On Isometric Immersions of Riemannian Manifolds in Euclidean Space

by

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The purpose of this paper is to present complete proofs of the existence and rigidity theorems[2] for isometric immersions announced in [7]. (For earlier results on existence of isometric immersions, see [1], [2], and [5].) Our existence theorem is an analogue of the result of Hirsch [4] for smooth immersions and states that a simply connected Riemannian manifold M can be isometrically immersed in euclidean space if and only if there is a suitably equipped candidate for a normal bundle over M. The rigidity theorem asserts that the normal bundle with its additional structure essentially determines the immersion up to a rigid motion. These two theorems contain as special cases the classical results for hypersurfaces.

The idea of the proof of the existence theorem is based upon the following observation. Suppose $\varphi : M \to R^{n+k}$ is an isometric immersion of a Riemannian n-manifold M with normal bundle E. Then φ can be extended to an immersion $\tilde{\varphi}$ of a tubular neighborhood W of the zero section in E into R^{n+k}. Since W is immersed as open subset of R^{n+k}, the metric induced by $\tilde{\varphi}$ on W is flat and, of course, induces the given metric on M (as the zero section).

Conversely, suppose E is a k-plane bundle over M and that some tubular neighborhood W of the zero section of E has a flat metric inducing the given metric on M. It is then not difficult to prove (see Lemma 2.3 below) that, if M is simply connected, W can be isometrically immersed in R^{n+k} and thus so can M. What we do below is to find

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2. It was pointed out to me recently that the existence theorem given here is a reformulation of (the ambient space euclidean case of) Theorem 5, page 202 of Bishop and Crittenden, "Geometry on Manifolds," Academic Press, New York and London, 1964. The proofs are, however, quite different.

conditions, in terms of structure on the bundle under which a tubular neighborhood of the zero section has a flat metric.

The proof of the rigidity theorem uses the same ideas in that the general case is reduced to the codimension zero flat case.

The organization of the paper is as follows. In section 1, we state the principal results of the paper. In section 2, we show how additional structure on a bundle can be used to define a metric on a tubular neighborhood of the zero section of the bundle, state the important properties of the metric, and derive the existence and rigidity theorems from these properties. The last three sections of the paper are devoted to proving that the metric defined in section 2 has the stated properties.

Finally, I would like to express my gratitude to Rolph Schwarzenberger for helpful conversations during the early stages of this work.

1. *Statements of results.*

In this section, we state the principal results of the paper. Throughout the paper, all objects considered (maps, manifolds, bundles, etc.) will be differentiable of class C^∞.

Let $\varphi : M \to R^{n+k}$ be an isometric immersion of a Riemannian n-manifold M into euclidean space, E its normal bundle, and $T = TM$ the tangent bundle of M. As the normal bundle of an isometric immersion, E possesses additional structure consisting of a *bundle metric* $(\,,)$, a *connection D*, and a *second fundamental form A* (which is a section in Hom $(T \otimes E, T)$). The bundle metric on E is induced from the usual metric on R^{n+k} in the obvious way and the connection and second fundamental form obtained by projecting the usual connection in R^{n+k} onto the normal and tangent planes of M respectively. (For a more complete description of this structure, see [7].) It is easily seen that the connection D is compatible with the metric $(\,,)$ so that

$$X(N, N') = (D_X N, N') + (N, D_X N') \tag{1.1}$$

and also that A is self adjoint in the sense that

$$(A_X N, Y) = (X, A_Y N) \tag{1.2}$$

where X, Y are tangent vectors at $u \in M$, N, N' normal fields. (Since no confusion seems likely, we use the notation $(\,,)$ both for the bundle metric in E and for the Riemannian metric on M.)

We define the *second fundamental tensor associated with A* to be the section B in Hom $(T \otimes T, E)$ defined by

$$(B(X, Y), N) = (A_X N, Y) \tag{1.3}$$

for tangent vectors X, Y on M and normal vector N. It follows immediately from (1, 2) that B is symmetric.

