DIFFERENTIAL FORMS AND THEIR INTEGRALS

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Introduction

In geometry there are, essentially, three points of view[1]

- the pointwise geometry, sometimes called linear (or multilinear) algebra,
- the local geometry, sometimes called analysis,
- the global geometry, usually called topology.

The interplay of the three points of view is one of the beauties of geometry. In these notes we want to present, in the classical context of differential forms and their integrals, an example of such an interplay.

The result we will be focusing is the Theorem of de Rham, that states that integration gives an isomorphism between the de Rham cohomology and the dual of the singular homology (with real coefficients). We will prove this Theorem in the case of open sets of Euclidean spaces, which is, really, the significant case. The extension of the proof presented here to the case of manifolds is very simple. Naturally, on the way, we will introduce all necessary concepts.

The choices we made for the subject and the presentation attend the basic needs

- relevance: it is a relevant theory both in classical and modern mathematics,
- prerequisite: just a basic knowledge of linear algebra and calculus of several variables,
- introduction to more advanced topics: we hope to give the reader a painless introduction to more advanced topics as algebraic topology and partial differential equation between others.

These notes where prepared for a short course given by the second author at the "I Colóquio de Matemática da Região Nordeste" that will take place at The Federal University of Sergipe, Brazil, from 28/02 to 04/03 of 2011, but they grow up from courses delivered by the authors at various levels. The lectures where addressed to an audience of undergraduate students. We tank the organization for the invitation.

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[1] Quoting freely from A. Weinstein, Journal of Differential Geometry, 1970.

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CHAPTER 1

The de Rham cohomology for open sets of $\mathbb{R}^n$

1. Exterior forms

Let $\mathbb{E}$ be a finite dimensional real vector space and $\mathbb{E}^*$ its dual. We will identify, as usual, $\mathbb{E}$ with $(\mathbb{E}^*)^* := \mathbb{E}^{**}$.

1.1. Definition. A *tensor of type* (p, q) in $\mathbb{E}$ is a multilinear[1] map:

$$t\colon \underbrace{\mathbb{E}^* \times \cdots \times \mathbb{E}^*}_{p \text{ times}} \times \underbrace{\mathbb{E} \times \cdots \times \mathbb{E}}_{q \text{ times}} \longrightarrow \mathbb{R}$$

We will denote by $\mathbb{E}_{(p,q)}$ the space of these tensors. This is a real vector space with the obvious operations of sum of multilinear maps (summing the values) and product by a scalar (multiplying the values by the scalar).

1.2. Examples.

- $\mathbb{E}_{(0,1)} = \mathbb{E}^*$, $\mathbb{E}_{(1,0)} = \mathbb{E}^{**} = \mathbb{E}$.
- A scalar product in $\mathbb{E}$ is an element of $\mathbb{E}_{(0,2)}$.
- It is convenient to define $\mathbb{E}_{(0,0)} := \mathbb{R}$.

We will be interested mainly in tensors of type $(0, q)$. To simplify the notations we will set $\mathbb{E}_q := \mathbb{E}_{(0,q)}$. Beside adding tensors, we can multiply them.

1.3. Definition. Given $\omega \in \mathbb{E}_p$, $\tau \in \mathbb{E}_q$, we define the *tensor product* $\omega \otimes \tau \in \mathbb{E}_{p+q}$ as

$$\omega \otimes \tau(x_1, \ldots, x_{p+q}) := \omega(x_1, \ldots, x_p)\tau(x_{p+1}, \ldots x_{p+q}).$$

It is easy to see that the tensor product is associative and distributive (Exercise 9.1).

1.4. Proposition. *Let* $\{\omega_1, \ldots, \omega_n\}$ *be a basis of* $\mathbb{E}_1 = \mathbb{E}^*$. *Then the set* $\{\omega_{i_1} \otimes \cdots \otimes \omega_{i_q} : i_1, \ldots, i_q \in \{1, \ldots, n\}\}$ *is a basis of* $\mathbb{E}_q$.

Proof. Let $\{e_1, \cdots, e_n\}$ be the dual basis, i.e., $\omega_i(e_j) = \delta_{ij}$. Then:

$$\sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}(e_{j_1}, \ldots, e_{j_q}) = a_{j_1 \cdots j_q}.$$

It follows, by a standard argument, that the the elements of the set in question are linearly independent. Conversely, given $\omega \in \mathbb{E}_q$ we define $a_{i_1 \cdots i_q} = \omega(e_{i_1}, \ldots, e_{i_q})$. It is easy to check that $\omega = \sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}$, and this concludes the proof.

□

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[1] i.e. linear in each variable.

We will be interested in special elements of $\mathbb{E}_q$. Let $\Sigma(p)$ be the group of permutation of $\{1, \dots, p\} \subseteq \mathbb{N}$. If $\pi \in \Sigma(p)$, we will denote by $|\pi|$ the sign of π, i.e. $|\pi| = 1$ if π is the product of an even number of transpositions and $|\pi| = -1$ otherwise.

1.5. Definition. Let $\omega \in \mathbb{E}_p$. We will say that

- ω is *a symmetric form* if $\quad \omega(x_1, \dots, x_p) = \omega(x_{\pi(1)}, \dots, x_{\pi(p)}), \quad \forall\, \pi \in \Sigma(p)$.
- ω is an *exterior form* [2] if $\quad \omega(x_1, \dots, x_p) = |\pi|\omega(x_{\pi(1)}, \dots, x_{\pi(p)}), \quad \forall\, \pi \in \Sigma(p)$.

We will denote by $\Sigma^p(\mathbb{E})$ the space of symmetric tensors in $\mathbb{E}_p$ and with $\Lambda^p(\mathbb{E})$ the space of exterior p-forms. These are subspaces of $\mathbb{E}_p$. Clearly $\Lambda^0(\mathbb{E}) = \mathbb{R} = \Sigma^0(\mathbb{E}), \quad \Lambda^1(\mathbb{E}) = \mathbb{E}_1 = \mathbb{E}^* = \Sigma^1(\mathbb{E})$.

We will be mostly interested in exterior forms and we will describe now the basic examples.

1.6. Example. Let $\{e_1, \dots, e_n\}$ be a fixed basis of $\mathbb{E}$ and $\{\phi_1, \dots, \phi_n\}$ be the dual basis. Let us fix indexes $1 \leq i_1 < \dots < i_p \leq n$ and define:

$$\Phi_{(i_1,\dots,i_p)}(x_1, \dots, x_p) := \det(\phi_{i_j}(x_k)).$$

In other words we consider the matrix whose k^{th} column is given by the coordinates of x_k in the fixed basis, and compute the determinant of the sub matrix obtained considering only the lines $(i_1, \dots, i_p)$ of the original matrix. The $\Phi_{(i_1,\dots,i_p)}$'s are exterior p-forms since the determinant is multilinear in the columns and, permuting the columns it changes sign according to the parity of the permutation. As we will see (Proposition 1.22 and Remark 1.20), these forms are a basis of $\Lambda^p(\mathbb{E})$.

1.7. Remark. By Example 1.6 p-forms are, essentially, determinants of $p \times p$ matrices and, therefore, "p-dimensional (oriented) volume elements". So they appear as the natural integrands of the multiple (oriented) integrals. These statement will be made precise in the next chapter.

The tensor product of exterior forms is not, in general, an exterior form. But we can "alternate" the tensor product in order to obtain an exterior form.

Define the linear operator

$$A : \mathbb{E}_p \longrightarrow \mathbb{E}_p, \qquad A(\tau)(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\pi \in \Sigma(p)} |\pi| \tau(x_{\pi(1)}, \dots, x_{\pi(p)}).$$

1.8. Proposition.

1. *If* $\tau \in \mathbb{E}_p, \quad A(\tau) \in \Lambda^p(\mathbb{E})$.
2. *If* $\tau \in \Lambda^p(\mathbb{E}), \quad A(\tau) = \tau$.

In particular $A^2 = A$.

Proof. If $p = 1$ there is nothing to prove, so we will assume $p > 1$. For $i, j \in \{1, \dots, p\}$, we will denote by (ij) the element of $\Sigma(p)$ that interchanges i and j and leaves the other integers fixed. If $\pi \in \Sigma(p)$, we set $\pi' = \pi \circ (ij)$. Then $|\pi'| = -|\pi|$ and

$$A(\tau)(x_1, \dots, x_j, \dots, x_i, \dots, x_p) = \frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi(1)}, \dots, x_{\pi(j)}, \dots, x_{\pi(i)}, \dots, x_{\pi(p)}) =$$

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[2] The terms alternating tensor or skew symmetric tensor are also used in the literature.

$$\frac{1}{p!}\sum_{\pi}|\pi|\tau(x_{\pi'(1)},\ldots,x_{\pi'(i)},\ldots,x_{\pi'(j)},\ldots,x_{\pi'(p)}) =$$

$$\frac{1}{p!}\sum_{\pi'}-|\pi'|\tau(x_{\pi'(1)},\ldots,x_{\pi'(i)},\ldots,x_{\pi'(j)},\ldots,x_{\pi'(p)}) = -A(\tau)(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_p)$$

It is easy to see that the equation above implies that $A(\tau)\in\Lambda^p(\mathbb{E})$ (see Exercise **??**). Moreover, if $\tau\in\Lambda^p(\mathbb{E})$,

$$A(\tau)(x_1,\ldots,x_p) = \frac{1}{p!}\sum_{\pi}|\pi|\tau(x_{\pi(1)},\ldots x_{\pi(p)}) = \frac{1}{p!}\sum_{\pi}|\pi|^2\tau(x_1,\ldots x_p) = \tau(x_1,\ldots,x_p)$$

and this proves the second claim.

□

Observe that, in general, $A(\phi\otimes\psi)\neq A(\phi)\otimes A(\psi)$. However we have

1.9. LEMMA. *If $\phi_1,\ldots,\phi_p\in\mathbb{E}^*$, then:*

$$A(\phi_1\otimes\cdots\otimes\phi_p) = \frac{1}{p!}\sum_{\sigma\in\Sigma(p)}|\sigma|\phi_{\sigma(1)}\otimes\cdots\otimes\phi_{\sigma(p)}.$$

PROOF.

$$A(\phi_1\otimes\cdots\otimes\phi_p)(x_1,\ldots,x_p) = \frac{1}{p!}\sum_{\sigma\in\Sigma(p)}|\sigma|\phi_1\otimes\cdots\otimes\phi_p(x_{\sigma(1)},\ldots,x_{\sigma(p)}) =$$

$$\frac{1}{p!}\sum_{\sigma\in\Sigma(p)}|\sigma|\phi_1(x_{\sigma(1)})\cdots\phi_p(x_{\sigma(p)}) = \frac{1}{p!}\sum_{\sigma\in\Sigma(p)}|\sigma|\phi_{\sigma(1)}(x_1)\cdots\phi_{\sigma(p)}(x_p).$$

□

Using the operator A we can define *product of exterior forms.*

1.10. DEFINITION. The *exterior (or wedge) product* is defined as the map

$$\wedge:\Lambda^p(\mathbb{E})\times\Lambda^q(\mathbb{E})\longrightarrow\Lambda^{p+q}(\mathbb{E}),\quad \wedge(\omega,\tau):=\omega\wedge\tau=\frac{(p+q)!}{p!q!}A(\omega\otimes\tau).$$

(The reason for the coefficient $\frac{(p+q)!}{p!q!}$ will be discuss in Remark 1.21.)

It is easy to prove that the exterior product is distributive (see Exercise 9.2). It is also true that it is associative, but this fact is a little bit tricky. The proof involves a characterization of the kernel of A. For this, although not strictly necessary[3], we start introducing some algebraic concepts.

1.11. DEFINITION. An *algebra* over the reals is a real vector space $\mathbb{E}$ together with a bilinear map, the *product*, $b:\mathbb{E}\oplus\mathbb{E}\longrightarrow\mathbb{E}$.

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[3] We could hide those concepts in the proof, but we prefer to expose them, also to be free to use them in what follows.

Examples of such a structure are

- The real or complex numbers with the usual multiplication. They are associative and commutative algebras.
- The set of real (or complex) valued functions defined on an open set $U \subseteq \mathbb{R}^n$, with the usual sum and product of functions. This is an associative and commutative algebra.
- The spaces $M(n, \mathbb{K})$ of $n \times n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with the usual product of matrices. They are associative but non commutative algebras (if $n > 1$!).
- The *tensor algebra* $\mathbb{E}_* = \oplus_{p \geq 0} E_p$ with the tensor product (suitably extended).
- The *exterior algebra* $\Lambda^*(\mathbb{E}) = \oplus_{p \geq 0} \Lambda^p(\mathbb{E})$ with the wedge product (suitably extended).

1.12. Definition. An algebras homomorphism $h : \mathbb{E} \longrightarrow \mathbb{E}'$ between the algebras $\mathbb{E}$ and $\mathbb{E}'$ is a linear map such that the image of the product of two elements in $\mathbb{E}$ is the product of the images (in $\mathbb{E}'$).

1.13. Definition. An ideal $\mathcal{I}$ of an algebra $\mathbb{E}$ is a vector subspace of $\mathbb{E}$ such that if $x \in \mathcal{I}, y \in \mathbb{E}$, then $b(x, y),\ b(y, x) \in \mathcal{I}$

It is not difficult to see that if $\mathcal{I}$ is an ideal of $\mathbb{E}$, the quotient vector space $\mathbb{E}/\mathcal{I}$ has a natural product (and hence a structure of algebra) such that the quotient map is an algebras homomorphism. Moreover, given an algebras homomorphism $h : \mathbb{E} \longrightarrow \mathbb{E}'$, the kernel of h, $\ker h$, is an ideal and, in fact, every ideal is the kernel of an algebras homomorphism.

We go back now to the case of our interest. We want to characterize the kernel of the operator A extended, by linearity, to the tensor algebra. The point is that A is not an algebras homomorphism, hence we can not guarantee, a priori, that $\ker A$ is an ideal. Then we start by proving that $\ker A$ is, in fact, an ideal.

Consider the ideal $\mathcal{I} \subseteq \mathbb{E}_*$ generated by $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$. This is the vector subspace of $\mathbb{E}_*$ generated by elements of the form $\tau \otimes \phi \otimes \phi, \psi \otimes \psi \otimes \eta,\ \phi, \psi \in \mathbb{E}^*, \tau, \eta \in \mathbb{E}_*$ or, alternatively, the intersection of all ideals containing the elements of the form $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$.

1.14. Theorem. *$\ker A = \mathcal{I}$.*

Proof. It is easily seen that $\mathcal{I} \subseteq \ker A$. We will prove that $\ker A \subseteq \mathcal{I}$. Consider the quotient algebra $\mathbb{E}_*/\mathcal{I}$. Denote by $\cdot$ the product in this quotient and by $\pi : \mathbb{E}_* \longrightarrow \mathbb{E}_*/\mathcal{I}$ the projection map, which is an algebra homomorphism. First observe that, if $\phi, \psi \in \mathbb{E}^*$:

$$0 = \pi((\phi + \psi) \otimes (\phi + \psi)) = \pi(\phi \otimes \phi + \phi \otimes \psi + \psi \otimes \phi + \psi \otimes \psi) = \pi(\phi \otimes \psi) + \pi(\psi \otimes \phi),$$

i.e. $\pi(\phi \otimes \psi) = -\pi(\psi \otimes \phi)$. Therefore, for $\phi_1, \ldots, \phi_p \in \mathbb{E}^*$ and $\sigma \in \Sigma(p)$, we have

$$\pi(\phi_{\sigma(1)}, \otimes \ldots, \otimes \phi_{\sigma(p)}) = \pi\left(\phi_{\sigma(1)}\right) \cdots \pi(\phi_{\sigma(p)}) = |\sigma| \pi(\phi_1) \cdots \pi(\phi_p) = |\sigma| \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

Hence

$$\pi(A(\phi_1 \otimes \cdots \otimes \phi_p)) = \pi(\frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \pi(\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(p)})) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma|^2 \pi(\phi_1 \otimes \cdots \otimes \phi_p) = \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

So any element in $\ker A$ is in $\mathcal{I} := \ker\ \pi$. $\square$

1.15. Corollary. *Let* $\omega \in \mathbb{E}_p,\ \tau \in \mathbb{E}_q$. *If* $A(\omega) = 0,\quad A(\omega \otimes \tau) = 0 = A(\tau \otimes \omega)$.

Proof. It follows from the fact that $\ker A$ is an ideal. □

At this point we can prove the announced result

1.16. Proposition. *The wedge product is associative.*

Proof. First we observe that:

$$A(A(\omega \otimes \eta) \otimes \theta)) = A(\omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)).$$

In fact, by 1.8, $A(A(\eta \otimes \theta) - \eta \otimes \theta) = 0$ and, by 1.15, we have that:

$$0 = A(\omega \otimes [A(\eta \otimes \theta) - \eta \otimes \theta]) = A(\omega \otimes A(\eta \otimes \eta) - \omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)) - A(\omega \otimes \eta \otimes \theta),$$

which proves the second equality. The first one is proved in a similar way.

Therefore, if $\omega \in \Lambda^k(\mathbb{E}), \eta \in \Lambda^l(\mathbb{E}), \theta \in \Lambda^m(\mathbb{E})$, we have:

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!} A((\omega \wedge \eta) \otimes \theta) = \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(\omega \otimes \eta \otimes \theta),$$

and the associativity follows from the associativity of the tensor product. □

1.17. Example. Let $\phi_1, \phi_2 \in \mathbb{E}^*, x_1, x_2 \in \mathbb{E}$. Then:

$$\phi_1 \wedge \phi_2(x_1, x_2) = 2\frac{1}{2}(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)) = \det[\phi_i(x_j)].$$

More generally, an induction on p gives:

1.18. Proposition. *Let* $\phi_i \in \mathbb{E}^*, x_j \in \mathbb{E}\quad i, j = 1, \ldots, p$. *Then:*

$$\phi_1 \wedge \cdots \wedge \phi_p(x_1, \ldots, x_p) = \det[\phi_i(x_j)].$$

In particular if $\sigma \in \Sigma(p),\ \phi_1 \wedge \cdots \wedge \phi_p = |\sigma| \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}$.

1.19. Remark. Observe that, by 1.16, the form $\phi_1 \wedge \cdots \wedge \phi_p$ is well defined.

1.20. Remark. In the Example 1.6 the form $\Phi_{i_1,\ldots,i_p}$ is just $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$.

