned} \sigma_{\alpha\alpha_1}\kappa_\alpha(v) = \sigma_{\alpha\alpha_1}(\beta'_\alpha(v),\ \delta'_\alpha(v)) &= (\varphi_{\alpha\alpha_1}\beta'_\alpha(v),\ \psi_{\alpha\alpha_1}\delta'_\alpha(v)) \\ &= (\beta'_{\alpha_1}\sigma'_{\alpha\alpha_1}(v),\ \delta'_{\alpha_1}\sigma'_{\alpha\alpha_1}(v)) \\ &= \kappa_{\alpha_1}\sigma'_{\alpha\alpha_1}(v), \quad v \in K'_\alpha. \end{aligned}$$

Finally we show that (κ_α) induces $1\colon G \longrightarrow G$; but this is an immediate consequence of the relation $\rho_\alpha\kappa_\alpha = \rho'_\alpha$, which holds since $\rho_\alpha\kappa_\alpha(v) = \rho_\alpha(\beta'_\alpha(v),\ \delta'_\alpha(v)) = \theta_\alpha\beta'_\alpha(v) = \rho'_\alpha(v)$, $v \in K'_\alpha$. Thus the theorem is completely proved.

We offer some commentary on this theorem. If one compares it with Definition 2.12 (or with Theorem 2.5) one finds a close resemblance. The collection

$$\{G_\alpha\} \xleftarrow{\beta_\alpha} \{K_\alpha\} \xrightarrow{\delta_\alpha} \{H_\alpha\} \tag{5.8}$$

has the same type of *universal property*, relative to $\gamma\colon G \longrightarrow H$, as the direct limit has relative to a direct system of groups. Indeed, part (i) of Theorem 5.1 asserts that (5.8) possesses a certain property and part (ii) asserts that it is universal for this property. Thus we can regard the proof of Theorem 5.1 as demonstrating the existence of such a universal solution (5.8) just as we provided in Section 2 a construction of a direct limit (see Theorem 2.19). It is thus possible to imitate the proof of Theorem 2.14 to show the *uniqueness* of (5.8) satisfying the requirements of Theorem 5.1; it is, in fact, a characteristic feature of all definitions by means of universal properties that such definitions automatically imply the uniqueness (up to canonical equivalence) of the objects being defined. The reader should refer to texts on category theory for further discussion of this type of definition in universal algebra.

The uniqueness of (5.8) allows us to refer to it as the *canonical realization* of $\gamma : G \to H$. It is possible to make this realization *functorial*; this will be described in a subsequent paper*. It is of some interest to note that if $(\gamma_\alpha) : \{G_\alpha\} \longrightarrow \{H_\alpha\}$ induces $\gamma : G \longrightarrow H$, then

$$\{G_\alpha\} \xleftarrow{1} \{G_\alpha\} \xrightarrow{(\gamma_\alpha)} \{H_\alpha\}$$

is not necessarily the canonical realization of γ. For this to be true it is necessary and sufficient that, for each α, $\eta_\alpha : H_\alpha \longrightarrow H$ is one-one. Thus in this last case, when η_α is one-one, a realization $(\gamma_\alpha) : \{G_\alpha\} \longrightarrow \{H_\alpha\}$, $\gamma : G \longrightarrow H$, is unique (if it exists) and is the canonical realization.

Remark. The pull-back (see (5.2), (5.3)) is, of course, another example of a definition by means of a solution of a universal problem. Indeed, the pull-back may be regarded as the *inverse* limit of a system (of groups) over the directed set consisting of 3 elements, which may be represented diagrammatically by

$$\begin{array}{ccc} & & \cdot \\ & & \downarrow \\ \cdot & \rightarrow & \cdot \end{array}$$

where the arrow means that the domain precedes the range in the given order. It is thus a feature of the proof of Theorem 5.1 that we demonstrate a special case of the remarkable fact that pull-backs commute with direct limits in the category of groups. The general problem of when direct limits (colimits) and inverse limits (limits) commute is discussed in B. Eckmann and P. J. Hilton (loc. cit.).

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* Peter Hilton. The category of direct systems and functors on groups (Battelle Research Report).