As usual, we denote the *Riemannian curvature tensor* by R and define $\bar{R}$, the *curvature of E relative to D*, by the equation

$$\bar{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X, Y]} N.$$

In this context, the *Gauss equations* are

$$R(X, Y)Z = A_X B(Y, Z) - A_Y B(X, Z)$$

and

$$\bar{R}(X, Y)N = B(A_X N, Y) - B(X, A_Y N)$$

and the *Codazzi-Mainardi equation*

$$\nabla_X A_Y N - \nabla_Y A_X N - A_{[X, Y]} N = A_Y D_X N - A_X D_Y N.$$

In the above, ∇ is the Levi-Civita connection on M, X, Y, and Z are tangent vector fields on M and N is a section in E (a normal field).

It is well known (see, for example, Hicks [3], p. 76) that the Gauss and Codazzi-Mainardi equations are satisfied in the above situation. Thus, the existence of a k-plane bundle over a Riemannian manifold with the additional structure described above for which the Gauss and Codazzi-Mainardi equation hold is a necessary condition for the existence of an isometric immersion in R^{n+k}. Our existence theorem below asserts that, if M is simply connected, this condition is also sufficient.

We call a k-plane bundle over a manifold a *Riemannian k-plane bundle* if it is equipped with a bundle metric and compatible connection. If E is any k-plane bundle over a Riemannian manifold M, a *second fundamental form* in E is a section A in Hom $(T \otimes E, T)$ satisfying (1.2). If E is a Riemannian vector bundle with a second fundamental form A, we define the *associated second fundamental tensor B* as in (1.3).

We can now state our main results:

Existence Theorem. *Let M be a simply connected Riemannian n-manifold with a Riemannian k-plane bundle E over M equipped with a second fundamental form A and associated second fundamental tensor B. Then, if the Gauss and Codazzi-Mainardi equations are satisfied, M can be isometrically immersed in R^{n+k} with normal bundle E.*

Rigidity Theorem. *Let $\varphi, \varphi' : M \longrightarrow R^{n+k}$ be isometric immersions of a connected Riemannian n-manifold with normal bundles E, E' equipped as above with bundle metrics, connections, and second fundamental forms. Suppose there is an isometry $f : M \longrightarrow M$ that can be covered by a bundle* map $\tilde{f} : E \longrightarrow E'$ *which preserves the bundle metrics, the connections, and the second fundamental forms. Then there is a rigid motion* F *of* R^{n+k} *such that* $F \circ \varphi = \varphi' \circ f$.

Suppose now that the normal bundle E to an isometric immersion is trivial and choose an orthonormal framing $\eta = \{N_1, \ldots, N_k\}$ of E. (That is, $N_1, \ldots, N_k$ are normal fields on M which form an orthonormal base for the normal space at each pont of M.) Define 1-forms $\omega^{\alpha\beta}$, $1 \leq \alpha,\ \beta \leq k$, on M by

$$D_X N_\alpha = \sum_{\beta=1}^{k} \omega^{\alpha\beta}(X) N_\beta \tag{1.4}$$

and sections $L_1, \ldots, L_k$ in Hom (T, T) by

$$L_\alpha X = A_X N_\alpha . \tag{1.5}$$

It follows immediately from (1.1) and (1.2) that

$$\omega^{\alpha\beta} = -\omega^{\beta\alpha} \tag{1.6}$$

and that

$$(L_\alpha X, Y) = (X, L_\alpha Y) \tag{1.7}$$

for $1 \leq \alpha, \beta \leq k$. Furthermore, the Gauss and Codazzi-Mainardi equations become

$$R(X, Y)Z = \sum_{\beta=1}^{k} [(L_\beta Y, Z) L_\beta X - (L_\beta X, Z) L_\beta Y], \tag{1.8}$$

$$\bar{R}(X, Y) N_\alpha = \sum_{\beta=1}^{k} [(L_\alpha X, L_\beta Y) - (L_\alpha Y, L_\beta X)] N_\beta , \tag{1.9}$$

$$\nabla_X L_\alpha Y - \nabla_Y L_\alpha X - L_\alpha [X, Y] = \sum_{\beta=1}^{k} [\omega^{\alpha\beta}(X) L_\beta Y - \omega^{\alpha\beta}(Y) L_\beta X], \tag{1.10}$$

where α ranges from 1 to k. Of course, in (1.9),

$$\bar{R}(X, Y) N_\alpha = \sum_{\beta=1}^{k} [(d\omega^{\alpha\beta} - \sum_{\gamma=1}^{k} \omega^{\alpha\gamma} \wedge \omega^{\gamma\beta})(X, Y)] N_\beta$$

so (1.8), (1.9), and (1.10) invole only the forms $\omega^{\alpha\beta}$, the section L_α, and invariants of the manifold M.