1.21. Remark. The coefficient $\dfrac{(p+q)!}{p!q!}$ in 1.10 is convenient both for avoiding coefficients in 1.18 and for a geometric reason: let $\mathbb{E}$ be an inner product space, $\{e_1, \ldots, e_n\}$ an orthonormal basis and $\{\phi_1, \ldots, \phi_n\}$ the dual basis (so $\phi_i(e_j) = \langle e_i, e_j \rangle = \delta_{ij}$). Given vectors $x_1, \ldots, x_n \in \mathbb{E},\ \phi_1 \wedge \cdots \wedge \phi_n(x_1, \ldots, x_n)$ is the "volume" of the parallelepiped of edges the $x_i's$. The coefficient above is such that the "unit cube", i.e. the parallelepiped spanned by the e_i's has volume 1. We will be more precise at the end of this section (see Definition 1.26).

The following Proposition is proved, essentially, as Proposition 1.4.

1.22. Proposition. *Let* $\{\phi_1, \ldots, \phi_n\}$ *be a basis for* $\mathbb{E}^*$. *Then*

$$\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$$

is a basis of $\Lambda^p(\mathbb{E})$. *In particular* $\Lambda^p(\mathbb{E})$ *has dimension* $\binom{n}{p}$ *and* $\Lambda^p(\mathbb{E}) = \{0\}$, *if* $p > n$.

1.23. PROPOSITION. *The algebra* $\Lambda^*(\mathbb{E})$ *is graded commutative*[4] , *i.e. if* $\omega \in \Lambda^p(\mathbb{E}),\ \tau \in \Lambda^q(\mathbb{E})$

$$\omega \wedge \tau = (-1)^{pq} \tau \wedge \omega.$$

In particular the square of a form of odd degree is zero.

PROOF. It is easily seen that the claim is true for products of decomposable elements (i.e. elements of the form $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$). The general case follows from the fact that such forms span, by Proposition 1.22, the exterior algebra. ☐

1.24. REMARK. There is a restriction, in Proposition 1.22, on the set of indexes with respect to Proposition 1.4 and this is due to the graded commutativity of the exterior algebra.

Let $L : \mathbb{E} \longrightarrow \mathbb{F}$ be a linear map. Recall that the *transpose* of L is the map

$$L^* : \mathbb{F}^*(= \mathbb{F}_1) \longrightarrow \mathbb{E}^*(= \mathbb{E}_1), \quad L^*(\phi)(x) := \phi(Lx).$$

This map extends to a linear map

$$\mathbb{E}_p(L) : \mathbb{F}_p \longrightarrow \mathbb{E}_p, \quad \mathbb{E}_p(L)(\omega)(x_1, \dots, x_p) = \omega(L(x_1), \dots, L(x_p)).$$

It is simple to see that if $\omega \in \Lambda^p(\mathbb{F})$ then $\mathbb{E}_p(L)(\omega) \in \Lambda^p(\mathbb{E})$. So we get, by restriction, a linear map

$$\Lambda^p(L) := \mathbb{E}_p(L)|_{\Lambda^p(\mathbb{F})} : \Lambda^p(\mathbb{F}) \longrightarrow \Lambda^p(\mathbb{E}),$$

and, by additivity, a linear map $\Lambda^*(L) : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$.

When clear from the context we will write L_p^*, or just L^*, for $\Lambda^p(L)$ and $\Lambda^*(L)$.

1.25. PROPOSITION. $L^*(\omega \wedge \tau) = L^*(\omega) \wedge L^*(\tau)$. *This means that L induces a graded algebra morphism* $L^* : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$. *Moreover we have the following properties, called the* funtorial properties[5]

(1) $(\mathbb{1}_{\mathbb{E}})^* = \mathbb{1}_{\Lambda^*(\mathbb{E})}$.
(2) *If* $L : \mathbb{E} \longrightarrow \mathbb{F}$ *and* $T : \mathbb{F} \longrightarrow \mathbb{G}$ *are linear maps, then* $(T \circ L)^* = L^* \circ T^*$.

PROOF. To prove the first assertion, we just observe that, if $\phi_i \in \mathbb{E}^*, x_j \in \mathbb{E},\ i, j = 1, \dots, p$, we have:

$$L_p^*(\phi_1 \wedge \cdots \wedge \phi_p)(x_1, \dots, x_p) = \det[\phi_i(Lx_j)] = \det[L^*(\phi_i)(x_j)] = L^*(\phi_1) \wedge \cdots \wedge L^*(\phi_p)(x_1, \dots, x_p).$$

Since $\Lambda^p(\mathbb{E})$ is spanned by elements of the form $\phi_1 \wedge \cdots \wedge \phi_p$ (see 1.22), the conclusion follows by linearity. The functorial properties are obvious. ☐

Let $\mathbb{E}$ be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$. Then we have a canonical isomorphism[6]

$$\flat : \mathbb{E} \longrightarrow \mathbb{E}^*, \quad \flat(x)(y) = \langle x, y \rangle,$$

and therefore an inner product in $\mathbb{E}^*$ that makes $\flat$ an isometry.

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[4] An algebra $\mathbb{E}$, with product $b : \mathbb{E} \oplus \mathbb{E} \longrightarrow \mathbb{E}$ is a *graded algebra* if there is a sequence of vector subspaces $\mathbb{E}_i$ such that $\mathbb{E} = \oplus \mathbb{E}_i$ and $b(\mathbb{E}_i \oplus \mathbb{E}_j) \subseteq E_{i+j}$. Such an algebra is said to be *graded commutative* if for $\omega \in \mathbb{E}_p,\ \tau \in \mathbb{E}_q,\ b(\omega, \tau) = (-1)^{pq} b(\tau, \omega)$.

[5] In the language of category theory this means that the low that associate to a finite dimensional real vector space $\mathbb{E}$ the graded algebra $\Lambda^*(\mathbb{E})$ and to a linear maps $L : \mathbb{E} \longrightarrow \mathbb{F}$ the map L^* is a contravariant functor from the category of finite dimensional real vector spaces and linear maps, to the category of algebras and their morphisms.

[6] Sometimes called the *musical isomorphism.* It inverse is often denoted by $\sharp$.

We define an inner product in $\Lambda^p(\mathbb{E})$ declaring orthonormal a basis of the type $\{\omega_{i_1}\wedge\cdots\wedge\omega_{i_p} : i_1 < \cdots < i_p\}$ where $\{\omega_i\}$ is an orthonormal basis of $\mathbb{E}^*$. Observe that:

$$\langle \phi_1 \wedge \cdots \wedge \phi_p, \psi_1 \wedge \cdots \wedge \psi_p\rangle = \det(\langle \phi_i, \psi_j\rangle).$$

In fact, the formula above, extended by bi-linearity, defines the inner product with respect to which $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} : i_1 < \cdots < i_p\}$ is orthonormal.

We recall that two bases of a n-dimensional real vector space $\mathbb{E}$ are *equioriented* if the matrix that gives the change of bases has positive determinant. This relation is an equivalence relation and the set of bases of $\mathbb{E}$ is divided into two equivalence classes. The choice of one of these classes is the choice of an *orientation* on $\mathbb{E}$. $\mathbb{E}$ is *oriented* if such a choice has been made and the bases in the chosen class will be called *positive.* Naturally an orientation in $\mathbb{E}$ induces an orientation on $\mathbb{E}^*$, declaring positive the bases that are dual of positive bases of $\mathbb{E}$.

1.26. Definition. Let $\mathbb{E}$ be a n-dimensional *oriented* inner product space and $\{\omega_1, \dots, \omega_n\}$ a positive orthonormal basis of $\mathbb{E}^*$. The *volume form* of $\mathbb{E}$ is the n-form $v = \omega_1 \wedge \cdots \wedge \omega_n$.

1.27. Lemma. *The volume form is well defined, i.e. does not depend on the choice of the basis.*

Proof. Let $\{\omega_i\}$, $\{\phi_j\}$ be bases of $\mathbb{E}^*$ and $A = (a_{ij})$ such that $\phi_k = \sum a_{kj}\omega_j$. Then

$$\phi_1 \wedge \cdots \wedge \phi_n = \sum_{\sigma\in\Sigma(n)} |\sigma| a_{1\sigma(1)} \cdots a_{n\sigma(n)}\omega_1 \wedge \cdots \wedge \omega_n = \det(A)\omega_1 \wedge \cdots \wedge \omega_n.$$

If the bases are orthonormal and positive, $A \in SO(n)$. In particular $\det(A) = 1$. ☐

1.28. Definition. Let $\mathbb{E}$ be a n-dimensional oriented inner product space. The *Hodge (star) operator* is the operator

$$*_p : \Lambda^p(\mathbb{E}) \longrightarrow \Lambda^{(n-p)}(\mathbb{E}), \qquad *_p(\eta)(x_1, \dots, x_{(n-p)}) := \langle \eta \wedge \flat(x_1) \wedge \cdots \wedge \flat(x_{(n-p)}), v\rangle,$$

where v is the volume form. When clear from the context, we will write simply $*$ instead of $*_p$.

1.29. Remark. Let $\{\omega_i\}$ be a positive orthonormal basis for $\mathbb{E}^*$. Then the Hodge operator may be defined extending by linearity the map:

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-p}},$$

where $\{i_1, \dots, i_p, j_1, \dots j_{n-p}\}$ is an even permutation of $\{1, \dots, n\}$.

The following properties are easily established

1.30. Proposition. *$*$ is a linear isometry and $*_{n-p} \circ *_p = (-1)^{p(n-p)} \mathbb{1}_{\Lambda^p(\mathbb{E})}$.*

2. Vector fields and differential forms

2.1. Definition. Let U be an open set of $\mathbb{R}^n$. A *vector field* on U is a smooth[7] map $X : U \longrightarrow \mathbb{R}^n$. We will denote by $\mathcal{H}(U)$ the space of vector fields on U.

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[7] By smooth we will always mean C^∞.

2.2. Remark. Let X be a vector field. We want to think of $X(x)$ as a vector *based at* x. This is the reason why we use different names for the same thing[8]. We can make this point more precise as follows:

- The *tangent space of* U *at* $x \in U$ is the vector space
$$T_xU = \{(x, v) : v \in \mathbb{R}^n\}$$
with the obvious operations on the second component.
- The *tangent bundle* of U is
$$TU = \cup_{x\in U} T_xU = U \times \mathbb{R}^n.$$

A vector field on U should be defined as a smooth map $\tilde{X} : U \longrightarrow TU$ of the form $\tilde{X}(x) = (x, X(x))$, $X : U \longrightarrow \mathbb{R}^n$. Naturally, in our context, we are just complicating notations, but this point of view, that seems silly now, will come in handy when the concepts we are discussing in this chapter are extended to the case of differentiable manifolds.

An other approach to vector fields that will be useful later is the following.

Let $\mathcal{F}(U)$ be the algebra of smooth real valued functions defined in U (with the usual operations of sum and product of functions).

2.3. Definition. A *derivation of* $\mathcal{F}(U)$, (resp. a *derivation at* $x \in U$) is an $\mathbb{R}$-linear map $Y : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ (resp. $Y(x) : \mathcal{F}(U) \longrightarrow \mathbb{R}$), such that:
$$Y(fg) = Y(f)g + fY(g) \quad \text{(resp. } Y(x)(fg) = Y(x)(f)g(p) + f(p)Y(x)(g)) \;\; \forall\, f, g \in \mathcal{F}(U).$$

Both the set of derivations and the set of derivations at x have a natural structure of real vector space. We will denote by $\mathcal{D}er(U)$ and $\mathcal{D}er_x(U)$ these spaces. Observe that $\mathcal{D}er(U)$ is infinite dimensional (if $n > 0$!) while, as we will see soon, $\mathcal{D}er_x(U)$ is n-dimensional.

2.4. Example. Let $v \in \mathbb{R}^n$, $x \in U$. Given $f \in \mathcal{F}(U)$, we will denote by $v(x)f$ the usual directional derivative of f, at x, in the v direction, i.e.
$$v(x)(f) := \frac{\mathrm{d}}{\mathrm{d}t} f(x + tv)|_{(t=0)}.$$
Then $v(x) : \mathcal{F}(U) \longrightarrow \mathbb{R}$ is a derivation at x. When $v = e_i$, the i^{th} vector of the canonical basis of $\mathbb{R}^n$, we will use the standard notation
$$e_i(x)f := \frac{\partial f}{\partial x_i}(x).$$

If $X \in \mathcal{H}(U)$, we define a derivation $X \in \mathcal{D}er(U)$, by $X(f)(x) := X(x)(f)$. It is easily seen that $X(f)(x) \in \mathcal{F}(U)$ so X is, in fact, a derivation in $\mathcal{D}er(U)$.

Some simple but basic facts are the following:

2.5. Lemma. *Let* $f \in \mathcal{F}(U)$ *and* $X_x \in \mathcal{D}er_x(U)$.

- *If* f *vanishes on an open neighborhood* $V \subseteq U$, *then* $X_x(f) = 0$. *In particular, if two functions* $f, g \in \mathcal{F}(U)$ *coincide in a neighborhood of* x, $X_xf = X_xg$.

---

[8] B. Russel used to say that "Mathematics is the art of calling different things with the same name and the same thing with different names".

- *If f is constant in a neighborhood of x, $X_x f = 0$.*
- *If f is (locally) a product of functions vanishing at $x, X_x f = 0$.*

Proof. Let $\phi \in \mathcal{F}(U)$ be a function vanishing in a neighborhood V_1 of x and identically 1 outside V (see Lemma 7.3 for the existence of such functions). Then $f = \phi f$ and

$$X_x(f) = (X_x\phi)f(x) + \phi(x)X_x f = 0.$$

The second claim follows from $1 \cdot 1 = 1$ and the definition of a derivation. The third one is also immediate. □

Let $x \in \mathbb{R}^n$. Consider the set

$$\tilde{\mathcal{F}_x} := \{(f, V) : V \text{ is a neighborhood of } x, \; f \in \mathcal{F}(V)\}.$$

2.6. Definition. The algebra of *germs of smooth functions at x*, $\mathcal{F}_x$, is the quotient of $\tilde{\mathcal{F}_x}$ by the equivalence relation $(f, U) \sim (g, V) \iff f = g$ in a neighborhood of x (contained in $U \cap V$). The operations are the usual sum and product of functions (which are defined in the intersections of the domains).

We will denote by $\mathcal{D}_x$ the space of derivations of $\mathcal{F}_x$. Lemma 2.5 imply, in particular, that an element of $\mathcal{D}er_x$ induces a derivation of $\mathcal{F}_x$. The advantage of this point of view is that we do not have to worry about the domain of definition of a function.

As we have seen, a vector defines a derivation at x and hence an element of $\mathcal{D}_x$. We will see next that all derivations in $\mathcal{D}_x$ are of this type.

2.7. Theorem. *Given $p \in \mathbb{R}^n$ and a derivation $X_p \in \mathcal{D}_p$, there exist a unique vector $v \in \mathbb{R}^n$ such that $X_p = v(p)$. In particular $\mathcal{D}_p \cong T_p\mathbb{R}^n \cong \mathcal{D}er_p(U)$.*

Proof. Let $f \in \mathcal{F}_p$. Consider, in a suitable neighborhood of p, the Taylor formula

$$f(x_1, \dots, x_n) = f(p) + \sum_1^n \frac{\partial f}{\partial x_i}(p)(x_i - x_i(p)) + \Phi(x),$$

where $\Phi(x)$ is product of two functions vanishing at p.

Applying X_p to both sides and using Lemma 2.5 we have:

$$X_p(f) = \sum_1^n X_p(x_i)\frac{\partial f}{\partial x_i}(p).$$

Therefore:

$$X = \sum_1^n X(x_i)\frac{\partial}{\partial x_i}(p),$$

and the map that associates to e_i the derivation $\dfrac{\partial}{\partial x_i}(p)$ extends to an isomorphism of $\mathbb{R}^n$ (or, better T_pU) onto $\mathcal{D}_p$. □

In what follows we will identify T_pU with $\mathcal{D}_p$ and $\mathcal{H}(U)$ with $\mathcal{D}er(U)$.

The composition of two derivations is not, in general, a derivation. However the commutator of two derivations is a derivation (see Exercise 9.22). This fact suggest the following

2.8. Definition. Let $X, Y \in \mathcal{D}er(U)$. The *Lie product* of X and Y is the commutator $[X, Y] := X \circ Y - Y \circ X$.

The following properties are easy to prove and we will leave the details to the reader (Exercise 9.23).

2.9. Proposition. *The Lie product* $[\,\cdot,\cdot\,] : \mathcal{H}(U) \times \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ *is a* $\mathbb{R}$*-bilinear map. Moreover*

(1) $[X, Y] = -[Y, X]$,
(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacoby identity).*

2.10. Remark. An algebra which satisfies the properties in Proposition 2.9 is called a *Lie algebra.*

2.11. Definition. A *differential p-form* on an open set $U \subseteq \mathbb{R}^n$ is a smooth map $\omega : U \longrightarrow \Lambda^p(\mathbb{R}^n) \cong \mathbb{R}^{\binom{n}{p}}$. When clear from the context we will just say that ω is a differential form or simply a form.

2.12. Remark. According to Remark 2.2 we can complicate the definition in order to have one that make sense in the context of smooth manifold. Consider the *bundle of exterior p-forms*

$$\Lambda^p(U) := \cup_{x \in U} \Lambda^p(T_x U)$$

that can be identified with $U \times \Lambda^p(\mathbb{R}^n)$. Then a differential p-form is a smooth map $\tilde{\omega} : U \longrightarrow \Lambda^p(U)$ such that $\tilde{\omega}(x) \in \Lambda^p(T_x U)$, i.e, $\tilde{\omega}(x) = (x, \omega(x))$, $\omega(x) \in \Lambda^p(\mathbb{R}^n)$, modulo the identification.

We will denote by $\Omega^p(U)$ the set of differential p-forms on U. $\Omega^p(U)$ has an obvious structure of real vector space. Moreover we can multiply a differential form by a function and this operation is associative and distributive, in the appropriate sense, i.e. $\Omega^p(U)$ is a *module over* $\mathcal{F}(U)$.

A differential form $\omega \in \Omega^p(U)$ induces a $\mathcal{F}(U)$-multilinear map, denoted by the same symbol,

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U), \quad \omega(X_1, \ldots, X_p)(x) = \omega(x)(X_1(x), \ldots, X_p(x)).$$

Conversely, we have

2.13. Theorem. *A* $\mathbb{R}$*-multilinear map*

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U),$$

is induced by a differential form if and only if it is $\mathcal{F}(U)$*-multilinear.*

Proof. Clearly, if ω is induced by a form, it is $\mathcal{F}(U)$-multilinear. Suppose that ω is $\mathcal{F}(U)$-multilinear. Let $x \in U, X_i \in T_x U$. Extend the X_i's to vector fields $\tilde{X}_i \in \mathcal{H}(U)$, $\tilde{X}_i(y) = \sum_j a_{ij}(y) e_j$, and define:

$$\omega(x)(X_1, \ldots, X_p) := \omega(\tilde{X}_1, \ldots, \tilde{X}_p)(x).$$

In order to show that the above equality defines a form it is sufficient to show that it does not depend on the extensions. In fact, by $\mathcal{F}(U)$-multilinearity,

$$\omega(\tilde{X}_1, \ldots, \tilde{X}_p)(x) = \sum_{i_1, \ldots, i_p = 1}^{n} a_{1i_1}(x) \cdots a_{pi_p}(x) \omega(e_{i_1}, \ldots, e_{i_p}).$$

□

2.14. Example. Since $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$, $\Omega^0(U) = \mathcal{F}(U)$.

The basic example of a differential form is the following. Let $f \in \mathcal{F}(U)$. Then the differential of f is the the 1-form

$$(\mathrm{d}f)(x)(X) := X(x)(f), \qquad X \in \mathcal{D}er(U).$$

In particular, we can consider the coordinate functions $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. At each point $x \in U$, the differentials at x, $\mathrm{d}x_i(x)$ [9] are a basis of $\Lambda^1(\mathbb{R}^n)$. Therefore $\{\mathrm{d}x_{i_1}(x) \wedge \cdots \wedge \mathrm{d}x_{i_p}(x) : \ 1 \leq i_i < \cdots < i_p \leq n\}$ is a basis of $\Lambda^p(\mathbb{R}^n)$. So we have

2.15. Proposition. *Let $\omega \in \Omega^p(U)$. Then ω can be written in a unique way as:*

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1,\ldots,i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p},$$

where $\omega_{i_1,\ldots,i_p} \in \mathcal{F}(U)$.

2.16. Example. If $f \in \mathcal{F}(U)$, $\ \mathrm{d}f = \displaystyle\sum_1^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i$.

2.17. Remark. As a real vector space, $\Omega^p(U)$ is infinite dimensional (if $n > 0$!), but as a $\mathcal{F}(U)$-module, it is a free module of dimension $\binom{n}{p}$.

Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth function, $F(x) = (F_1(x), \ldots, F_m(x))$. Then $\mathrm{d}F(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map and we have an induced map $F^* : \Lambda^p(\mathbb{R}^m) \longrightarrow \Lambda^p(\mathbb{R}^n)$. This map induces a linear map:

$$F^* : \Omega^p(V) \longrightarrow \Omega^p(U), \qquad F^*(\omega)(X_1, \ldots, X_p)(x) := \omega(\mathrm{d}F(x)(X_1), \ldots, \mathrm{d}F(x)(X_p)).$$

If $x_1, \ldots, x_n$, $y_1, \ldots y_m$ are the canonical coordinates in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, we have

$$F^*(\mathrm{d}y_i) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j, \tag{1}$$

and therefore, if $\omega = \displaystyle\sum_{i_1,\ldots,i_p} \omega_{i_1,\ldots,i_p} \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_p}$,

$$F^*(\omega)(x) = \sum_{i_1,\ldots,i_p} \omega_{i_1,\ldots,i_p}(F(x)) F^*(\mathrm{d}y_{i_1}) \wedge \ldots \wedge F^*(\mathrm{d}y_{i_1}).$$

We have the *functorial properties*:

- $\mathbb{1}_U^* = \mathbb{1}_{\Omega^p(U)}$,
- If $F_1 : U_1 \longrightarrow U_2$ e $F_2 : U_2 \longrightarrow U_3$ are smooth maps, $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism.

2.18. Example. Let $U \subseteq \mathbb{R}^n$ and $j : U \longrightarrow U \times \mathbb{R}^m, j(x_1 \ldots, x_n) = (x_1 \ldots, x_n, 0 \ldots, 0)$, be the inclusion. If $\omega = f(x_1, \ldots, x_{n+m}) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, i_1 < \cdots < i_p$, $\ j^*\omega = 0$, if $i_p > n$, and $j^*\omega = f(x_1, \ldots x_n, 0, \ldots, 0) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$ is $i_p \leq n$.

---

[9] Since $\mathrm{d}x_i = x_i, \mathrm{d}x_i$ is the form that associate to a vector its i^{th} coordinate, in the canonical basis.

3. The de Rham cohomology

Differentiating a function can be viewed as a $\mathbb{R}$-linear map:

$$\mathrm{d} : \Omega^0(U) = \mathcal{F}(U) \longrightarrow \Omega^1(U).$$

We will extend now this operation to higher dimensional forms.

3.1. Theorem. *There exists a unique family of* $\mathbb{R}$ *linear operators* $\mathrm{d}^p : \Omega^p(U) \longrightarrow \Omega^{p+1}(U)$, $p = 0, \dots, n$, *such that:*

1. $\mathrm{d}^0 = \mathrm{d}$ *(the usual differential).*
2. $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0.$
3. *If* $\omega \in \Omega^p(U),\ \tau \in \Omega^q(U), \mathrm{d}^{p+q}\omega \wedge \tau = \mathrm{d}^p\omega \wedge \tau + (-1)^p \omega \wedge \mathrm{d}^q\tau.$

Moreover, if $F : U \longrightarrow V$ *is a smooth map and* $\omega \in \Omega^p(V)$, $\mathrm{d}^p F^*\omega = F^*\mathrm{d}^p\omega$.

When clear from the context we will write simply d *for* d^p.

Proof. Let us suppose that such a family exists. If $\omega = f(x)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$, we have:

$$\mathrm{d}\omega = (\mathrm{d}f) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} + f\,\mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}).$$

Now, from (1), $\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$, and, from (2) and (3)

$$\mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) = \sum_{i_1 < \cdots < i_p} \pm \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}\mathrm{d}x_{i_j} \wedge \cdots \wedge \mathrm{d}x_{i+p} = 0.$$

Therefore, if $\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1 \dots i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$,

$$\mathrm{d}\omega = \sum_k \sum_{i_1 < \cdots < i_p} \frac{\partial \omega_{i_1 \dots i_p}}{\partial x_k}\,\mathrm{d}x_k \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

This shows that if such a family exist, it is unique. Conversely, if we define d^p by the formula above we obtain a family of operators that, as it is easily seen, has the desired properties.

The last claim follows from

$$F^*(\mathrm{d}y_i) = \sum_j \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j = \mathrm{d}(y_i \circ F) = \mathrm{d}(F^*(y_i))$$

and the fact that F^* is an algebras morphism. ☐

The operator d is called the *de Rham differential* or *exterior differential* or simply the *differential.*

3.2. Remark. The following facts are useful and easy to verify:

1. d is a local operator, i.e. if $\omega \equiv \tau$ in an open set U, then $\mathrm{d}\omega = \mathrm{d}\tau$ in U.
2. d may be defined, without the use of coordinates, by the formula:

$$\mathrm{d}\omega(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \dots, \hat{X}_i, \dots X_p) + \sum_{i<j} (-i)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).$$

It is easily seen that the expression on the right hand side of (2) is $\mathcal{F}(U)$-multilinear and so, by Theorem 2.13, it is a differential form (see Exercise 9.25).

So we have a sequence of vector spaces and $\mathbb{R}$-linear maps:

$$0 \longrightarrow \Omega^0(U) \xrightarrow{\mathrm{d}^0} \Omega^1(U) \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{n-1}} \Omega^n(U) \longrightarrow 0$$

which is a *cochain complex*, i.e. $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$, or, equivalently, $\operatorname{Im} \mathrm{d}^{p-1} \subseteq \ker\ \mathrm{d}^p$ (see next section the definition and basic properties of cochain complexes). This sequence is called the *de Rham complex of U*. We define

- $Z^p(U) := \ker\ \mathrm{d}^p$, *the space of p-cocycles* or *closed p-forms.*
- $B^p(U) := \operatorname{Im}\ \mathrm{d}^{p-1}$, *the space p-coboundaries* or *exact p-forms.*
- $H^p(U) := Z^p(U)/B^p(U)$, *the p-dimensional (de Rham) cohomology of U.*

Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth function. As we already observed, F induces a map $F^* : \Omega^p(V) \longrightarrow \Omega^p(U)$. Since, by Theorem 3.1, $F^* \circ \mathrm{d} = \mathrm{d} \circ F^*$, F^* maps closed forms to closed form and exact forms to exact forms. Therefore it induces a $\mathbb{R}$-linear map, that we will still denote by F^*:

$$F^* : H^p(V) \longrightarrow H^p(U).$$

The basic functorial properties are easily verified:

- $\mathbb{1}_U^* = \mathbb{1}_{H^p(U)}$,
- If $F_1 : U_1 \longrightarrow U_2$ and $F_2 : U_2 \longrightarrow U_3$ are smooth maps, then $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism. So the de Rham cohomology is a (differential) topological invariant of U.

4. Algebraic aspects of cohomology

The construction of the de Rham cohomology fits into a general algebraic setting called *homological algebra.* In this section we will discuss some elementary facts that will be used in these notes. For simplicity we will restrict to the case of real vector spaces (not necessarily finite dimensional) although most of the matter could be extended to the case of modules over commutative rings (see Remarks 4.9 and 4.20).

The objects we will study are sequences of (real) vector spaces and linear maps of the type

$$\mathcal{E} := \{(\mathbb{E}^p, \mathrm{d}^p) : \mathrm{d}^p : \mathbb{E}^p \longrightarrow \mathbb{E}^{p+1}\}.$$

When we introduce "objects" it is a good strategy to introduce "morphisms" between such objects, i.e. maps that preserves the structure of the objects.

4.1. Definition. A *morphism* $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, between two sequences is a sequence of linear maps $\phi_p : \mathbb{E}^p \longrightarrow \mathbb{F}^p$ such that the diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{E}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\phi_p"] & \mathbb{E}^{p+1} \arrow[r] \arrow[d, "\phi_{p+1}"] & \cdots \\
\cdots \arrow[r] & \mathbb{F}^p \arrow[r, "\mathrm{d}^p"] & \mathbb{F}^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

commute, i.e. $\mathrm{d}^p \circ \phi_p = \phi_{p+1} \circ \mathrm{d}^p$ (we are using the same symbols d^p for the linear maps in the two sequences).

The morphism is an *isomorphism* if all ϕ_p are vector spaces isomorphisms.

We have some special sequences.

4.2. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *exact at* $\mathbb{E}^p$ if $\operatorname{Im} \mathrm{d}^{p-1} = \ker \mathrm{d}^p$. The sequence is an *exact sequence* if it is exact at all $\mathbb{E}^p$.

4.3. Examples.

(1) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F}$ is exact at $\mathbb{E}$, if and only if ϕ is injective.
(2) A sequence of the type $\mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact at $\mathbb{F}$ if and only if ϕ is surjective.
(3) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact if and only if ϕ is an isomorphism.

4.4. Definition. A sequence of the type:

$$\{0\} \longrightarrow \mathbb{E} \longrightarrow \mathbb{F} \longrightarrow \mathbb{G} \longrightarrow \{0\}$$

is called a short sequence.

4.5. Proposition. *A short* exact *sequence*

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \xrightarrow{\psi} \mathbb{G} \longrightarrow \{0\}$$

is isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{i} \mathbb{E} \oplus \mathbb{G} \xrightarrow{\pi} \mathbb{G} \longrightarrow \{0\},$$

where $i(v) = (v, 0)$ *and* $\pi(v, w) = w$.

Proof. Let $\tilde{\mathbb{G}}$ be a complement[10] of $\operatorname{Im} \phi = \ker \psi$, i.e $\mathbb{F} = \varphi(\mathbb{E}) \oplus \tilde{\mathbb{G}}$. The map $\psi \mid_{\tilde{\mathbb{G}}} \colon \tilde{\mathbb{G}} \longrightarrow \mathbb{G}$ is an isomorphism. Therefore the map $k \colon \mathbb{F} \longrightarrow \mathbb{E} \oplus \mathbb{G}$, $k(v + w) = (\varphi^{-1}(v), \psi(w))$ $(v \in \varphi(\mathbb{E}), w \in \tilde{\mathbb{G}})$ is the required isomorphism. ☐

The following result appears often in the applications

4.6. Lemma. [The five Lemma] *Consider the diagram:*

$$\begin{tikzcd}
\mathbb{E}_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \mathbb{E}_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & \mathbb{E}_3 \arrow[r, "f_3"] \arrow[d, "\phi_3"] & \mathbb{E}_4 \arrow[r, "f_4"] \arrow[d, "\phi_4"] & \mathbb{E}_5 \arrow[d, "\phi_5"] \\
\mathbb{F}_1 \arrow[r, "g_1"] & \mathbb{F}_2 \arrow[r, "g_2"] & \mathbb{F}_3 \arrow[r, "g_3"] & \mathbb{F}_4 \arrow[r, "g_4"] & \mathbb{F}_5
\end{tikzcd}$$

If the squares commute, the lines are exact and the ϕ_i*'s are isomorphisms for* $i = 1, 2, 4, 5$ *then* ϕ_3 *is an isomorphism.*

Proof. Suppose $\phi_3(e_3) = 0$. Then $\phi_4(f_3(e_3)) = g_3(\phi_3(e_3)) = 0$. Therefore $f_3(e_3) = 0$ and, by exactness of the first line, $e_3 = f_2(e_2)$. Now $g_2(\phi_2(e(2)) = \phi_3(e_3) = 0$, and therefore $\phi_2(e_2) = g_1(\mu_1)$, for some $\mu_1 \in \mathbb{F}_1$, by exactness of the second line. Since ϕ_1 is surjective, there exists $e_1 \in \mathbb{E}_1$ such that $\phi_1(e_1) = \mu_1$. Finally

$$0 = f_2(f_1(e_1)) = f_2(\phi_2^{-1} g_1 \phi_1(e_1)) = f_2(e_2) = e_3$$

and therefore ϕ_3 is injective. We will show now that ϕ_3 is surjective. Let $\mu_3 \in \mathbb{F}_3$, $\mu_4 = g_3(\mu_3)$ and $e_4 \in \phi_4^{-1}(\mu_4)$. Now $\phi_5(f_4(e_4)) = g_4(\mu_4) = 0$ and therefore $f_4(e_4) = 0$, since ϕ_5 is injective. In particular there exists $e_3 \in \mathbb{E}_3$ such that $f_3(e_3) = e_4$. Let $\overline{\mu}_3 = \phi_3(e_3)$ and $\omega = \mu_3 - \overline{\mu}_3$. Now $g_3(\omega) = 0$ and

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[10] Recall that a complement of a subspace is obtained starting from a basis $\{e_\alpha\}$ of the subspace and completing it to a basis of the ambient space with elements $\{f_\beta\}$ and considering the subspace spanned by the $\{f_\beta\}$.

therefore $\omega = g_2(\mu_2)$. Let $e_2 = \phi_2^{-1}(\mu_2)$. We have $\phi_3(f_2(e_2)) = g_2(\phi_2(e_2)) = \omega = \phi(e_3) - \mu_3$ and therefore $\mu_3 = \phi_3(e_3 - f_2(e_2)) \in \operatorname{Im} \phi_3$.

□

4.7. Remark. We observe that in the proof of Theorem 4.6 we use only that $\phi_2,\ \phi_4$ are isomorphisms, ϕ_1 is surjective and ϕ_5 is injective. However, in general, the lemma is used as it is stated.

A more general and very important class of sequences is the class of cochain complexes.

4.8. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *semiexact* or a *cochain complex* if $\operatorname{Im} \mathrm{d}^{p-1} \subseteq \ker \mathrm{d}^p, \quad \forall p$. Equivalently, it is a cochain complex if $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$.

If $\mathcal{E}$ is a cochain complex we define:

- $Z^p(\mathcal{E}) := \ker \mathrm{d}^p$, the *group of p-dimensional cocycles*,
- $B^p(\mathcal{E}) := \operatorname{Im} d^{p-1}$, the *group of p-dimensional coboundaries*,
- $H^p(\mathcal{E}) := Z^p(\mathcal{E})/B^p(\mathcal{E})$, the *p-dimensional cohomology group.*

4.9. Remark. Naturally $Z^p(\mathcal{E}),\ B^p(\mathcal{E}),\ H^p(\mathcal{E})$ are vector spaces. The use of the term “group” is due to the fact that they can be defined in the more general context of complexes of Abelian groups, or modules over a commutative ring.

The cohomology gives a measure of how much the complex is not an exact sequence.

4.10. Example. The de Rham complex $\cdots \longrightarrow \Omega^p(U) \xrightarrow{\mathrm{d}^p} \Omega^{(p+1)}(U) \longrightarrow \cdots$ is a cochains complex whose cohomology is the de Rham cohomology $H^p(U)$.

Consider now a morphism between two cochain complexes, $\phi : \mathcal{E} \longrightarrow \mathcal{F}$. The commutativity condition implies that cocycles are sent to cocycles and coboundaries to coboudaries. In particular ϕ induces linear maps

$$\phi_p^* : H^p(\mathcal{E}) \longrightarrow H^p(\mathcal{F}).$$

When clear from the context we will write simply ϕ^*.

The following “functorial” properties are easily verified:

- $\mathbb{1}^* = \mathbb{1}$,
- $(\phi \circ \psi)^* = \phi^* \circ \psi^*$,

It is convenient to consider also sequences with “decreasing indexes”, i.e. a sequence of the type:

$$\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}.$$

If such a sequence is semiexact, we will call it a *chain complex.* For such a chain complex we define:

- $Z_p(\mathcal{E}) := \ker \partial_p$, the *group of p-dimensional cycles.*
- $B_p(\mathcal{E}) := \operatorname{Im} \partial_{p+1}$, the *group of p-dimensional boundaries.*
- $H_p(\mathcal{E}) := Z_p(\mathcal{E})/B_p(\mathcal{E})$, the *p-dimensional homology group.*

As in the case of cochains, a morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, between two chain complexes sends cycles to cycles and boundaries to boundaries, so it induces a sequence of maps $\phi_{*,p} : H_p(\mathcal{E}) \longrightarrow H_p(\mathcal{F})$ and the functorial properties are easily verified. When clear from the context we will write simply ϕ_*.

4.11. Remark. Naturally chain and cochain complexes are, essentially, the same objects. For example, changing the index p by $-p$ we pass from a chain complex to a cochain complex. But a more interesting approach is *duality* and we will discuss this now.