Conversely, suppose a Riemannian manifold M is equipped with 1-forms $\omega^{\alpha\beta}$ and sections L_α in Hom (T, T), $1 \leq \alpha,\ \beta \leq k$, satisfying (1.6) and (1.7). Then, we can define a metric, a compatible connection

D, and a second fundamental form A in the trivial k-plane bundle E over M by choosing a framing $N_1, \ldots, N_k$ of E, specifying it as orthonormal, and using (1.4) and (1.5) to define D and A. These observations lead to the following:

Corollary 1. *A simply connected Riemannian n-manifold M can be isometrically immersed in* R^{n+k} *with a trivial normal bundle if and only if there are 1-forms* $\omega^{\alpha\beta}$ *on M and sections* L_α *in Hom* (T, T), $1 \leq \alpha, \beta \leq k$, *satisfying (1.6) through (1.10).*

Corollary 2. *Let* $\varphi, \bar{\varphi} : M \longrightarrow R^{n+k}$ *be isometric immersions of the connected Riemannian n-manifold M with trivial normal bundles E and* $\bar{E}$. *Let* $f : M \longrightarrow M$ *be an isometry and suppose we can choose framings* η of E, $\bar{\eta}$, of $\bar{E}$ such that

$$f^* \bar{\omega}^{\alpha\beta} = \omega^{\alpha\beta} \qquad d f L_\alpha = \bar{L}_\alpha\, d f$$

for $1 \leq \alpha, \beta \leq k$ *where* $\omega^{\alpha\beta}$, L_α, $\bar{\omega}^{\alpha\beta}$, $\bar{L}_\alpha$ *are defined as above in terms of* η *and* $\bar{\eta}$. *Then there is a rigid motion F of* R^{n+k} *such that* $F \circ \varphi = \bar{\varphi} \circ f$.

If $k = 1$ in the above corollaries, they reduce to the classical existence and rigidity theorems for hypersurfaces.

2. The induced metric.

Let $\pi : E \rightarrow M$ be a Riemannian k-plane bundle with second fundamental form A and $TE \simeq H \oplus V$ be the decomposition into horizontal and vertical subbundles determined by the connection in E. As is well known, $H \simeq \pi^* TM$ and $V \simeq \pi^* E$ in a natural way so H and V inherit metrics from TM and E (both of which are denoted again by $(\,,\,)$). We define a (possibly singular) metric $\langle\,,\,\rangle$ on TE by setting $\langle\,,\,\rangle = (\,,\,)$ on V, designating H and V orthogonal, and defining

$$\langle Z, Z' \rangle = (Z + \tilde{A}_Z Y,\ Z' + \tilde{A}_{Z'}\, Y) \tag{2.1}$$

for Z, Z' horizontal tangent vectors at Y in E and $\tilde{A}$ the section in Hom $(H \otimes E,\ H)$ determined by A. (Here we treat Y as both a point in the manifold E and a vector in the vector bundle E.)

Now let $\varphi : M \rightarrow R^{n+k}$ be an isometric immersion with normal bundle E equipped with its induced bundle metric, connection D and second fundamental form A. Let $\tilde{\varphi} : E \rightarrow R^{n+k}$ be the obvious extension of φ taking the fibers of E linearly into the normal spaces to M.

Theorem 2.1. *The (possibly singular) metric on E induced by $\tilde{\varphi}$ from the usual metric on R^{n+k} is the one described above in* (2.1).

The proof of this theorem is given in section 3. We now derive the rigidity theorem from it.

Let $\varphi, \varphi' : M \to R^{n+k}$ be isometric immersions with normal bundles E, E' and $\tilde{f} : E \to E'$ a bundle map convering an isometry f of M as in the rigidity theorem. Let $\tilde{\varphi} : E \to R^{n+k}$, $\tilde{\varphi}' : E' \to R^{n+k}$ be extentions of φ and φ' as above and $W \subset E$, $W' \subset E'$ tubular neighborhoods of the zero sections on which $\tilde{\varphi}$ and $\tilde{\varphi}'$ are immersions and with $\tilde{f}\, W = W'$. Since $\tilde{f}$ preserves the additional structure in E and E', it follows from Theorem 2.1 that $\tilde{f} : W \to W$ is an isometry extending $f : M \to M$. (Of course, W and W' are Riemannian flat in the induced metrics.) Let $U \subset W$ be a coordinate ball on which both $\tilde{\varphi}$ and $\tilde{\varphi}' \circ \tilde{f}$ are (isometric) embeddings. Since U is flat, there is a rigid motion F of R^{n+k} with $F \circ \tilde{\varphi} | U = \tilde{\varphi}' \circ \tilde{f} | U$. It now follows easily that $F \circ \tilde{\varphi} = \tilde{\varphi}' \circ \tilde{f}$ so, in particular, $F \circ \varphi = \varphi' \circ f$.