Let $\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}$ be a chain complex. We define the *dual complex* $\mathcal{E}^* = \{(\mathbb{E}^p, \mathrm{d}^p)\}$ where $\mathbb{E}^p := (\mathbb{E}_p)^*$ is the dual space, and $\mathrm{d}^p = (\partial_p)^*$ is the transpose of ∂_p. It is simple to show that $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$ so $\mathcal{E}^*$ is, in fact, a cochain complex. We will denote with H_p (resp. H^p) the homology of $\mathcal{E}$ (resp. the cohomology of $\mathcal{E}^*$). Consider the bi-linear map

$$b : \mathbb{E}^p \times \mathbb{E}_p \longrightarrow \mathbb{R}, \quad b(\phi, c) = \phi(c).$$

It is easily seen that this map induces a bi-linear map

$$\tilde{b} : H^p \times H_p \longrightarrow \mathbb{R}, \quad \tilde{b}([\phi], [c]) = \phi(c),$$

and therefore a linear map

$$K : H^p \longrightarrow [H_p]^*, \quad K([\phi])([c]) = \phi(c).$$

4.12. Theorem. *The map K is an isomorphism.*

Proof. We start observing that we have two short exact sequences

$$\{0\} \longrightarrow Z_p \longrightarrow \mathbb{E}_p \xrightarrow{\partial_p} B_{p-1} \longrightarrow \{0\}, \quad \{0\} \longrightarrow B_{p-1} \longrightarrow Z_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\} \tag{2}$$

where the non labeled maps are the obvious ones. By Proposition 4.5, we have the decompositions

$$\mathbb{E}_p \cong Z_p \oplus B_{p-1}, \quad Z_{p-1} \cong B_{p-1} \oplus H_{p-1} \tag{3}$$

Claim: *K is surjective.* Let $[\phi] \in [H_p]^*$. Consider the map $\phi \circ \pi : Z_p \longrightarrow \mathbb{R}$, where $\pi : Z_p \longrightarrow H_p$ is the quotient map. Using the first decomposition in (3), we can extend this map to a map $\tilde{\phi} : \mathbb{E}_p \longrightarrow \mathbb{R}$ with $\tilde{\phi} = 0$ on B_{p-1}. Let $e \in \mathbb{E}_p$. Then $\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial(e)) = 0$, hence $\tilde{\phi}$ is a cocycle and $K([\tilde{\phi}]) = [\phi]$.
Claim: *K is injective.* Let $\psi \in Z^p$ be such that $\psi(c) = 0 \ \forall \ c \in Z_p$. The map $\phi = \psi \circ \partial^{-1} : B_{p-1} \longrightarrow \mathbb{R}$ is well defined since, by the first sequence in (2), the difference of two elements in $\partial^{-1}(B_{p-1})$ is a cycle. Using the decompositions in (3), we can extend ϕ to a map $\tilde{\phi} : E_{p-1} \longrightarrow \mathbb{R}$. Now, $\forall e \in E_p$, we have:

$$\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial e) = \psi \circ \partial^{-1}(\partial e) = \psi(e).$$

Hence $[\psi] = [\mathrm{d}\tilde{\phi}] = 0$. □

We will study now when two maps between cochain (resp. chain) complexes induces the same map in cohomology (resp. homology).

4.13. Definition. An *algebraic homotopy* between two morphisms $\phi, \psi : \mathcal{E} \longrightarrow \mathcal{F}$ of cochain (resp. chain) complexes is a family of maps $K_p : \mathbb{E}^p \longrightarrow \mathbb{F}^{p-1}$ (resp. $K_p : \mathbb{E}_p \longrightarrow \mathbb{F}_{p+1}$), such that:

$$\phi - \psi = \mathrm{d} \circ K + K \circ \mathrm{d} \quad (\text{resp. } \phi - \psi = \partial \circ K + K \circ \partial).$$

If there exists such an algebraic homotopy, we will say the the two morphisms are (algebraically) *homotopic.*

From the very definition of induced morphisms we have:

4.14. Proposition. *Two algebraically homotopic maps induce the same morphism in cohomology (resp. in homology).*

Consider now a short exact sequence of cochain complexes:

$$\{0\} \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow \{0\}.$$

In particular ϕ_i is injective and ψ_i is surjective. In general, at cohomology level, ϕ^* is not injective and ψ^* is not surjective. In any case, we still have a good relation between the cohomology groups of the three complexes.

4.15. Theorem. [Algebraic Mayer-Vietoris Theorem] *In the situation above there exists a family of linear maps $\Delta_p^* : H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$ such that the sequence:*

$$\cdots \longrightarrow H^p(\mathcal{E}) \xrightarrow{\phi^*} H^p(\mathcal{F}) \xrightarrow{\psi^*} H^p(\mathcal{G}) \xrightarrow{\Delta_p^*} H^{p+1}(\mathcal{E}) \longrightarrow \cdots$$

is a (long) exact sequence.

Proof. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{E}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\phi_p"] & \mathbb{E}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\phi_{p+1}"] & \mathbb{E}^{p+2} \arrow[r] \arrow[d, "\phi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{F}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\psi_p"] & \mathbb{F}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\psi_{p+1}"] & \mathbb{F}^{p+2} \arrow[r] \arrow[d, "\psi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{G}^p \arrow[r, "\mathrm{d}^p"] \arrow[d] & \mathbb{G}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d] & \mathbb{G}^{p+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact and the rows are the cochain complexes under consideration. The idea is to construct a map from $\mathbb{G}^p$ to $\mathbb{E}^{p+1}$. A natural choice would be $(\phi_{p+1})^{-1} \circ \mathrm{d}^p \circ \psi_p^{-1}$. The fact is that this map is not well defined. Let see how we can overcome this problem. Consider a cocycle $c \in \mathbb{G}^p$. Since ψ_p is surjective, there exists $b \in \mathbb{F}^i$ such that $c = \psi_p(b)$. The element $\mathrm{d}^p(b) \in \mathbb{F}^{p+1}$ is in $\ker \psi_{p+1}$ since the diagrams commute and c is a cocycle. Since $\ker \psi_{p+1} = \operatorname{Im} \phi_{p+1}$ we have $\mathrm{d}^p(b) = \phi_{p+1}(a)$ for some $a \in \mathbb{E}^{p+1}$ and this a is unique since ϕ_{p+1} is injective. Observe that $\mathrm{d}^{p+1}(a) = 0$, since $\phi_{p+2}(d^{p+1}(a)) = \mathrm{d}^{p+1}(\phi_{p+1}(a)) = \mathrm{d}^{p+1} \circ \mathrm{d}^p(b) = 0$ and ϕ_{i+2} is injective. Therefore a is a cocycle. We define: $\Delta_p^*\colon H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$, $\Delta_p^*([c]) = [a]$. We have to show that $[a]$ is well defined. The first choice we made was $b \in \mathbb{F}^p$. If b' is an other choice, i.e. $\psi^p(b') = \psi^p(b)$, then $b - b' \in \ker \psi_p = \operatorname{Im} \phi_p$. Therefore $b' - b = \phi_p(a')$, for some $a' \in \mathbb{E}^p$, and $b' = b + \phi_p(a')$. So, changing b by $b + \phi_p(a')$, we change a by $a + \mathrm{d}^p(a')$ and this does not change $[a]$. Next we shall show that $[a]$ does not depend on the choice of $c \in [c]$. Consider $c + \mathrm{d}^p(c')$. Since $c' = \psi_{p-1}(\tilde{b})$, for some $\tilde{b} \in \mathbb{F}^{p-1}$, we have $c + \mathrm{d}^{p-1}(c') = c + \mathrm{d}^{p-1}(\psi_{p-1}(\tilde{b})) = c + \psi_p(\mathrm{d}^{p-1}(\tilde{b})) = \psi_p(b + \mathrm{d}^{p-1}(\tilde{b}))$. Therefore b is substituted by $b + \mathrm{d}^{p-1}(\tilde{b})$, and this does not change $\mathrm{d}^p(b)$ and, therefore, $[a]$.

It is easy to see that Δ^*_p is linear. We leave to the reader the task of proving exactness. ☐

4.16. Remark. The map Δ^*_p is well defined in cohomology but *not* at cocycles level.

4.17. Definition. The sequence in Theorem 4.15 is called the (algebraic) *Mayer-Vietoris sequence.* The maps Δ^*_p, often denoted just by Δ^*, are the *Mayer-Vietoris coboundaries.*

4.18. Remark. Naturally we have a similar sequence in homology, associated to a short exact sequence of chain complexes. The similar maps Δ^p_* are called the *Mayer-Vietoris boundaries.* We leave the details to the reader.

An important aspect of the Mayer-Vietoris (co)boundaries is that they are "natural" in the following sense (Exercise 9.19)

4.19. Proposition. *A map between short exact sequences of (co)chain complexes induces a morphism between the associated Mayer-Vietoris exact sequences, i.e. the Mayer-Vietoris (co)boundaries commutes with the induced maps.*

4.20. Remark. As suggested in Remark 4.9, instead of chain and cochain complexes of vector spaces we could consider chain and cochain complexes of Abelian groups (or modules over a commutative ring). Almost all we have done in this section extends to the case of complexes of abelian groups. The "almost" refers to two exceptions:

- Proposition 4.5 does not hold in this more general setting. For example the sequence of abelian groups

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}, \quad \cdot 2(a) := 2a,$$

is a short exact sequence, but it is *not* isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}.$$

A short exact sequence of Abelian groups that verify Proposition 4.5 is called a *split short exact sequence.* A sufficient condition for splitting is given by the following simple fact

4.21. Proposition. *A short exact sequence of Abelian groups*

$$\{0\} \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \{0\}$$

splits if and only if there is a map $r : C \longrightarrow B$ *such that* $\psi \circ r = \mathbb{1}_C$. *This always happens if* C *is free* [11].

- We can consider "duality" in the context Abelian groups. If G is such a group, $G^* := Hom(G, \mathbb{Z})$ is the group of homomorphisms of G in $\mathbb{Z}$. Therefore we can define the dual of a chain complex of Abelian groups. However Theorem 4.12 does not holds in this context. In fact, one of the points in the proof was that the sequence of vector spaces

$$\{0\} \longrightarrow B_{p-1} \longrightarrow \mathbb{Z}_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\}$$

---

[11] A *free Abelian group* G is an Abelian group that admits a basis, i.e. a subset $\mathcal{B} \subseteq G$ such that for any Abelian group H and map $\phi : \mathcal{B} \longrightarrow H$, there exists a homeomorphism $\tilde{\phi} : G \longrightarrow H$, extending ϕ.

splits. As observed above, this is not the case, in general, for short exact sequences of Abelian groups. However, if H_{p-1} is a free Abelian group, then the sequence splits, by Proposition 4.21, and the Theorem holds true. In the general case there is still a relation between the homology of a chain complex of Abelian groups and the cohomology of the dual complex, known as *the Universal Coefficients Theorem.*

5. Basic properties of the de Rham cohomology

The natural problem that cohomology attacks is the problem of (indefinite) integration, i.e. the problem of solving the equation $d\omega = \beta$, for a given $\beta \in \Omega^{p+1}(U)$. A necessary condition for the existence of a solution ω is $d\beta = 0$. In general the problem has two aspects:

- *The local problem*: given $x \in U,\ \beta \in \Omega^{p+1}(U)$ do there exist a neighborhood $V \subseteq U$ of x and a solution $\omega \in \Omega^p(V)$ of the equation $d\omega = \beta|V$? In this case, as we will see, the condition $d\beta = 0$ is also sufficient.
- *The global problem*: given $\beta \in \Omega^{p+1}(U)$, does there exist a solution $\omega \in \Omega^p(U)$ of the equation $d\omega = \beta$? In this case, the condition $d\beta = 0$ is not any more sufficient and the answer will depend on the particular β and/or on the *topology* of U.

We will start with some simple examples.

5.1. Example. For $U = \mathbb{R}^0$ we have:

$$H^p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

5.2. Example. Let $U = \coprod_\alpha U_\alpha$ be the union of disjoint open sets U_α. Then $\Omega^p(U) = \prod_\alpha \Omega^p(U_\alpha)$ (direct product) and the differential preserves the decomposition, i.e. if $\omega = \{\omega_\alpha\},\ d\omega = \{d\omega_\alpha\}$. It follows that:

$$H^p(U) \cong \prod_\alpha H^p(U_\alpha).$$

5.3. Example. Let us analyze the 0-dimensional cohomology. In this case, the only exact 0-form is the zero form so $H^0(U)$ is the space of closed 0-forms, i.e. functions in $\mathcal{F}(U)$ with zero differential. Such a function is locally constant, in particular constant on the connected component of U. It follows that $H^0(U)$ is the direct product of copies of $\mathbb{R}$, as many as the connected components of U.

Let us give a further look at the 0-dimensional cohomology. Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open *connected* sets, and $F : U \longrightarrow V$ a smooth map. As we observe in 5.3, the zero dimensional cohomology of U is the space of constant functions, and the same for V. Given a 0-form $f \in \Omega^0(V) = \mathcal{F}(V),\ \ F^*(f) = f \circ F$ and therefore $F^* : H^0(V) \longrightarrow H^0(U)$ is an isomorphism. Modulo the identification of the zero dimensional cohomology groups with $\mathbb{R}$, $F^* = \mathbb{1} : \mathbb{R} \longrightarrow \mathbb{R}$.

We want to look now at the induced maps in higher dimensional cohomology groups. The question is the following: When two smooth maps $F_i : U \longrightarrow V,\ i = 0, 1$ induce the same morphism in cohomology? We will give a sufficient condition in terms of *homotopy.*

5.4. Definition. Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open sets and $F_i : U \longrightarrow V,\ i = 0, 1$ be smooth functions.

- A *homotopy* between the two functions is a smooth map[12]
$$H : U \times [0,1] \subseteq \mathbb{R}^{n+1} \longrightarrow V,$$
such that $H(x,i) = F_i(x), i = 0,1$.
- We will say that the two functions are *homotopic* if there exist a homotopy between them. In this case we will write $F_0 \sim F_1$.
- We will say that U and V are *homotopy equivalent* if there exist functions $F : U \longrightarrow V,\ G : V \longrightarrow U$, such that $G \circ F \sim \mathbb{1}_U,\quad F \circ G \sim \mathbb{1}_V$.
- We will say that U is *contractible* if U is homotopy equivalent to $\mathbb{R}^0$.

5.5. Remark. Given an homotopy $H : U \times [0,1] \longrightarrow V$, there is a smooth function $\overline{H} : U \times \mathbb{R} \longrightarrow V$, such that $\overline{H}(x,i) = F_i(x), i = 0,1$. In fact, if $\lambda : \mathbb{R} \longrightarrow [0,1]$ is a smooth function such that $\lambda(t) = 0$ if $t \leq 0$, $\lambda(t) = 1$ if $t \geq 1$, just take $\overline{H}(x,t) = H(x,\lambda(t))$ (see Lemma 7.3 for a proof of the existence of such λ).

A homotopy between two functions may be viewed as a curve in the space of smooth maps joining the two functions. Also may be viewed as a "smooth deformation" of one function to the other.

5.6. Theorem. [Homotopy invariance for cohomology] *If $F_i : U \longrightarrow V, i = 0,1$ are two homotopic smooth function, then $F_0^* = F_1^* : H^p(V) \longrightarrow H^p(U)$, for all p.*

Proof. By Remark 5.5 we can suppose that there is a homotopy $H : U \times \mathbb{R} \longrightarrow V$. Let $j_i : U \longrightarrow U \times \mathbb{R},\ \ i = 0,1,\ j_i(x) = (x,i)$, be the canonical inclusions. We claim that it is sufficient to prove that $j_0^* = j_1^*$. In fact, if so, we have:

$$F_0^* = (H \circ j_0)^* = j_0^* \circ H^* = j_1^* \circ H^* = (H \circ j_1)^* = F_1^*.$$

To prove that $j_0^* = j_1^*$ we will construct an algebraic homotopy between j_0^* and j_1^* (at the cochain level, see Definition 4.13 and Proposition 4.14), i.e. an $\mathbb{R}$-linear map $\tilde{H} : \Omega^p(U \times \mathbb{R}) \longrightarrow \Omega^{p-1}(U)$ such that:

$$\tilde{H}\mathrm{d}\omega + \mathrm{d}\tilde{H}\omega = j_1^*\omega - j_0^*\omega. \tag{4}$$

Let us construct such a map. If $\omega \in \Omega^p(U \times \mathbb{R})$, $\omega = \mathrm{d}t \wedge \alpha + \beta$, with:

$$\alpha = \sum_{i_1<\dots<i_{p-1}} \alpha_{i_1,\dots,i_{p-1}}(x,t)\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}},\quad \beta = \sum_{j_1<\cdots<j_p} \beta_{j_1,\dots,j_p}(x,t)\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}.$$

We define:

$$\tilde{H}(\omega) = \sum_{i_1<\dots<i_{p-1}} \left(\int_0^1 \alpha_{i_1,\dots,i_{p-1}}(x,t)\mathrm{d}t\right)\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}}.$$

Then:

$$\begin{aligned}\mathrm{d}\omega = -\mathrm{d}t \wedge \mathrm{d}\alpha + \mathrm{d}\beta = -\mathrm{d}t \wedge \sum_{j,i_1<\cdots<i_p} \frac{\partial \alpha_{i_1\dots i_{p-1}}}{\partial x_j}\, \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}} +\\ +\mathrm{d}t \wedge \sum_{j_1<\cdots<j_p} \frac{\partial \beta_{j_1,\dots,j_p}}{\partial t}\, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} + \gamma\end{aligned}$$

---

[12] A map $f : V \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$, defined in a *non necessarily open subset* $V \subseteq \mathbb{R}^N$ is smooth, if for all $p \in V$, f extends to a smooth map defined in an open neighborhood of p.

where γ does not contain terms with $\mathrm{d}t$. So:

$$\tilde{H}\mathrm{d}\omega = \sum_{j_1<\cdots<j_p}\left(\int_0^1 \frac{\partial\beta_{j_1,\dots,j_p}}{\partial t}\,\mathrm{d}t\right)\mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_p} \quad - \\ \sum_{j,i_1<\cdots<i_p}\left(\int_0^1 \frac{\partial\alpha_{i_1\dots i_{p-1}}}{\partial x_j}\quad \mathrm{d}t\quad\right)\mathrm{d}x_j\wedge\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_{p-1}},$$

$$\mathrm{d}\tilde{H}\omega = \sum_{j,i_1<\cdots<i_p}\left(\int_0^1 \frac{\partial\alpha_{i_1\dots i_{p-1}}}{\partial x_j}\,\mathrm{d}t\right)\mathrm{d}x_j\wedge\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_{p-1}}.$$

Therefore (see also Example 2.18):

$$\tilde{H}\mathrm{d}\omega+\mathrm{d}\tilde{H}\omega = \sum_{j_1<\cdots<j_p}\left(\int_0^1 \frac{\partial\beta_{j_1,\dots,j_p}}{\partial t}\,\mathrm{d}t\right)\mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_p} =$$

$$= \sum_{j_1<\cdots<j_p}[\beta_{j_1,\dots,j_p}(x,1)-\beta_{j_1,\dots,j_p}(x,0)]\mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_p} = j_1^*\omega - j_0^*\omega.$$

□

From 5.6, and the funtorial properties, we have:

5.7. Corollary. *If $U\subseteq\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$ are homotopically equivalent open sets, then they have isomorphic cohomology.*

In particular we have the so called *Poincaré Lemma*:

5.8. Corollary. [Poicaré Lemma] *If U is a star shaped* [13]*open set in $\mathbb{R}^n$, every closed p form, $p\geq 1$, is exact.*

5.9. Remark. Theorem 5.6 allows to define the map induced in cohomology by a *continuous map.* In fact, as we will see in the Appendix, a continuous map $F: U\longrightarrow V$ is homotopic, via a continuous homotopy $H: U\times[0,1]\longrightarrow V$, to a smooth map $\tilde{F}: U\longrightarrow V$ and if there is a continuous homotopy between two smooth maps, there is a smooth one. So $F^* := \tilde{F}^*$ is well defined and invariant by continuous homotopies.