In order to prove the existence theorem, we need the following result, the proof of which is given in the last two sections of the paper.

Theorem 2.2. *Let E be a Riemannian k-plane bundle equipped with a second fundamental form A and let W be a tubular neighborhood of the zero section of E on which the metric* $\langle \, , \rangle$ *defined in* (2.1) *is nonsingular. Then, if the Gauss and Codazzi-Mainardi equations are satisfied, this metric on W has curvature zero.*

The existence theorem now follows from the observation that the zero section $M \to W$ is an isometric embedding and the following lemma.

Lemma 2.3. *Suppose* W_1 *and* W_2 *are flat Riemannian n-manifolds,* W_1 *connected and simply connected and* W_2 *complete. Then there is an isometric immersion of* W_1 *into* W_2.

As this lemma is essentially well known, we include only a rough sketch of the proof.

Let $U \subset W_1$ be a flat coordinate neighborhood of $x \in W_1$ small enough so that it can be isometrically embedded in some flat coordinate neighborhood of W_2. For any $y \in W_1$ choose a path α from x to y, cover α by small flat coordinate neighborhoods, and extend the isometric embedding to an isometric immersion along α to y. This extenstion can always be accomplished since W_2 is complete.

If α' is another path from x to y which coincides with α except on a flat coordinate neighborhood, it is easily seen that the immersion defined in terms of α' agrees at y with the immersion defined in terms of α. Now, since W_1 is simply connected, for any path β from x to y, we can choose a sequence of paths $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_r = \beta$ such that

α_i coincides with α_{i+1}, $0 \leq i < r$, except on a flat coordinate neighborhood. This completes the proof of Lemma 2.3.

3. The proof of Theorem 2.1.

Before giving the proof of Theorem 2,1, we introduce some notational conventions which will be used throughout the remainder of the paper.

As above, $\pi : E \to M$ will be a Riemannian k-plane bundle equipped with a second fundamental form over a Riemannian n-manifold M. We let U be a coordinate neighborhood in M with local coordinates $u = (u^1, \ldots, u^n)$ and $Y_1, \ldots, Y_k$ a system of orthonormal sections in E over U. With this notation, the Levi-Civita connection ∇ on M, the connection D in E, and the second fundamental form A can be expressed (over U) as

$$\nabla_{\partial_i} \partial_j = \Gamma^l_{ij}\, \partial_l ,$$

$$D_{\partial_i} Y_\alpha = G^\beta_{i\alpha}\, Y_\beta ,$$

$$A_{\partial_i} Y_\alpha = H^j_{i\alpha}\, \partial_j ,$$

where here and in what follows, $\partial_i = \dfrac{\partial}{\partial u^i}$, the indices i, j, l, ... will run from 1 to n and $\alpha, \beta, \gamma, \ldots$ from 1 to k. We also employ the Einstein summation convention on repeated indices. In this context, equations (1.1) and (1.2) become

$$G^\beta_{i\alpha} + G^\alpha_{i\beta} = 0 \tag{3.1}$$

and

$$g_{jl} H^l_{i\alpha} = g_{il} H^l_{j\alpha} \tag{3.2}$$

where $(\partial_i , \partial_j) = g_{ij}$.

Using the local coordinates on U and the vector fields $Y_1, \ldots, Y_k$, we can define local coordinates $(u, y) = (u^1, \ldots, u^n, y^1, \ldots, y^k)$ in $\pi^{-1} U$ by letting (u, y) correspond to $y^\alpha Y_\alpha(u)$. In terms of these coordinates, the horizontal lifts of $\dfrac{\partial}{\partial u^i}$ on U to the point (u, y) in E are the vectors

$$Z_i = \frac{\partial}{\partial u^i} - y^\alpha G^\beta_{i\alpha} \frac{\partial}{\partial y^\beta} . \tag{3.3}$$

Of course, the $\dfrac{\partial}{\partial y^\alpha}$ are vertical vector fields on $\pi^{-1} U$.

Now to the proof of Theorem 2.1. Let $\varphi(x) = (\varphi^1(x), \dots, \varphi^{n+k}(x))$ be the isometric immersion and $A_{ia} = \dfrac{\partial \varphi^a}{\partial u^i}$ on U. (For the remainder of this section, the indices $a, b, c, \dots$ will run from 1 to $n+k$.) Considering the sections Y_α as normal fields, let $B_{\alpha a}$ be defined by $Y_\alpha = B_{\alpha a}\dfrac{\partial}{\partial x^a}$ where $x^1, \dots, x^{n+k}$ are the usual coordinate in R^{n+k} and let C_{ai} be defined by

(3.4) $$C_{ai}A_{ib} + B_{\alpha a}B_{\alpha b} = \delta_{ab}\,.$$

Then $\nabla_{\partial_i}\partial_j$ is the tangential component of

$$\frac{\partial A_{ja}}{\partial u^i}\frac{\partial}{\partial x^a} = \frac{\partial A_{ja}}{\partial u^i}(C_{al}A_{lb} + B_{\alpha a}B_{\alpha b})\frac{\partial}{\partial x^b}$$

which is

$$\frac{\partial A_{ja}}{\partial u^i}C_{al}A_{lb}\frac{\partial}{\partial x^b} = \frac{\partial A_{ja}}{\partial u^i}C_{al}\,d\varphi\frac{\partial}{\partial u^l}$$

From this it follows that

$$\Gamma^l_{ij} = \frac{\partial A_{ja}}{\partial u^i}C_{al}\,.$$

In a similar fashion, we can show that

(3.5) $$G^\beta_{i\alpha} = \frac{\partial B_{\alpha a}}{\partial u^i}B_{\beta a}$$

and

(3.6) $$H^j_{i\alpha} = \frac{\partial B_{\alpha a}}{\partial u^i}C_{aj}\,.$$

Define $\psi : U \times R^k \longrightarrow R^{n+k}$, $\psi(u, y) = (\psi^1(u, y), \dots, \psi^{n+k}(u, y))$, by $\psi^a(u, y) = \varphi^a(u) + y^\alpha B_{\alpha a}$. Then

$$d\psi\left(\frac{\partial}{\partial u^i}\right) = \left(A_{ia} + y^\beta\frac{\partial B_{\beta a}}{\partial u^i}\right)\frac{\partial}{\partial x^a},$$

$$d\psi\left(\frac{\partial}{\partial y^\alpha}\right) = B_{\alpha a}\frac{\partial}{\partial x^a},$$

so that, using (3.4) and (3.5),

$$d\psi Z_i = \left[A_{ia} + y^\beta\left(\frac{\partial B_{\beta a}}{\partial u^i} - \frac{\partial B_{\beta b}}{\partial u^i}(\delta_{ba} - C_{bl}A_{la})\right)\right]\frac{\partial}{\partial x^a}$$

Setting

(3.7) $$P_{il} = (\delta_{il} + y^\beta H^l_{i\beta})$$

and using (3.6), we have

$$d\psi Z_i = P_{il}A_{la}\frac{\partial}{\partial x^a}\,.$$

It now follows easily that

$$\left\langle\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle = \delta_{\alpha\beta}, \quad \left\langle\frac{\partial}{\partial y^\alpha}, Z_i\right\rangle = 0,$$

and

$$\langle Z_i, Z_j\rangle = P_{il}P_{jm}A_{la}A_{ma} = P_{il}P_{jm}g_{lm}$$

which is exactly the local form of the metric defined in (2.1).

Remark. It should be clear now that the techniques of this paper can be applied to study isometric immersions of Riemannian manifolds in spheres and hyperbolic spaces. For example, the analogue of the metric defined in (2.1) for isometric immersions in a sphere of radius r is given in local form on TW at (u, y) by

$$\left\langle\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle = \frac{r^2}{|y|^2}\left(y^\alpha y^\beta\left(1 - \frac{\sin^2|y|}{|y|^2}\right) + \delta_{\alpha\beta}\sin^2|y|\right),$$

$$\langle Z_i, Z_j\rangle = P'_{il}P'_{jm}g_{lm},$$

$$\left\langle Z_i, \frac{\partial}{\partial y^\alpha}\right\rangle = 0,$$

where $|y|^2 = y^\gamma y^\gamma$ and

$$P'_{il} = \delta_{il}\cos|y| + \frac{r}{|y|}H^l_{i\gamma}y^\gamma\sin|y|\,.$$

A similar expression holds in the hyperbolic case.