A basic method to compute the cohomology of an open set $U\subseteq\mathbb{R}^n$ is to write U as union of two, possibly simpler open sets U_1, U_2, and look for relations between the cohomology of U, U_i and $V := U_1\cap U_2$.

5.10. Lemma. *Consider the sequence :*

$$\{0\}\longrightarrow\Omega^p(U)\overset{(j_1^*,j_2^*)}{\longrightarrow}\Omega^p(U_1)\oplus\Omega^p(U_2)\overset{(k_1^*-k_2^*)}{\longrightarrow}\Omega^p(V)\longrightarrow\{0\},$$

where $j_i: U_i\longrightarrow U$ and $k_i: V\longrightarrow U_i$ are the inclusions. Then the sequence is a short exact sequence of cochain complexes.

---

[13] A subset $U\subseteq\mathbb{R}^n$ is star shaped if there exists $p\in U$ such that, for all $q\in U$, the segment joining p and q is contained in U. Star shaped subsets are contractible since the map $H(q,t) := tp+(1-t)q$ is a homotopy between $\mathbb{1}_U$ and the constant map $F(q)=p$.

PROOF. Observe that $j_i^*\omega = \omega|_{U_i}$ and, if $(\omega_1,\omega_2) \in \Omega^p(U_1) \oplus \Omega^p(U_2)$, $(k_1^* - k_2^*)(\omega_1,\omega_2) = \omega_1|_V - \omega_2|_V$ (see Example 2.18). So the exactness of the sequence is obvious, except for the surjectivity of $(k_1^* - k_2^*)$. To prove that $(k_1^* - k_2^*)$ is surjective we consider a partition of unity dominated by the covering $\{U_1, U_2\}$, i.e. smooth functions $\phi_i : U \longrightarrow [0,1]$, $i = 1, 2$ such that:

$$\phi_1(x) + \phi_2(x) = 1 \ \ \forall x \in U, \qquad \operatorname{supp}(\phi_i) := \overline{\{x \in U : \phi_i(x) > 0\}} \subseteq U_i$$

(see Theorem 7.2 for a proof of the existence of partitions of unity).

Given $\omega \in \Omega^p(V)$, we define:

$$\omega_i(x) = \begin{cases} \phi_j(x)\omega(x) & \text{if } x \in V \\ 0 & \text{if } x \in U_i \setminus V \end{cases}$$

where $i \neq j$. ω_i is well defined since ϕ_j vanishes outside $\overline{U_j}, j \neq i$. Moreover,

$$(k_1^* - k_2^*)(\omega_1, -\omega_2) = \omega_1|_V + \omega_2|_V = \phi_1\omega + \phi_2\omega = \omega.$$

Therefore $(k_1^* - k_2^*)$ is surjective. ☐

At this point Theorem 4.15 gives:

5.11. THEOREM. [Mayer Vietoris sequence for de Rham cohomology] *There exists a sequence of linear maps* $\Delta_p^* : H^p(V) \longrightarrow H^{p+1}(U)$*, such that the sequence below is exact:*

$$\cdots \longrightarrow H^p(U) \xrightarrow{(j_1^*, j_2^*)} H^p(U_1) \oplus H^p(U_2) \xrightarrow{(k_1^* - k_2^*)} H^p(V) \xrightarrow{\Delta^*} H^{i+1}(U) \longrightarrow \cdots$$

5.12. DEFINITION. The sequence above is called the *Mayer-Vietoris sequence for the de Rham cohomology* and the maps Δ_p^* are called the *Mayer-Vietories coboundaries.*

5.13. EXAMPLE. Let us apply the Mayer-Vietoris sequence to compute the cohomology of $\mathbb{R}^n \setminus \{0\}$. $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to $\Sigma_n := \mathbb{R}^n \setminus \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq \epsilon\}$. Hence the cohomology of the two spaces are isomorphic, by Corollary 5.7. We will compute the cohomology of the latter.

Consider the open sets:

$$U_1 = \{(x_1, \ldots, x_n) \in \Sigma_n : x_n > -\epsilon/2\}, \quad U_2 = \{(x_1, \ldots, x_n) \in \Sigma_n : x_n < \epsilon/2\}.$$

The following facts are easy to prove:

- $\Sigma_n = U_1 \cup U_2$.
- U_i is contractible, $i = 1, 2$.
- $U_1 \cap U_2$ is homotopy equivalent to $\Sigma_{(n-1)}$.

We will proceed by induction on n. If $n = 1$, Σ_1 is the disjoint union of two contractible sets, hence by Corollary 5.7 and Example 5.3 we have:

$$H^p(\Sigma_1) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

Consider $n = 2$. Since Σ_2 and the U_i's are connected, $H^0(\Sigma_2) \cong H^0(U_i) \cong \mathbb{R}$. Consider the Mayer-Vietoris sequence:

$$\begin{aligned}&\{0\} \longrightarrow H^0(\Sigma_2) \longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(\Sigma_1) \longrightarrow H^1(\Sigma_2) \longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow \\ &\longrightarrow \cdots \longrightarrow H^{p-1}(\Sigma_1) \longrightarrow H^p(\Sigma_2) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow \cdots .\end{aligned}$$

The first row reduces to:

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(\Sigma_2) \longrightarrow \{0\}.$$

Hence $H^1(\Sigma_2) \cong \mathbb{R}$. [14] From the second row we get $H^p(\Sigma_2) = \{0\}$ if $p > 1$.

For the general case we work by induction. Suppose $n \geq 3$ and

$$H^p(\Sigma_{n-1}) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-2 \\ \{0\} & \text{if } p \neq 0, n-2 \end{cases}$$

Consider again the Mayer-Vietoris sequence:

$$H^{p-1}(\Sigma_n) \longrightarrow H^{p-1}(U_1) \oplus H^{p-1}(U_2) \longrightarrow H^{p-1}(\Sigma_{n-1}) \longrightarrow H^p(\Sigma_n) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow$$

If $p > 1$ we have $H^p(\Sigma_n) \cong H^{p-1}(\Sigma_{n-1})$, and, for $p = 1$ we get

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(\Sigma_n) \longrightarrow \{0\}.$$

Hence

$$H^p(\Sigma_n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ \{0\} & \text{if } p \neq 0, n-1 \end{cases}$$

5.14. Remark. For further reference, we observe that Σ_n is homotopy equivalent to the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

6. An application: the Jordan-Alexander duality Theorem

It is convenient, as we will see, in order to avoid special arguments for the 0-dimensional case and to have cleaner statements, to introduce the *reduced cohomology.* Define:

$$\Omega^{-1}(U) := \mathbb{R} \qquad \mathrm{d}^{(-1)} : \Omega^{-1}(U) \longrightarrow \Omega^0(U), \qquad \mathrm{d}^{(-1)}(a) := a \in \Omega^0(U).$$

Then the sequence:

$$\{0\} \longrightarrow \Omega^{-1}(U) \xrightarrow{d^{(-1)}} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow \cdots$$

is a cochain compex called the *augmented de Rham complex.*

6.1. Definition. The *reduced de Rham cohomology* of U, $\tilde{H}^p(U)$, is the cohomology of the augmented de Rham complex.

6.2. Remark. It is clear that $\tilde{H}^{-1}(U) = \{0\}$, $H^0(U) \cong \tilde{H}^0(U) \oplus \mathbb{R}$ and $\tilde{H}^p(U) = H^p(U)$, if $p > 0$. In particular $\tilde{H}^p(U) = \{0\}$, $\forall\ p \geq 0$, if U is contractible.

---

[14] The first arrow is injective so the kernel of the second one, as well as the image, are 1-dimensional. Hence the kernel of the third one is also 1-dimensional and the conclusion follows.

The basic properties, as homotopy invariance and the Mayer-Vietoris exact sequence, continue to hold for the reduced cohomology and we will leave the proof to the reader (see Exercise 9.20).

We will discuss now a nice application of the Mayer-Vietoris argument, the so called *Jordan-Alexander duality principle*, that has, as a simple consequence, the celebrated Jordan closed curve Theorem. We will follow closely [**?**] and [**?**].

Let $F_i, \quad i = 1, 2$ be closed subsets of $\mathbb{R}^n$. Suppose that there exists a homeomorphism $\phi : F_1 \longrightarrow F_2$. It is natural to ask if there exists some relation between the complementary sets $\mathbb{R}^n \setminus F_i$. The illusion that they are homeomorphic or, at least, homotopy equivalent is soon frustrated. For example consider $F_1 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x\| = 2\}$ and $F_2 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x - (3, 0)\| = 1\}$. The complement of F_1 is homotopy equivalent to the disjoint union of a point and two circles, while the complement of F_2 is homotopy equivalent to the disjoint union of two points and the wedge [15] of two circles. It is easily seen that those space *are not* homotopy equivalent.

6.3. Remark. The fact that the complements of two homeomorphic closed set are not homotopy equivalent is important in several contexts. For examples in Knot Theory. Recall that a knot in $\mathbb{R}^3$ is a function $\gamma : S^1 \longrightarrow \mathbb{R}^3$ which is an homeomorphism onto its image. Two knots are equivalent if there exists an isotopy, i.e. a homotopy through homeomorphisms, which takes one into the other. One of the most important invariants for equivalence classes of knots is the fundamental group of the complement of the image. Now, the images of two knots are homeomorphic and if the complements would be homotopy equivalent, they would have isomorphic fundamental group and so the invariant would be trivial.

There is, however, an interesting relation between the complements of homeomorphic closed set:

6.4. Theorem. [Jordan Alexander duality Theorem]. *Let $F_i, i = 1, 2$, be closed sets in $\mathbb{R}^n$ and $\phi : F_1 \longrightarrow F_2$ an homeomorphism. Then:*

$$\tilde{H}^k(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^k(\mathbb{R}^n \setminus F_2).$$

Proof. We will consider $\mathbb{R}^n$ as the subspace of vectors in $\mathbb{R}^{n+k}$ with the last k coordinates zero. The proof of the Theorem will be an easy consequence of the following two Lemmas.

6.5. Lemma. *Let $F \subsetneq \mathbb{R}^n$ be a closed subset. Then $\tilde{H}^{i+1}(\mathbb{R}^{n+1} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad i \geq -1$.*

Proof. Consider the subsets of $\mathbb{R}^{n+1}$:

- $Z_+ := \mathbb{R}^{n+1} \setminus F \times \{t \in \mathbb{R} : t \leq 0\}$.
- $Z_- := \mathbb{R}^{n+1} \setminus F \times \{t \in \mathbb{R} : t \geq 0\}$.
- $Z := Z_+ \cup Z_- = \mathbb{R}^{n+1} \setminus F$.
- $Z_+ \cap Z_- \sim \mathbb{R}^n \setminus F$.

The orthogonal projection of Z_+ onto the hyperplane $x_{n+1} = 1$ is an homotopy equivalence. Hence the reduced cohomology of Z_+ vanishes in all dimensions. The same is true for Z_- and the Lemma follows from the Mayer-Vietoris sequence for the reduced cohomology:

$$\tilde{H}^i(Z_+) \oplus \tilde{H}^i(Z_-) = \{0\} \longrightarrow \tilde{H}^i(Z_+ \cap Z_-) \longrightarrow \tilde{H}^{i+1}(Z) \longrightarrow \tilde{H}^{i+1}(Z_+) \oplus \tilde{H}^{i+1}(Z_-) = \{0\}.$$

---

[15] Recall that the wedge of two topological spaces is the space obtained from the disjoint union identifying a fixed point in the first space with one in the second one.

□

6.6. Corollary. *If* $F \subseteq \mathbb{R}^n$ *is a closed set, then* $\tilde{H}^{i+k}(\mathbb{R}^{n+k} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad \forall\, i \geq -k.$

6.7. Lemma. *Let* $F_i \subseteq \mathbb{R}^n, i = 1, 2$ *be closed subsets and* $\phi : F_1 \longrightarrow F_2$ *an homeomorphism. Then* $\mathbb{R}^{2n} \setminus F_1 \times \{0\}$ *is homeomorphic to* $\mathbb{R}^{2n} \setminus \{0\} \times F_2$.

Proof. Let $\psi = \phi^{-1}$. The homeomorphisms ϕ, ψ extend, by Tietze's Theorem, to continuous maps $\Phi, \Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Define:

- $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n},\ L(x, y) = (x, y - \Phi(x))$.
- $R : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n},\ R(x, y) = (x - \Psi(y), y)$.

The maps L, R are homeomorphisms. In fact $L^{-1}(x, y) = (x, y + \Phi(x)), R^{-1}(x, y) = (x + \Psi(y), y)$. Consider $\Gamma := \{(x, y) \in \mathbb{R}^{2n} : x \in F_1,\ y = \phi(x)\} = \{(x, y) \in \mathbb{R}^{2n} : y \in F_2,\ x = \psi(y)\}$. We have $L(F_1 \times \{0\}) = \Gamma = R(\{0\} \times F_2)$ and therefore a homeomorphism:

$$\mathbb{R}^{2n} \setminus F_1 \times \{0\} \xrightarrow{L} \mathbb{R}^{2n} \setminus \Gamma \xrightarrow{R^{-1}} \mathbb{R}^{2n} \setminus \{0\} \times F_2.$$

□

The proof of the Theorem is, at this point, immediate:

$$\tilde{H}^i(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_2) \cong \tilde{H}^i(\mathbb{R}^n \setminus F_2).$$

□

As an immediate consequence of the Jordan-Alexander duality we have get the celebrated Jordan curve Theorem:

6.8. Theorem. [Jordan curve Theorem] *Let* $\gamma : S^1 \longrightarrow \mathbb{R}^2$ *be a homeomorphism onto its image*[16]. *Then* $\mathbb{R}^2 \setminus \gamma(S^1)$ *has exactly two connected components.*

Proof. Consider the unit circle $S^1 \subseteq \mathbb{R}^2$. It is clear that the complement of S^1 in $\mathbb{R}^2$ has exactly two connected components and therefore $\tilde{H}^0(\mathbb{R}^2 \setminus S^1) \cong \mathbb{R}$. By the duality principle $\tilde{H}^0(\mathbb{R}^2 \setminus \gamma(S^1)) \cong \mathbb{R}$ and therefore the complement of $\gamma(S^1)$ in $\mathbb{R}^2$ has also exactly two connected components. □

6.9. Remark. It is clear that the argument in the proof of Theorem 6.8 may be extended to the case of a closed hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ any time we have a "model", i.e. a close hypersurface homeomorphic to M^n and information on the complement of the model. For example this happens in the case of closed oriented surfaces in $\mathbb{R}^3$ or for the case of closed hypersurfaces of $\mathbb{R}^{n+1}$, homeomorfic to a sphere.

7. Appendix A: partitions of unity and smooth approximations of continuous functions

Partitions of unity is a basic tool that allows to glue together locally defined objects (such as functions, forms etc.) to obtain a globally defined one. In this appendix we will prove the existence of partitions of unity and apply the result to to prove that continuous functions may be approximate by smooth ones. We start with the basic definition.

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[16] Such a map is usually called a *Jordan curve.*

7.1. DEFINITION. Let $U \subset \mathbb{R}^n$ be an open set and V_α an open covering of U. A *partition of unity dominated by the covering* V_α is a family of smooth functions $\lambda_i : \mathbb{R}^n \longrightarrow [0,1]$ such that:

1. For all i there exist α such that $\operatorname{supp}(\lambda_i) := \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subseteq V_\alpha$.
2. For all $x \in U$ there exist a neighborhood U_x of x such that $U_x \cap \operatorname{supp}(\lambda_i) = \emptyset$ for all but finitely many of the λ_i's.
3. For $x \in U$, $\sum_i \lambda_i(x) = 1$ (observe that, by (2), the sum is finite).

Our aim is to prove the following result:

7.2. THEOREM. *Let $U \subset \mathbb{R}^n$ be an open set and V_α an open covering of U. Then there exist a partition of unity dominated by V_α.*

PROOF. We will use the following notations:

$$B(p,r) = \{x \in \mathbb{R}^n : \|x\| < r\}, \quad D(x,r) = \{x \in \mathbb{R}^n : \|x\| \leq r\} = \overline{B(p,r)} .$$

We will start with some preliminary results.