As above, the rigidity theorem is a direct consequence of the form of this metric. (Of course, rigid motion is replaced by isometry for spheres and hyperbolic spaces.) Although we have not carried out the computations, it seems reasonable to expect that, with suitably modified Gauss and Codazzi-Mainardi equations, an analogue of Theorem 2.2 can be proved for immersions in spheres and hyperbolic spaces.

4. The Levi-Civita connection on W.

In this section, we define an affine connection on the tubular neighborhood W of the zero section in E and prove that it is the Levi-Civita connection associated with the metric of (2.1) *when the Gauss and Codazzi-Mainardi equations hold.* The proof of Theorem 2.2 will then be complete when we prove, as we do in the next section, that the curvature of this connection is zero.

It might be worth mentioning here that, whereas we define the connection below in terms of local coordinates, it could easily have been done globally. This would involve defining it directly only on those vector fields on W which are induced from sections in TM and E via the equivalences $H \simeq \pi^* TM$, $V \simeq \pi^* E$ (using the lifts of the connections ∇ and D via these same equivalences) and extending to arbitrary vector fields in the obvious way. Our reason for adopting the local approach rather than the global is that the proofs in this and the next section seem to be far easier in the local context.

Before defining the connection, we introduce a bit of notation. As in the previous section, we will be working with local coordinates (u, y) in W. Since no confusion seems likely, we will denote $\frac{\partial}{\partial y^\alpha}$ by Y_α and use the symbol ∇ for the connection described below. (The Levi-Civita connection on M will occur from now on only in the form Γ^l_{ij}.) Let P_{ij} be defined as in (3.7) and let Q_{ij} be the inverse for P_{ij} so that

$$Q_{il} P_{lj} = P_{il} Q_{lj} = \delta_{ij} \tag{4.1}$$

We can now define the connection ∇ on W as follows:

$$\nabla_{Y_\alpha} Y_\beta = 0$$
$$\nabla_{Y_\alpha} Z_i = H^l_{i\alpha} Q_{lm} Z_m$$
$$\nabla_{Z_i} Y_\alpha = H^l_{i\alpha} Q_{lm} Z_m + G^\beta_{i\alpha} Y_\beta$$
$$\nabla_{Z_i} Z_j = (\Gamma^l_{im} P_{jm} + Z_i(P_{jl})) Q_{lr} Z_r - H^l_{i\alpha} g_{lm} P_{jm} Y_\alpha$$

where Z_i (defined in (3.3)) span the horizontal subbundle locally and Y_α span the vertical subbundle locally.

Remark. This connection was obtained originally by pulling back the usual connection on R^{n+k} to W via the map ψ defined in the previous section.

Note that, since we prove below that the connection defined above is the unique Levi-Civita connection associated with the metric of (2.1), the local formula above do actually define a global connection.

We now prove ∇ symmetric. This involves three cases:

$$\nabla_{Y_\alpha} Y_\beta - \nabla_{Y\beta} Y_\alpha = [Y_\alpha, Y_\beta],$$
$$\nabla_{Y_\alpha} Z_i - \nabla_{Z_i} Y_\alpha = [Y_\alpha, Z_i],$$

and

$$\nabla_{Z_i} Z_j - \nabla_{Z_j} Z_i = [Z_i, Z_j].$$

In the first of these all terms vanish and the second follows from the easily checked fact that

$$[Y_\alpha, Z_i] = -G^\beta_{i\alpha} Y_\beta .$$

In order to prove the third equation, note that

$$[Z_i, Z_j] = y^\alpha \left(\frac{\partial G^\beta_{i\alpha}}{\partial u^j} - \frac{\partial G^\beta_{j\alpha}}{\partial u^i} + G^\gamma_{i\alpha} G^\beta_{j\gamma} - G^\gamma_{j\alpha} G^\beta_{i\gamma} \right) Y_\beta$$