7.3. LEMMA. *Given $\delta_1, \delta_2 \in \mathbb{R}$, $0 < \delta_1 < \delta_2$, and $p \in \mathbb{R}^n$, there exist a smooth function $\phi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi(x) = 0$ in $B(p,\delta_1)$ and $\phi(x) = 1$ in $\mathbb{R}^n \setminus B(p,\delta_2)$.*

PROOF. We can suppose, up to a translation, $p = 0$. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

It is easily seen that, at $t = 0$, the left and right derivatives of f, of any order, vanish. So f is a smooth function. The the function

$$\phi(x) = \frac{f(\|x\|^2 - \delta_1^2)}{f(\|x\|^2 - \delta_1^2) + f(\delta_2^2 - \|x\|^2)}$$

is well defined, since the denominator of the right hand side never vanishes, it is smooth since it is a composition of smooth functions, has values in $[0,1]$, vanishes for $\|x\| \leq \delta_1$ and it is identically 1 for $\|x\| \geq \delta_2$. □

7.4. COROLLARY. *Let $K \subseteq \mathbb{R}^n$ be a compact set and $V \subseteq \mathbb{R}^n$ an open set with $K \subseteq V$. Then there exist a smooth function $\psi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\psi(x) = 1$, if $x \in K$ and $\psi(x) = 0$ if $x \notin V$.*

PROOF. For any $p \in K$ consider $\delta(p)$ such that $D(p, 2\delta(p)) \subseteq V$. Then there is a finite number of points, $p_1, \ldots, p_r \in K$, such that $K \subseteq \bigcup D(p_i, \delta(p_i))$. By Corollary 7.4, for each i we have a function $\phi_i : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi_i(x) = 0$, $x \in D(p_i, \delta(p_i))$ and $\phi(y) = 1$, $y \notin D(p_i, 2\delta(p_i))$. Then the function

$$\psi(x) = 1 - \phi_1(x) \cdots \phi_r(x)$$

has the required properties. □

7.5. LEMMA. *There exist a continuous proper function*[17] $\phi : U \longrightarrow [0,\infty)$.

---

[17] A function is *proper* if the inverse image of a compact set is compact.

Proof. Since $\mathbb{R}^n$ is homeomorphic to the open ball $B(0,1)$, we can suppose that U is bounded. For $x \in U$, define $d(x)$ to be the distance of x to the boundary of U. Then $d : U \longrightarrow \mathbb{R}$ is a *positive* continuous function. Consider $\phi : U \longrightarrow [0,\infty)$, $\phi(x) = d(x)^{-1}$. Then ϕ is continuous and for all $n \in \mathbb{N}, \phi^{-1}[0,n]$ is a closed bounded set in U, hence compact. So ϕ is proper. ☐

We will prove now Theorem 7.2. Consider a proper function $\phi : U \longrightarrow [0,\infty)$ and set

$$A_n = \phi^{-1}[n, n+1], \qquad V_n = \phi^{-1}(n - \frac{1}{2}, n + \frac{3}{2}).$$

Then A_n is compact and may be covered with a finite number of balls $B_{k,n}$ such that each disk $D_{k.n} := \overline{B_{k,n}}$ is contained in some $V_\alpha \cap V_n$. For each such disk we have a smooth function $\phi_{k,n} : U \longrightarrow [0,1]$ vanishing outside $V_\alpha \cap V_n$ and identically 1 in $D_{k,n}$. It is clear from the construction that the A_n's cover U and so, for all $x \in U$, there is at least one of the $\phi_{n.k}$'s not vanishing at x. Also $V_n \cap V_{n+2} = \emptyset$ so the supports of the $\phi_{n,k}$ are a locally finite covering and $\sum_{k,n} \phi_{k,n}(x) < \infty$, $\forall x \in U$. So the family of functions

$$\lambda_{n,k} = \frac{\phi_{n,k}}{\sum_{i,j} \phi_{i,j}}$$

is a well defined partition of unity dominated by the covering V_α. ☐

We will prove now that continuous functions may be approximate by smooth functions, a fact that we already mentioned in Remark 5.9. The proof is a good example of how to use partition of unity.

7.6. Theorem. *Let $U \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ be open sets, $F : U \longrightarrow W$ a continuous function and $\epsilon : U \longrightarrow \mathbb{R}$ a continuous, positive function. Suppose that F is smooth on a closed set $A \subseteq U$. Then there exists a smooth function $G : U \longrightarrow W$ such that $\|F(x) - G(x)\| < \epsilon$, $\forall x \in U$ and $F(x) = G(x)$ if $x \in A$. Moreover we can choose such a G such that there exist a homotopy $H : U \times [0,1]$ between F and G, with $H(x,t) = F(x)$, $\forall x \in A$.*

Proof. Let us suppose, first, $W = \mathbb{R}^m$. We recall that F smooth on A means that for all $x \in A$ there exists a neighborhood V_x of x and a smooth extension h_x of $F|_{V_x \cap A}$. For $x \in U$ we consider a neighborhood V_x of x and a function $h_x : V_x \longrightarrow \mathbb{R}^m$ with the following conditions:

1. If $x \in A$, h_x is a smooth extension of $F|_{V_x \cap A}$.
2. If $x \notin A$, $V_x \cap A = \emptyset$ and $h_x(y) = F(x)$, $\forall y \in V_x$.
3. $\forall y \in V_x, \|F(y) - F(x)\| < \frac{\epsilon(x)}{2}$, $\|h_x(y) - F(x)\| < \frac{\epsilon(x)}{2}$, $\|x - y\| < \frac{\epsilon(x)}{2}$.

Consider a smooth partition of unity, λ_α, dominated by the covering V_x. Then $\forall\ \alpha$ there exists $x = x(\alpha)$ with $\mathrm{supp}(\lambda_\alpha) \subseteq V_{x(\alpha)}$. For every α fix such a $x(\alpha)$ and set

$$G(z) = \sum_\alpha \lambda_\alpha(z) h_{x(\alpha)}(z).$$

Then G is a smooth function since in a neighborhood of a point is a finite sum of smooth functions. If $z \in A$, let $\lambda_{\alpha_1}, \dots \lambda_{\alpha_k}$ be the functions of the partition non vanishing at z. Then the $h_{x(\alpha_j)}(z) = F(z)$, by condition (1) and (2) on the covering V_x. Therefore $G(z) = \sum \lambda_{\alpha_j}(z) F(z) = F(z)$ and G is an extension of $F|_A$. In general we have:

$$\|G(y) - F(y)\| \leq \| \sum \lambda_\alpha(y) h_{x(\alpha)}(y) - \sum \lambda_\alpha(y) F(x(\alpha)\| + \| \sum \lambda_\alpha(y) F(x(\alpha)) - \sum \lambda_\alpha(y) F(y)\| \leq$$

$$\leq \sum \lambda_\alpha(y)\|h_{x(\alpha)}(y) - F(x(\alpha))\| + \sum \lambda_\alpha(y)\|F(x(\alpha)) - F(y)\| < \epsilon.$$

Hence G is an ϵ approximation of F.

Finally $H(x,t) = tF(x) + (1-t)G(x)$ is the required homotopy.

If $W \subseteq \mathbb{R}^m$ the same argument works, choosing the V_x with the additional condition that $F(V_x)$ is contained in an open disk contained in W. ☐

7.7. Corollary. *If two smooth maps are homotopic via a continuous homotopy, then they are homotopic via a smooth one.*

8. Appendix B: tensor product of vector spaces

We can take a slightly different approach to tensors and we will discuss this approach now.

8.1. Definition. Let $\mathbb{E}, \mathbb{F}$ be two real vector spaces (not necessarily finite dimensional). Consider the vector space freely generated by $\{(x,y) : x \in \mathbb{E}, y \in \mathbb{F}\}$ and the subspace generated by the elements of the type:

- $(x_1 + x_2, y) - (x_1, y) - (x_2, y), \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2), \quad x_i \in \mathbb{E},\ y_i \in \mathbb{F}.$
- $r(x,y) - (rx, y), \quad r(x,y) - (x, ry), \quad x \in \mathbb{E},\ y \in \mathbb{F},\ r \in \mathbb{R}.$

The quotient space is called the *tensor product* of $\mathbb{E}$ and $\mathbb{F}$ and will be denoted by $\mathbb{E} \otimes \mathbb{F}$. The class of (x,y) in $\mathbb{E} \otimes \mathbb{F}$ will be denoted by $x \otimes y$.

In other words we can think of $\mathbb{E} \otimes \mathbb{F}$ as the space of finite (formal) linear combinations of elements of the type $x \otimes y$ with the "calculus rules"

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$
- $r(x \otimes y) = rx \otimes y = x \otimes ry.$

The following facts are easily verified

8.2. Proposition.

1. $\mathbb{E} \otimes \mathbb{F} \cong \mathbb{F} \otimes \mathbb{E}, \quad \mathbb{E} \otimes \mathbb{R} \cong \mathbb{E}.$
2. $(\mathbb{E} \otimes \mathbb{F}) \otimes \mathbb{P} \cong \mathbb{E} \otimes (\mathbb{F} \otimes \mathbb{P}).$
3. $\mathbb{E} \otimes (\mathbb{F} \oplus \mathbb{P}) \cong \mathbb{E} \otimes \mathbb{F} \oplus \mathbb{E} \otimes \mathbb{P}.$
4. *If* $\{e_i\}, \{f_j\}$ *are bases for* $\mathbb{E}, \mathbb{F}$ *respectively, then* $\{e_i \otimes f_j\}$ *is a basis for* $\mathbb{E} \otimes \mathbb{F}$*. In particular, if* $\mathbb{E}$, $\mathbb{F}$ *are finite dimensional,* $\dim(\mathbb{E} \otimes \mathbb{F}) = \dim(\mathbb{E})\dim(\mathbb{F})$.
5. *If* $\mathbb{E}$ *is finite dimensional,* $\mathbb{E}^* \otimes \mathbb{E}^* \cong \mathbb{E}_2$.

Let $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ the bi-linear extension of $\pi(x,y) = x \otimes y$.

8.3. Proposition. *The following* universal property of the tensor product *holds:*

• (UP$\otimes$) *If* $\mathbb{K}$ *is a vector space and* $b : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{K}$*, is a bilinear map, there exists a unique linear map* $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ *such that* $l \circ \pi = b$.

Proof. Set $l(x \otimes y) = b(x,y)$. By the "calculus rules", l extend to a linear map of $\mathbb{E} \otimes \mathbb{F}$ into $\mathbb{K}$ such that $l \circ \pi = b$. If $l' : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ is a linear map with $l' \circ \pi = b$, then $l'(x \otimes y) = b(x,y) = l(x \otimes y)$. Since the elements of the type $x \otimes y$ spans $\mathbb{E} \otimes \mathbb{F}$, $l = l'$. ☐

Objects defined by *universal properties* are *unique*

8.4. Proposition. *If* $\mathbb{H}$ *is a vector space and* $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ *is a bi-linear map such that* UP$\otimes$ *is verified for* $(\tilde{\pi}, \mathbb{H})$, *then* $\mathbb{H} \cong \mathbb{E} \otimes \mathbb{F}$.

Proof. From the universal property for $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ it follows that there is a unique linear map $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{H}$ such that $l \circ \pi = \tilde{\pi}$. By the universal property of $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ it follows that there is a unique map $l' : \mathbb{H} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ such that $l' \circ \tilde{\pi} = \pi$. Now, $l \circ l' : \mathbb{H} \longrightarrow \mathbb{H}$ is such that $\tilde{\pi} \circ (l \circ l') = \tilde{\pi}$. But also $\tilde{\pi} \circ \mathbb{1} = \tilde{\pi}$. Hence, by uniqueness, $(l \circ l') = \mathbb{1}$. Analogously $l' \circ l = \mathbb{1}$, hence l and l' are inverse isomorphisms. ☐

The important feature of the tensor product is that it allows *to transform a bi-linear problem in a linear one*, which is, in general, easier to solve.

9. Exercises

9.1. Prove that the tensor product of tensors is associative and distributive.

9.2. Prove that the exterior product is distributive with respect to the sum.

9.3. Prove Proposition 1.22.

9.4. Prove that $\phi_1, \dots, \phi_p \in \mathbb{E}^*$ are linearly independent if and only if $\phi_1 \wedge \dots \wedge \phi_p \neq 0$.

9.5. Prove that two sets of linearly independent elements of $\mathbb{E}^*$, $\{\phi_1, \dots, \phi_p\}$ and $\{\psi_1, \dots, \psi_p\}$ span the same subspace of $\mathbb{E}^*$, if and only if $\phi_1 \wedge \dots \wedge \phi_p = d\ \psi_1 \wedge \dots \wedge \psi_p,\ d \in \mathbb{R}$. In this case, d is the determinant of the matrix that gives the change of basis.

9.6. Let $\omega \in \Lambda^*(\mathbb{E}),\ \omega = \sum_0^n \omega_i,\ \omega_i \in \Lambda^i(\mathbb{E})$. Prove that ω is invertible in $\Lambda^*(\mathbb{E})$[18] if and only if $\omega_0 \neq 0$.

9.7. Let $\mathbb{E}$ be a n-dimensional vector space. Let $\pi : \mathbb{E}^* \times \dots \times \mathbb{E}^* \longrightarrow \Lambda^p(\mathbb{E})$ the p-linear extension of $(\phi_1, \dots, \phi_p) \longrightarrow \phi_1 \wedge \dots \wedge \phi_p$. Prove that the following universal property of the exterior algebra holds:

- (UP$\wedge$) If $\mathbb{K}$ is a vector space and $b : \mathbb{E}^* \times \dots \times \mathbb{E}^* \longrightarrow \mathbb{K}$ is an alternated p-linear map, then there exists a unique linear map $l : \Lambda^p(\mathbb{E}) \longrightarrow \mathbb{K}$ such that $l \circ \pi = b$.

9.8. Prove that the universal property (UP$\wedge$) characterizes $\Lambda^p(\mathbb{E})$ i.e., given a vector space $\mathbb{L}$ and a p-linear map $\tilde{\pi} : \mathbb{E}^* \times \dots \times \mathbb{E}^* \longrightarrow \mathbb{L}$ such that $(\tilde{\pi}, \mathbb{L})$ verifies UP$\wedge$, then $\mathbb{L} \cong \Lambda^p(\mathbb{E})$.

9.9. Prove that $\Lambda^p(\mathbb{E}^*) \cong [\Lambda^p(\mathbb{E})]^*$.

9.10. Let $v \in \Lambda^n(\mathbb{E}) \setminus \{0\}$. Define a map:

$$b_v : \Lambda^p(\mathbb{E}) \times \Lambda^{(n-p)}(\mathbb{E}) \longrightarrow \mathbb{R}, \qquad b_v(\omega, \tau)v := \omega \wedge \tau.$$

Prove that b_v is non degenerate and hence defines an isomorphism $\tilde{b}_v : \Lambda^p(\mathbb{E}) \longrightarrow [\Lambda^{(n-p)}(\mathbb{E})]^*$.

9.11. Let $\phi_1, \dots, \phi_r \in \mathbb{E}^*$ be linearly independent. Let $\psi_1, \dots, \psi_r \in \mathbb{E}^*$ be such that $\sum_i \phi_i \wedge \psi_i = 0$. Prove that $\psi_i = \sum_j a_{ij} \phi_j$ with $a_{ij} = a_{ji}$.

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[18] i.e. there exists $\omega^{-1} \in \Lambda^*(\mathbb{E})$ such that $\omega \wedge \omega^{-1} = 1$.

9.12. A form $\omega \in \Lambda^p(\mathbb{E})$ is *decomposable* if $\omega = \phi_1 \wedge \cdots \wedge \phi_p,\ \phi_i \in \mathbb{E}^*$. By Proposition 1.22, any p-form is sum of decomposable forms.

(1) Show that, if $\dim(\mathbb{E}) = n$, any $(n-1)$-form is decomposable.
(2) Show that, if $\dim(\mathbb{E}) = 4$ and $\{\phi_1, \dots, \phi_4\}$ is a basis of $\mathbb{E}^*$, then $\phi_1\wedge\phi_2+\phi_3\wedge\phi_4$ is *not* decomposable.

9.13. Let $\mathbb{E}$ be a n-dimensional vector space. A vector space $G(\mathbb{E})$ with an associative product[19], denoted by $\wedge$, is called a *Grassman algebra for* $\mathbb{E}$ if:

(1) $G(\mathbb{E})$ contains a subspace isomorphic to $\mathbb{R} \oplus \mathbb{E}$ and is generated, as an algebra, by this subspace.
(2) $1 \wedge x = x, x \wedge x = 0,\ \forall x \in \mathbb{E}$,
(3) $\dim(G(\mathbb{E})) = 2^n$.

Prove that $G(\mathbb{E})$ is isomorphic to $\Lambda^*(\mathbb{E}^*)$.

9.14. Let $\phi \in \mathbb{E}^* \setminus \{0\}$ and $\omega \in \Lambda^p(\mathbb{E})$. Show that, if $\phi \wedge \omega = 0$, then there exists $\tau \in \Lambda^{p-1}$ such that $\omega = \phi \wedge \tau$. Conclude that the sequence:

$$\cdots \longrightarrow \Lambda^{p-1}(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^p(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^{p+1}(\mathbb{E}) \longrightarrow \cdots$$

is exact. (Hint: choose a basis containing ϕ.)

9.15. Let $\mathbb{L}$ be a finite dimensional real Lie algebra, i.e. a finite dimensional real vector space with a bi-linear map $[\ ,\ ] : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L},\ \ (X, Y) \longrightarrow [X, Y]$ such that, $\forall\ \ X, Y, Z \in \mathbb{L}$ we have:

(1) $[X, Y] = -[Y, X]$,
(2) $[[X, Y]Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

Define a map $\mathrm{d}^p : \Lambda^p(\mathbb{L}) \longrightarrow \Lambda^{p+1}(\mathbb{L})$,

$$\mathrm{d}^p(\omega)(X_1, \dots, X_{p+1}) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{P+1}).$$

Show, at least for $p = 1$, that $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$.

In particular the sequence above is a cochain complex and its cohomology is called the *cohomology of the Lie algebra* $\mathbb{L}$.

9.16. Let $\mathbb{L}$ be a Lie algebra. $\omega \in \Lambda^p(\mathbb{L})$ is said to be Ad-invariant if, $\forall\ \ Y, X_1, \dots, X_p \in \mathbb{L}$, we have:

$$\sum_i (-1)^{i-1} \omega([Y, X_i], X_1, \dots, \hat{X}_i, \dots, X_p) = 0.$$

(1) Show that if $\omega \in \Lambda^p(\mathbb{L})$ is Ad-invariant, $\mathrm{d}^p\omega = 0$.
(2) Show that *span* $\{[X, Y] : X, Y \in \mathbb{L}\} = \mathbb{L}$ if and only if the only Ad-invariant 1-form is the zero form.
(3) Show that if the only Ad-invariant 1-form is the zero form, the only Ad-invariant 2-form is the zero form.

REMARK: Under suitable hypothesis the cohomology of the Lie algebra is isomorphic to the space of Ad-invariant forms.

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[19] i.e a bilinear map $\wedge : G(\mathbb{E}) \times G(\mathbb{E}) \longrightarrow G(\mathbb{E}), \wedge(v, w) := v \wedge w$ such that $(v \wedge w) \wedge z = v \wedge (w \wedge z)$.