and

$$Z_i(P_{jl}) = y^\alpha \left(\frac{\partial H^l_{j\alpha}}{\partial u^i} - G^\beta_{i\alpha} H^l_{j\beta} \right)$$

by direct computation so that, using the definition of P_{jm} and the fact that $\Gamma^l_{ij} = \Gamma^l_{ji}$, we have

$$\nabla_{Z_i} Z_j - \nabla_{Z_j} Z_i = y^\alpha \left(\frac{\partial H^l_{j\alpha}}{\partial u^i} - \frac{\partial H^l_{i\alpha}}{\partial u^j} + \Gamma^l_{im} H^m_{j\alpha} - \Gamma^l_{jm} H^m_{i\alpha} - G^\beta_{i\alpha} H^l_{j\beta} + G^\beta_{j\alpha} H^l_{i\beta} \right) Q_{lr} Z_r$$
$$- (H^l_{i\alpha} g_{lm} P_{jm} - H^l_{j\alpha} g_{lm} P_{im}) Y_\alpha .$$

Now, the coefficient of Z_r vanishes by the Codazzi-Mainardi equation and the term in Y_α is exactly $[Z_i, Z_j]$ if we use the definition of P_{lm}, equation (4.1), and the second of the Gauss equations. Thus ∇ is symmetric.

In order to prove ∇ is compatible with the metric $\langle , \rangle$, we must show that

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_Y Z \rangle \tag{4.2}$$

where each of X, Y, and Z is a Y_α or a Z_i. If $X = Y_\alpha$ and at least one of Y or Z is a Y_β, all terms of (4.2) vanish. If $X = Z_i$ and at least one of Y or Z is a Y_α, both sides vanish by an easy computation. We consider in more detail the two remaining cases.

Suppose $X = Y_\alpha$, $Y = Z_i$, and $Z = Z_j$. Then

$$\begin{aligned} Y_\alpha\langle Z_i, Z_j\rangle &= Y_\alpha(P_{il}P_{jm}g_{lm}) \\ &= Y_\alpha(P_{il})P_{jm}g_{lm} + P_{il}Y_\alpha(P_{jm})g_{lm} \\ &= H^l_{i\alpha}P_{jm}g_{lm} + H^m_{j\alpha}P_{il}g_{lm}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \nabla_{Y_\alpha}Z_i, Z_j\rangle &= \langle H^l_{i\alpha}Q_{lm}Z_m, Z_j\rangle \\ &= H^l_{i\alpha}Q_{lm}P_{mr}P_{js}g_{rs} \\ &= H^l_{i\alpha}P_{js}g_{ls} \end{aligned}$$

using (4.1). Similarly $\langle Z_i, \nabla_{Y_\alpha}Z_j\rangle = H^l_{j\alpha}P_{is}g_{ls}$ so (4.2) holds in this case.

Finally, we check the case when all three of X, Y, and Z are horizontal. First of all,

$$\begin{aligned} Z_i\langle Z_j, Z_l\rangle &= Z_i(P_{jr}P_{ls}g_{rs}) \\ &= Z_i(P_{jr})P_{ls}g_{rs} + P_{jr}Z_i(P_{ls})g_{rs} + P_{jr}P_{ls}Z_i(g_{rs}) \end{aligned}$$

and, using (4.1), we see that

$$\begin{aligned} \langle \nabla_{Z_i}Z_j, Z_l\rangle + \langle Z_j, \nabla_{Z_i}Z_l\rangle &= (\Gamma^r_{im}P_{jm} + Z_i(P_{jr}))P_{ls}g_{rs} \\ &\quad + (\Gamma^r_{im}P_{lm} + Z_i(P_{lr}))P_{js}g_{rs}. \end{aligned}$$

The equality of these two expressions follows from the equation

$$Z_i(g_{rs}) = \frac{\partial}{\partial u^i}g_{rs} = \Gamma^m_{ir}g_{ms} + \Gamma^m_{is}g_{mr}$$

which holds since the Γ^l_{ij} define a connection compatible with the metric g_{rs}.

This concludes the proof of the fact that the connection defined above is the Levi-Civita connection associated with the metric on W.

5. The curvature of the connection on W.

We now complete the proof of Theorem 2.2 by showing that the curvature of the connection defined in the previous section is zero when the Gauss and Codazzi-Mainardi equations hold. This will be done by showing

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$$

is zero when each of X, Y, and Z is a Y_α or a Z_i. (We use R for the curvature here since no confusion seems likely.)

For convenience, we list here the symmetry properties of R needed in this section (See, for example, Milnor [6], p. 53).

(5.1) $$R(X, Y)Z + R(Y, X)Z = 0$$

(5.2) $$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

(5.3) $$\langle R(X, Y)Z, W\rangle + \langle R(X, Y)W, Z\rangle = 0$$

(5.4) $$\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle.$$

Now to the proof. First note that $R(Y_\alpha, Y_\beta)Y_\alpha = 0$ trivially and

$$\begin{aligned} R(Y_\alpha, Z_i)Y_\beta &= \nabla_{Y_\alpha}\nabla_{Z_i}Y_\beta \\ &= H^l_{i\beta}(Y_\alpha(Q_{lm}) + Q_{lr}H^s_{r\alpha}Q_{sm})Z_m. \end{aligned}$$

Since $Q_{lm}P_{ms} = \delta_{ls}$,

$$\begin{aligned} Y_\alpha(Q_{lm}P_{ms}) = 0 &= Y_\alpha(Q_{lm})P_{ms} + Q_{lm}Y_\alpha(P_{ms}) \\ &= Y_\alpha(Q_{lm})P_{ms} + Q_{lm}H^s_{m\alpha} \end{aligned}$$

from which it follows that $R(Y_\alpha, Z_i)Y_\beta = 0$. Using (5.1) and (5.2), we see that $R(X, Y)Z = 0$ wherever one of X, Y, Z is a Z_i and the other two Y_α and Y_β.

We now compute $R(Z_i, Z_j)Y_\alpha$. First note that, using (5.4) we have

$$\langle R(Z_i, Z_j)Y_\alpha, Y_\beta\rangle = \langle R(Y_\alpha, Y_\beta)Z_i, Z_j\rangle = 0$$

so, in computing $R(Z_i, Z_j)Y_\alpha$, we can *and will* ignore the vertical component. Proceeding, we have

$$\begin{aligned} R(Z_i, Z_j)Y_\alpha &= \nabla_{Z_i}\nabla_{Z_j}Y_\alpha - \nabla_{Z_j}\nabla_{Z_i}Y_\alpha \\ &= \nabla_{Z_i}(H^l_{j\alpha}Q_{lr}Z_r + G^\beta_{j\alpha}Y_\beta) - \nabla_{Z_j}(H^l_{i\alpha}Q_{lr}Z_r + G^\beta_{i\alpha}Y_\beta) \\ &= Z_i(H^l_{j\alpha}Q_{lr})Z_r - Z_j(H^l_{i\alpha}Q_{lr})Z_r \\ &\quad + H^l_{j\alpha}Q_{lm}(\Gamma^q_{ip}P_{mp} + Z_i(P_{mp}))Q_{qr}Z_r \\ &\quad - H^l_{i\alpha}Q_{lm}(\Gamma^q_{jp}P_{mp} + Z_j(P_{mp}))Q_{qr}Z_r \\ &\quad + G^\beta_{j\alpha}H^m_{i\beta}Q_{mr}Z_r - G^\beta_{i\alpha}H^m_{j\beta}Q_{mr}Z_r. \end{aligned}$$

Again, using $Q_{lr}P_{rs} = \delta_{ls}$, we have

$$Z_i(Q_{lr}) = -Q_{lm}Z_i(P_{ms})Q_{sr}$$

so

$$Z_i(H^l_{j\alpha}Q_{lr}) = \frac{\partial H^l_{j\alpha}}{\partial u^i}Q_{lr} - H^l_{j\alpha}Q_{lm}Z_i(P_{ms})Q_{sr}$$

with a similar expression for $Z_j(H^l_{i\alpha}Q_{lr})$. Thus, using the Codazzi-Mainardi equation, the above expression vanishes and $R(Z_i, Z_j)Y_\alpha = 0$.

Again from the symmetry of R we conclude that $R(X, Y)Z = 0$ whenever two of X, Y, Z are Z_i and Z_j and the other Y_α and also that the vertical component of $R(Z_i, Z_j)Z_l$ is zero. The computation of the horizontal component of $R(Z_i, Z_j)Z_l$ is tedious but straight forward and much like those given above. It vanishes as a consequence of the two Gauss equations. We leave the details as an exercise for the reader.

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