9.17. Let $\mathcal{E} = \{0\} \longrightarrow \mathbb{E}_n \longrightarrow \cdots \longrightarrow \mathbb{E}_0 \longrightarrow \{0\}$ be a chain complex. Assume that the $\mathbb{E}_i$'s are finite dimensional and let H_i be the homology groups of the complex. Prove that:

$$\sum_0^n (-1)^i \dim(\mathbb{E}_i) = \sum_0^n (-1)^i \dim(H_i).$$

The number above is called the *Euler characteristic of the complex.*

9.18. Let $\mathcal{E}, \mathcal{F}$ be chain complexes as in Exercise 9.17, and $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism. Prove that:

$$\sum (-1)^i \mathrm{trace}(\phi_i) = \sum (-1)^i \mathrm{trace}(\phi_{*,i}).$$

The number above is called the *Leftchetz number of* ϕ.

9.19. Show that the (algebraic) Mayer-Vietoris sequence (Theorem 4.15) is exact and the (co)boundaries are natural (Proposition 4.19).

9.20. Show that the Mayer-Vietoris sequence for the reduced cohomology (see Definition 6.2) is exact.

9.21. Consider the algebra $\mathcal{F}_p$ and $\mathcal{I}_p = \{[(f, V)] \in \mathcal{F}_p : f(p) = 0\}$.

(1) Prove that $\mathcal{I}_p$ is an ideal and, in fact, the unique maximal (non trivial) ideal of $\mathcal{F}_p$ (a ring with a unique maximal ideal is called a *local ring*).

(2) Let $\mathcal{I}_p^2$ be the ideal generated by products of elements of $\mathcal{I}_p$. Prove that the quotient $\mathcal{I}_p/\mathcal{I}_p^2$ is isomorphic, as a vector space, to $(\mathbb{R}^n)^*$.

9.22. Prove that the composition of derivations is *not*, in general, a derivation but the commutator (Lie product) of derivations is a derivation (see Proposition 2.9).

9.23. Prove Proposition 2.9.

9.24. Let $U \subseteq \mathbb{R}^n$ be an open set, and

$$X = \sum_k a_k(x) \frac{\partial}{\partial x_k}, \qquad Y = \sum_k b_k(x) \frac{\partial}{\partial x_k}$$

be smooth vector fields in U.

(1) Compute $[X, Y]$ $(:= X \circ Y - Y \circ X)$ in the basis $\dfrac{\partial}{\partial x_k}$.

(2) Let $f : U \longrightarrow V \subseteq \mathbb{R}^m$ be a smooth map, $\tilde{X}, \tilde{Y}$ vector fields in V such that $\mathrm{d}f(x)(X) = \tilde{X}(f(x))$, $\mathrm{d}f(x)(Y) = \tilde{Y}(f(x))$. Prove that $[\tilde{X}, \tilde{Y}](x) = \mathrm{d}f(x)([X, Y])$.

(3) Let $X_1, \ldots, X_p$ be linear independent vectors in $\mathbb{R}^n$. Show that there exist smooth vector fields $\tilde{X}_1, \ldots, \tilde{X}_p$ in $\mathbb{R}^n$ such that, for a fixed $x \in R^n$, $\tilde{X}_i(x) = X_i$ and $[\tilde{X}_i, \tilde{X}_j] = 0$ in $\mathbb{R}^n$.

9.25. (see Remark 3.2) Let U be an open set in $\mathbb{R}^n$ and $\omega \in \Omega^p(U)$. Prove that:

$$\mathrm{d}\omega(X_0, \ldots, X_p) = \sum_{k=0}^p (-1)^k X_i(\omega(X_0, \ldots, \hat{X}_k, \ldots X_p)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

9.26. Let $U \subseteq \mathbb{R}^n$ be an open set and $v = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ be the volume form. We will identify vectors fields and 1-forms via the "musical isomorphisms" $\flat : \mathcal{H}(U) \longrightarrow \Omega^1(U)$ and its inverse $\sharp : \Omega^1(U) \longrightarrow \mathcal{H}(U)$. Also $*$ will denote the Hodge operator. We define the classical differential operator of calculus:

- The *gradient* $\nabla : \mathcal{F}(U) \longrightarrow \mathcal{H}(U)$, $\nabla f := \sharp \mathrm{d}f = \sum \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}$.
- The *divergence* div $: \mathcal{H}(U) \longrightarrow \mathcal{F}(U)$, $\operatorname{div}\left(\sum X_i \frac{\partial}{\partial x_i}\right) = \sum \frac{\partial X_i}{\partial x_i}$.
- The (geometers) *Laplacian* $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, $\Delta f = -\operatorname{div}\nabla f$.
- The *rotational* rot $: \Omega^1(U) \longrightarrow \Omega^{n-2}(U)$ rot $\omega = *\mathrm{d}\omega$.

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Prove that:

(1) $(\Delta f) = -\mathrm{d} * (\mathrm{d}f) = -\sum_1^n \frac{\partial^2 f}{\partial x_i^2}$.
(2) $\Delta(fg) = g\Delta f + f\Delta g - \langle \nabla f, \nabla g\rangle$.
(3) ω is closed if and only if rot $\omega = 0$.
(4) rot $\nabla f = 0$.
(5) If $n = 3$ compute rot $\sum X_i \frac{\partial}{\partial x_i}$ and show that div rot $\omega = 0$.

9.27. Let $U \subseteq \mathbb{R}^n$ be an open set. Show that $H^n(U) = \{0\}$ if and only if $\forall\ f \in \mathcal{F}(U)$ there exists a vector field $X \in \mathcal{H}(U)$ such that $\operatorname{div} X = f$.
Remark: It can be shown that the Laplacian $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is surjective (this a non trivial fact). In particular the equation $\operatorname{div} X = f$ has solutions $\forall\ f \in \mathcal{F}(U)$. In particular $H^n(U) = \{0\}$.

9.28. Identify $\mathbb{R}^2$ with the complex line $\mathbb{C}$, $(x, y) \longrightarrow x + iy, i = \sqrt{-1}$. If $U \subseteq \mathbb{R}^2$ is an open set and $f : U \longrightarrow \mathbb{C}$, we will write $f(z) := f(x, y) = u(x, y) + iv(x, y), u, v \in \mathcal{F}(U)$. f is said to be *holomorphic* if it is C^1 and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(Cauchy-Riemann equations).}$$

It can be shown that an holomorphic function is smooth, and, more than that, complex analytic, i.e. it is locally the sum of its (complex) Taylor series.

(1) Show that the Cauchy-Riemann equations just say that the differential $\mathrm{d}f(z) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{C}$-linear (i.e. commutes with multiplication by $i = \sqrt{-1}$).
(2) Define *complex* 1-forms:

$$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y, \quad f\mathrm{d}z := (u + iv)\mathrm{d}z := (u\mathrm{d}x - v\mathrm{d}y) + i(u\mathrm{d}y + v\mathrm{d}x).$$

and the complex derivative $f'(z)$ by the identity $f'(z)\mathrm{d}z = \mathrm{d}f$. Prove that f is holomorphic if and only if the real and imaginary parts of $\mathrm{d}f$ are closed. In this case $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$.
(3) Prove that if $f = u + iv$ is holomorphic, then $u, v : U \longrightarrow \mathbb{R}$ are harmonic functions (i.e. $\Delta u = \Delta v = 0$).
(4) Show that, if U is star shaped, given an harmonic function $u : U \longrightarrow \mathbb{R}$, there exist an harmonic function $v : U \longrightarrow \mathbb{R}$ such that $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic. The function v is defined up to an additive constant and is called the *harmonic conjugate* of u.

9.29. Use Example 5.13 to prove the *Theorem of invariance of dimension*:
Theorem: If $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, then $n = m$.

CHAPTER 2

Integration and the singular homology of open sets of $\mathbb{R}^n$

In Remark 1.7 of Chapter 1, we observed that p-forms are "p-dimensional (oriented) volume elements" and hence, the natural integrands for the (oriented) multiple integrals. In this Chapter we will make this statement precise, we will introduce the singular homology of open sets in $\mathbb{R}^n$ and see how integration gives a duality between homology and the de Rham cohomology.

1. Integration on singular chains and Stokes Theorem

1.1. DEFINITION. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega = f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(U)$. Let $D \subseteq U$ be the closure of an open bounded set. We define

$$\int_D \omega = \int_D f(x_1, \dots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where the integral on the right hand side is the usual Riemann integral.

1.2. REMARK. The integral defined above is "oriented" in the sense that if $\omega_\sigma = f(x)\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)}$, $\sigma \in \Sigma(n)$, then

$$\int_D \omega = |\sigma| \int_D \omega_\sigma.$$

In particular the integral depends on an ordering the coordinates, i.e., depends the choice of an orientation in R^n, while the usual Riemann integral of a function does not depend on such a choice (see also Exercise 5.2).

In order to define the integral of a p-form, we first define the "domain of integration".

1.3. DEFINITION.

- A *p-simplex in* $\mathbb{R}^n$ is the convex hull[1] of $(p+1)$ points $\{v_0, \dots, v_p\} \subset \mathbb{R}^n$ in general position[2]. The points v_i are called the *vertexes* of the simplex. Any subset of $q+1$ (distinct) vertexes determine a q-simplex called a *face* of the original one.
- Let $\{e_1, \dots, e_p\}$ be the canonical basis of $\mathbb{R}^p$ and $e_0 = 0$. The *standard p-simplex*, $\Delta^p \subset \mathbb{R}^p$ is the simplex with vertexes $\{e_0, e_1, \dots, e_p\}$.
- A *differentiable singular p-simplex in U*, is a smooth map $\sigma : \Delta^p \longrightarrow U$ (i.e. σ extends to a smooth map of an open neighborhood of Δ^p) . When clear from the context we will omit the term differentiable.

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[1] We recall that the convex hull of a subset of $\mathbb{R}^n$ is the smallest convex set that contain the given set.

[2] The points $\{v_0, \dots, v_p\}$ are in general position if they are not contained in any affine subspace of dimension less than p. This is equivalent to the fact that the vectors $\{v_i - v_0 : i = 1, \dots p\}$ are linearly independent.

1.4. Remark. Given a p-simplex with vertexes $\{v_o, \ldots, v_p\}$, a point in the simplex can be written in a unique way in the form $v = \sum_{i=0}^{p} \lambda_i v_i$ with $\lambda_i \in [0,1] \subset \mathbb{R}$ and $\sum_{i=0}^{p} \lambda_i = 1$. The numbers λ_i are the *baricentric coordinate of* v.

1.5. Example. An important example of a singular simplex is the following: Let $\{v_0, \ldots, v_p\}$ be points of $\mathbb{R}^n$, not necessarily in general position. Define $L(v_0, \ldots, v_p)$ as the singular simplex of $\mathbb{R}^n$ that maps the point of Δ^p with baricentric coordinates $\{\lambda_0, \ldots, \lambda_p\}$ to the point $\sum_{i=0}^{p} \lambda_i v_i \in \mathbb{R}^n$. This simplex will be called the *linear simplex with vertexes* $\{v_0, \ldots, v_p\}$.

1.6. Definition. Let $\omega \in \Omega^p(U)$ be a differential p-form and $\sigma : \Delta^p \to U$ a singular p-simplex. Define:

$$\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega,$$

where the integral on the right hand side is in the sense of Definition 1.1.

1.7. Example. If $f \in \mathcal{F}(U)$ is a smooth function, i.e. a 0-form, and $p \in U$ a fixed point, i.e. a 0-simplex, then the integral of the form on the simplex is just $f(p)$.

1.8. Example. If $\omega = \sum \omega_i \mathrm{d}x_i \in \Omega^1(U)$ is a 1-form and $\sigma : \Delta^1 \longrightarrow U$ a smooth 1-simplex, then

$$\sigma^* \omega = \tilde{\omega}(t)\mathrm{d}t, \quad \text{with} \ \ \tilde{\omega}(t) = \sigma^* \omega(t)(\mathrm{d}t) = \omega(\sigma(t))(\mathrm{d}\sigma(t)(1)) = \omega(\sigma(t))(\dot{\sigma}(t)) = \sum_{i=1}^{n} \omega_i(\sigma(t))\dot{\sigma}_i(t),$$

where $\sigma_i(t) = \langle \sigma(t), e_i \rangle$ is the i^{th} coordinate of σ. Hence

$$\int_\sigma \omega = \int_0^1 \left[\sum_{i=1}^{n} \omega_i(\sigma(t))\dot{\sigma}_i(t) \right] \mathrm{d}t.$$

The fundamental result in the elementary integration theory is Stokes Theorem. It relates the integral of an n-form on a domain to the integral of a primitive on the boundary. We will define now the ingredients necessary to state this Theorem.

We will start introducing more general domains of integration for a p-form.

1.9. Definition. A *singular* p*-chain* is a (formal) finite linear combination of singular p-simplexes, with real coefficients.

The set $C_p(U)$ of all such p-chains is a real vector space, with the obvious operations.
If $\omega \in \Omega^p(U), c \in C_p(U),\ c = \sum a_i \sigma_i$, we define the integral of ω on c by:

$$I(c, \omega) := \int_c \omega := \sum a_i \int_{\sigma_i} \omega.$$

Next we have to define the boundary of a p chain. Intuitively, the boundary of a singular simplex will be the restriction of the simplex to the boundary of the standard p-simplex Δ^p (which is a chain and not a simplex). More precisely:

1.10. Definition. The *boundary operator* $\partial_p : C_p(U) \longrightarrow C_{p-1}(U)$ is defined as the linear extension of

$$\partial_p\ \sigma := \sum_0^p (-1)^i \sigma \circ F_i,$$

where σ is a singular p-simplex and $F_i : \Delta^{p-1} \longrightarrow \Delta^p$ is the linear simplex $F_i = L(e_0, \ldots, \hat{e_i}, \ldots, e_p)$.

1.11. Remark. The signs in the definition above guarantee that the $(p-1)$ faces of Δ^p are taken with the *induced orientations.*

1.12. Example. For a linear simplex, we have the formula:

$$\partial_p L(v_0, \dots, v_p) = \sum_{i=1}^{p} (-1)^i L(v_0, \dots, \hat{v_i}, \dots, v_p).$$

In our context we have the following version of the classical Stokes Theorem:

1.13. Theorem. [Stokes Theorem] *If $c \in C_{p+1}(U),\ \omega \in \Omega^p(U)$, then*

$$I(\partial c, \omega) := \int_{\partial c} \omega = \int_c \mathrm{d}\omega := I(c, \mathrm{d}\omega).$$

Proof. By linearity, it is sufficient to prove the Theorem when c is a singular simplex $\sigma : \Delta^{p+1} \longrightarrow U$. In this case

$$\int_\sigma \mathrm{d}\omega = \int_{\Delta^{p+1}} \sigma^* \mathrm{d}\omega = \int_{\Delta^{p+1}} \mathrm{d}\sigma^* \omega$$

(see Theorem 3.1 of Chapter 1 for the last equality). Also

$$\int_{\partial\sigma} \omega = \int_{\partial\Delta^{p+1}} \sigma^* \omega,$$

where $\partial\Delta^{p+1}$ is the linear chain $\sum_{i=0}^{p+1} (-1)^i L(e_0, \dots, \hat{e_i}, \dots e_{p+1}) \in C_p(\Delta^{p+1})$.

Now $\eta := \sigma^*\omega = \sum_i f_i(x_1, \dots, x_{p+1}) \mathrm{d}x_1 \wedge \cdots \hat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_{p+1}$. Again, by linearity, it is sufficient to prove the Theorem for each monomials. Since we can permute coordinate, up to sign, it is not restrictive to assume

$$\eta = f(x_1, \dots, x_{p+1}) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

Then:

$$\mathrm{d}\eta = (-1)^p \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p+1}.$$

Hence, by Fubini Theorem

$$\int_{\Delta^{p+1}} \mathrm{d}\eta = (-1)^p \int_{\Delta^{p+1}} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \cdots \mathrm{d}x_{p+1} = (-1)^p \int_{\Delta^p} \left[\int_0^{1-\sum_i^p x_i} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_{p+1} \right] \mathrm{d}x_1 \cdots \mathrm{d}x_p =$$

$$= (-1)^p \int_{\Delta^p} \left[f(x_1, \dots, x_p, 1 - \sum_{i=1}^{p} x_i) - f(x_1, \dots, x_p, 0) \right] \mathrm{d}x_1 \cdots \mathrm{d}x_p,$$

where Δ^p is the standard simplex $\{e_0, \dots e_p\} \subseteq \mathbb{R}^p \subseteq \mathbb{R}^{p+1}$.

Now $\partial\Delta^{p+1} = L(e_1, \dots e_{p+1}) + (-1)^{p+1} L(e_0, \dots, e_p) + \gamma$ where γ is a chain of linear simplexes that are faces of Δ^{p+1} containing both e_0 and e_{p+1}. Since on each of such faces at list one of the first p coordinates vanishes, $\eta = 0$ on γ. Hence:

$$\int_{\partial\Delta^{p+1}} \eta = \int_{L(e_1, \dots e_{p+1})} \eta + (-1)^{p+1} \int_{L(e_0, \dots e_p)} \eta =$$

$$= (-1)^p \int_{\Delta^p} f(x_1, \dots, x_p, 1 - \sum_{i=1}^{p} x_i) \mathrm{d}x_1 \cdots \mathrm{d}x_p + (-1)^{p+1} \int_{\Delta^p} f(x_1, \dots, x_p, 0) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \int_{\Delta^{p+1}} \mathrm{d}\eta.$$

□

2. Singular homology

We will now look a little deeper at the boundary operator.

2.1. LEMMA. $\partial_{(p-1)} \circ \partial_p = 0$.

PROOF. Let σ be a singular simplex. From (1.12), we have:

$$\partial_p(\sigma) = \sum_i (-1)^i \sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, e_p).$$

Therefore:

$$\partial_{(p-1)}\partial_p(\sigma) = \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \sigma \circ L(e_0, \ldots, \hat{e_j}, \ldots, \hat{e_i}, \ldots, e_p) +$$
$$+ \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{(j-1)} \sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_p).$$

Observe that the term $\sigma \circ L(e_0, \ldots, \hat{e_i}, \ldots, \hat{e_j}, \ldots, e_p)$, i, j fixed, appears twice in the above sum with opposite signs, and therefore $\partial_{(p-1)}\partial_p(\sigma) = 0$.

□

In particular the sequence:

$$\cdots \longrightarrow C_{(p+1)}(U) \overset{\partial_{(p+1)}}{\longrightarrow} C_p(U) \overset{\partial_p}{\longrightarrow} C_{(p-1)}(U) \overset{\partial_{(p-1)}}{\longrightarrow} \cdots,$$

is a chain complex and we define:

- $Z_p(U) := \ker \partial_p$ the group of *p-dimensional cycles.*
- $B_p(U) := \operatorname{Im} \partial_{(p+1)}$ the group of *p-dimensional boundaries.*
- $H_p(U) := Z_p(U)/B_p(U)$ the p^{th} *dimensional (singular) homology group.*

From Stokes Theorem 1.13 we get:

2.2. THEOREM. *If* $a \in Z_p(U)$, $I(a, \mathrm{d}\omega) = 0$. *If* $\sigma \in Z^p(U)$, $I(\partial b, \sigma) = 0$. *Therefore the operator* $I : C_p(U) \times \Omega^p(U) \to \mathbb{R}$ *induces an* $\mathbb{R}$*-bilinear operator:*

$$\tilde{I} : H_p(U) \times H^p(U) \longrightarrow \mathbb{R}, \qquad \tilde{I}([c], [\omega]) := I(c, \omega).$$

2.3. REMARK. The classical Theorem of de Rham, that we will prove later on, states that $\tilde{I}$ is non degenerate and induces an isomorphism:

$$dR : H^p(U) \longrightarrow [H_p(U)]^*,$$

called de *de Rham isomorphism.*

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a smooth map. Then F induces a linear map $F_* : C_p(U) \longrightarrow C_p(V)$, obtained extending by linearity the map which sends a singular simplex $\sigma : \Delta^p \longrightarrow U$ to the singular simplex $F \circ \sigma : \Delta^p \to V$. It is easy to check that F_* commutes with the boundary operator and hence it is a morphism between chain complexes. Therefore it induces a morphism in homology, that we will denote with the same symbol:

$$F_* : H_p(U) \longrightarrow H_p(V).$$

The following *functorial properties* are easily established[3]:

- $(1\!\!1_U)_* = 1\!\!1_{H_p(U)}$,
- $(G \circ F) = G_* \circ F_*$,

We will look now at a few examples that are the analogue, for homology, of Examples 5.2, 5.3 and 5.1 of Chapter 1.

2.4. Example. Let $U = \mathbb{R}^0$. Then there is a unique singular p-simplex, the constant one. His boundary is the alternated sum of $(p+1)$ elements, all equal to the (unique) $(p-1)$-simplex. Therefore the boundary operator is null if p is odd and it is the identity if p is even. The complex of singular chains is given by:

$$\longrightarrow C_{(2p+1)}(U) = \mathbb{R} \xrightarrow{0} C_{2p}(U) = \mathbb{R} \xrightarrow{1\!\!1} C_{(2p-1)}(U) = \mathbb{R} \xrightarrow{0} \cdots \xrightarrow{0} C_0(U) = \mathbb{R} \longrightarrow \{0\}.$$

Therefore:

$$H_p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

2.5. Remark. It could appear more natural and, in fact, some times would be more convenient, to define chains and homology using *singular cubes*, i.e., smooth maps of the unit cube $[0,1]^p \subseteq \mathbb{R}^p$ into U. Since a p-cube has always an even number of $(p-1)$-faces, this construction gives, for $U = \mathbb{R}^0$, a chain complex with p-dimensional chain group $\mathbb{R}$ and null boundary operators. So the homology would be isomorphic to $\mathbb{R}$ in all dimensions, which is not what we would like to have. However if we take the quotient of the complex of singular cubes by a suitable subcomplex, we obtain a new complex whose homology is the same as the homology of the complex of singular simplexes.

2.6. Example. Let $U = \coprod_\alpha U_\alpha$ be the disjoint union of the open sets U_α. Since Δ^p is connected, the image of a singular simplex is contained in some U_α. Therefore $C_p(U) = \bigoplus_\alpha C_p(U_\alpha)$ (direct sum) and the boundaries preserve the decomposition, i.e. if $c = \{c_\alpha\}$, $\partial c = \{\partial c_\alpha\}$. It follows that:

$$H_p(U) \cong \bigoplus_\alpha H_p(U_\alpha).$$

2.7. Remark. We observe explicitly that we are dealing with *finite* linear combinations of simplexes so we have a direct sum instead of a direct product, as in the case of cohomology. Furthermore, this is in agreement with the de Rham Theorem 2.3, since the dual of the direct sum of vector spaces is the direct product of the duals.

2.8. Example. Let us analyze the 0-dimensional homology. Let us suppose first that U is connected. A 0-simplex is a constant map, i.e. a point in U. Such a simplex is a cycle, by definition. On the other hand, given two points in U they may be joined by a smooth curve, i.e. a 1-simplex. The boundary of such simplex is the difference of the two points, so the two points are in the same homology class. It follows that $H_0(U) \cong \mathbb{R}$. Also, as in the case of cohomology, if $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are connected open sets and $F : U \longrightarrow V$ is a smooth map, the induced map $F_* : H_0(U) \longrightarrow H_0(V)$ is an isomorphism.

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[3] This means that the homology is a *covariant* functor from the category of open sets of $\mathbb{R}^n$ and smooth maps into the category of (graded) vector spaces and linear maps.

If U is not connected, lets say with connected components U_α, it follows from Example 2.6 that:

$$H_0(U) \cong \bigoplus_\alpha \mathbb{R}.$$

Next we will prove the *homotopy invariance for homology*:

2.9. THEOREM. *Let $F, G : U \to V$ be homotopic smooth maps. Then $F_* = G_*$.*

PROOF. Let $H : U \times [0,1] \to V$ be a homotopy between F and G. The strategy, analogous to the case of cohomology, is to construct an *algebraic homotopy* between the maps induced at level of chain complexes, i.e. a map $\tilde{H}_p : C_p(U) \longrightarrow C_{(p+1)}(V)$ such that:

$$\partial \circ \tilde{H} + \tilde{H} \circ \partial = G_* - F_*.$$

The theorem will follows since if $c \in Z_p(U)$, $G_*(c) - F_*(c) \in B_p(V)$ i.e., $[G_*(c)] = [F_*(c)]$ in $H_p(V)$.

Consider the product $\Delta^p \times [0,1] \subset \mathbb{R}^{p+2}$. If σ is a singular p-simplex of U, we consider the map $H \circ (\sigma \times \mathbb{1}) : \Delta^p \times [0,1] \longrightarrow V$. The problem is that $\Delta^p \times [0,1]$ is not a simplex. The strategy will be to subdivide $\Delta^p \times [0,1]$ into simplexes and to take a suitable alternated sum of the restrictions of $H \circ (\sigma \times \mathbb{1})$ to such simplexes.

Consider $v_i = (e_i, 0), w_i = (e_i, 1)$, and the linear $(p{+}1)$-simplexes $L(v_0, \dots, v_i, w_i, \dots w_p)$. If $\sigma : \Delta^p \longrightarrow U$ is a singular p-simplex, we define:

$$\tilde{H}(\sigma) = \sum_{i=0}^{p} (-1)^i H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, v_i, w_i, \dots, w_p),$$

and extend by linearity to a morphism $\tilde{H} : C_p(U) \longrightarrow C_{p+1}(V)$. We show now that the map is, in fact, an algebraic homotopy. Using 1.12 and the functorial properties, we get:

$$\begin{aligned}
\partial \tilde{H}(\sigma) &= \sum_{j<i} (-1)^i (-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots w_p) + \\
&+ \sum_{j \geq i} (-1)^i (-1)^{j+1} H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, v_i, w_i, \dots, \hat{w_j}, \dots, w_p), \\
\tilde{H}\partial(\sigma) &= \sum_{j<i} (-1)^{i-1} (-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots w_p) + \\
&+ \sum_{j \geq i} (-1)^i (-1)^j H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, v_i, w_i, \dots, \hat{w_j}, \dots, w_p).
\end{aligned}$$

The terms on the right hand side of the first equation with $i = j$ cancel except for the terms

$$H \circ (\sigma \times \mathbb{1}) \circ L(\hat{v_0}, w_0, \dots, w_p) = G \circ \sigma \quad \text{and} \quad - H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, v_p, \hat{w_p}) = -F \circ \sigma.$$

The rest of the sum is the opposite of the right hand side of the second equation, hence the conclusion. □

From Theorem 2.9 and the funtorial properties we have:

2.10. COROLLARY. *If $F : U \longrightarrow V$ is a homotopy equivalence, then $F_* : H_p(U) \to H_p(V)$ is an isomorphism. In particular, a contractible space has the same homology as $\mathbb{R}^0$ (see Example 2.4).*

2.11. Remark. As in the case of cohomology, the homotopy invariance allows to define the map induced in homology by a continuous map (see Remark 5.9 in Chapter 1).

We also have a Mayer-Vietoris type sequence for homology. Let $U_i \subseteq \mathbb{R}^n, i = 1, 2$ be open sets and define $U = U_1 \cup U_2, V = U_1 \cap U_2$. Consider the sequence of chain complexes (with the obvious boundary maps):

$$\{0\} \longrightarrow C_p(V) \overset{((j_1)_*,(j_2)_*)}{\longrightarrow} C_p(U_1) \oplus C_p(U_2) \overset{((k_1)_*-(k_2)_*)}{\longrightarrow} C_p(U) \longrightarrow \{0\},$$

where $j_i : V \to U_i, k_i : U_i \to U$ are the inclusions.

We wold like to proceed like in the case of cohomology. The problem we have here is that the sequence above is not exact. More precisely, $((k_1)_* - (k_2)_*)$ is not surjective, since a chain in U maight not be the sum of chains in U_i. To overcome this problem, we consider the chain complex $C_p(U_1 + U_2) \subseteq C_p(U)$ spanned by the singular simplexes of U_1 and U_2. Substituting $C_p(U)$ with this complex, we have a short exact sequence of chain complexes. The point that makes this work is the following Theorem (that we will not prove here).

2.12. Theorem. *The inclusion $C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology.*

Using Theorem 2.12 and Theorem 4.15, we have, as for cohomology:

2.13. Theorem. *There are morphisms $\Delta_* : H_p(U) \longrightarrow H_{(p-1)}(V)$ and a long exact sequence:*

$$\cdots \longrightarrow H_p(V) \overset{((j_1)_*,(j_2)_*)}{\longrightarrow} H_p(U_1) \oplus H_p(U_2) \overset{((k_1)_*-(k_2)_*)}{\longrightarrow} H_p(U) \xrightarrow{\Delta_*} H_{(p-1)}(V) \longrightarrow \cdots .$$

The (long) exact sequence above is called the *Mayer-Vietoris sequence in homology* and the maps Δ_p the *Mayer-Vietoris boundaries operators.*

3. The de Rham Theorem for open sets of $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be an open set. As we have seen, integration induces a linear map:

$$dR : H^p(U) \longrightarrow (H_p(U))^*, \quad dR([\omega])([c]) = \int_c \omega.$$

We have already announced that this map is an isomorphism and the aim of this section is to prove this fact. We will start with a Lemma of general character, useful in many situations.

3.1. Lemma. [4] *Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathcal{P}$ a statement about the open subsets $V \subseteq U$. Suppose that:*

1. *$\mathcal{P}$ is true for convex sets,*
2. *If $\mathcal{P}$ is true for disjoint sets, then it is true for their union,*
3. *If $\mathcal{P}$ is true for two sets and for their intersection, then it is true for their union.*

Then $\mathcal{P}$ is true for U.

Proof. First we observe that $\mathcal{P}$ is true for the union of n convex sets. In fact, for $n = 2$ it follows from (3) observing that the intersection of two convex sets is convex. Suppose that $\mathcal{P}$ is true for the union of

---

[4] The lemma is often called the "onion lemma" and the reason will be clear from the proof.

$(n-1)$ convex sets. Let $V_1, \ldots, V_n$ be convex sets and $V = V_1 \cup \ldots \cup V_{(n-1)}$. Then $\mathcal{P}$ is true for V_n and, by the inductive hypothesis, for V. But it is also true for $V \cap V_n$ since

$$V \cap V_n = (V_1 \cap V_n) \cup \ldots \cup (V_{(n-1)} \cap V_n)$$

is union of $(n-1)$ convex sets. From (3), $\mathcal{P}$ is true for the union of all the V_i's.

Let $\phi : U \longrightarrow [0, \infty)$ be a proper function (see Lemma 7.5 in Chapter 1). Define:

$$A_n = \phi^{-1}([n, n+1]).$$

Since ϕ is proper, A_n is compact and we can cover it with a finite number of open convex sets, $U_{\alpha,n}$, contained in $\phi^{-1}\left((n - \frac{1}{2}, n + \frac{3}{2})\right)$. Let $U_n = \cup_\alpha U_{\alpha,n}$. Now $\mathcal{P}$ is true for U_n, since it is a finite union of convex sets. Let us consider $U_{even} = \cup_n U_{2n}$ and $U_{odd} = \cup_n U_{2n+1}$. Then, by (2), $\mathcal{P}$ is true for U_{even} and U_{odd} since each one is disjoint union of sets for which $\mathcal{P}$ is true. Finally $U_{even} \cap U_{odd} = \cup_n U_{\alpha,2n} \cap U_{\beta,2n+1}$ and therefore is disjoint union of sets that are finite union of convex sets. Therefore, by (3), $\mathcal{P}$ is true for $U = U_{even} \cup U_{odd}$. ☐

We can prove now the de Rham Theorem.

3.2. Theorem. *The map $dR : H^p(U) \longrightarrow [H_p(U)]^*$ is an isomorphism.*

Proof. Since we will work with several open sets, it is convenient to denote with dR_V the de Rham map relative to the open set $V \subseteq U \subseteq \mathbb{R}^n$. We are going to use Lemma 3.1. Let us consider the statement:

$$\mathcal{P}(V) = dR_V : H^p(V) \longrightarrow [H_p(V)]^* \text{ is an isomorphism.}$$

Clearly the statement is true for convex sets. In fact they are contractible and we have to check the statement in dimension 0, which is trivial. Also, if it is true for a family of disjoint open sets, it is also true for their union (recall that the dual of the direct sum is the direct product).

Let us suppose that $\mathcal{P}$ is true for the open sets V, W and for $V \cap W$. Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(V \cap W) \arrow[r] \arrow[d, "dR_{V\cap W}"] & H^{p+1}(V \cup W) \arrow[r] \arrow[d, "dR_{V\cup W}"] & H^{p+1}(V) \oplus H^{p+1}(W) \arrow[r] \arrow[d, "dR_V \oplus dR_W"] & \cdots \\
\cdots \arrow[r] & (H_p(V \cap W))^* \arrow[r] & (H_{p+1}(V \cup W))^* \arrow[r] & (H_{p+1}(W))^* \oplus (H_{p+1}(W))^* \arrow[r] & \cdots
\end{tikzcd}$$

where the upper row is the Mayer-Vietoris sequence for cohomology and the lower row is the *dual* of the Mayer-Vietoris sequence in homology. Since integration commutes with induced maps, the diagram above is induced by a cochain complex morphism. In this situation the Mayer-Vietoris (co)boundaries are natural (see Proposition 4.19 of Chapter 1), hence the squares are commutative. Since $dR_{V\cap W}, dR_V \oplus dR_W$ are isomorphisms by hypothesis, it follows from the five Lemma (Lemma 4.6 of Chapter 1) that $dR_{V\cup W}$ is an isomorphism. So $\mathcal{P}$ verifies the hypothesis of Lemma 3.1 and hence $dR = dR_U$ is an isomorphism. ☐

3.3. Remark. Starting with the singular complex $\mathcal{C}(U) = \{C_p(U), \partial_p\}$, we can consider the *dual complex* $\mathcal{C}^*(U) = \{C_p(U)^*, \partial_p^*\}$ (see Remark 4.11 of Chapter 1). The cohomology of $\mathcal{C}^*(U)$ is called the *singular cohomology* of U and is isomorphic, by Theorem 4.12 of Chapter 1, to the dual of the singular homology of U. So the de Rham Theorem states that the singular cohomology and the de Rham cohomology are isomorphic. The de Rham cohomology $H^*(U) = \oplus_{p\geq 0} H^p(U)$ has a natural product, distributive, associative and graded commutative, induced by the exterior product of forms. In the singular cohomology is possible to introduce, by geometric arguments, a product, called the *cup product*, which is distributive, associative and graded commutative. The de Rham Theorem really says that dR is an isomorphism of *algebras.*

3.4. Remark. Singular homology is usually defined starting with *continuous simplexes* i.e., continuous maps $\sigma : \Delta^p \longrightarrow U$ [5]. The singular chain complex $\mathcal{C}^0(U) = \{C_p^0(U), \partial_p\}$ is defined in the obvious way, i.e. the spaces $C_p^0(U)$ are the vector spaces with basis the singular continuous simplexes and the boundary operator is defined just as in the smooth case. The basic properties, such as homotopy invariance and the Mayer-Vietoris exact sequence, are also proved just as in the smooth case. The inclusion $\mathcal{C}(U) \longrightarrow \mathcal{C}^0(U)$ is a morphism of chain complexes, so it induces a map between the homology groups. Using the same arguments as in the proof of the de Rham Theorem, it is easy to prove that actually, the maps induced in homology are isomorphisms.

4. Integration of 1-forms and some applications

Let $U \subseteq \mathbb{R}^n$ be an open set. If $\gamma : [a,b] \longrightarrow U$ is a smooth map, we can consider the smooth 1-simplex $\tilde{\gamma} = \gamma \circ L(a,b)$ where $L(a,b)$ is the linear 1-simplex $L(a,b) : \Delta^1 \longrightarrow [a,b]$, $L(a,b)(t) = (1-t)a + tb$. If $\omega \in \Omega^1(U)$ is a 1-form, we define

$$\int_\gamma \omega := \int_{\tilde{\gamma}} \omega = \int_0^1 \left[\sum \omega_i(\tilde{\gamma}(t))\dot{\tilde{\gamma}}_i(t)\right] \mathrm{d}t = \int_a^b \left[\sum \omega_i(\gamma(t))\dot{\gamma}_i(t)\right] \mathrm{d}t,$$

where the second integral is the integral of ω on the 1-simplex $\tilde{\gamma}$ and the last equality came from the formula of change of variable in 1-dimensional integrals (see also Example 1.8).

For the rest of this section, when clear from the context, *we will make no difference between the curve γ and the 1-simplex $\tilde{\gamma}$.*

Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow U$ be a piecewise smooth curve, i.e. a continuous curve such that there exists a partition $t_0 = a < t_1 < \cdots < t_k = b$ of $[a,b]$ such that $\gamma_i := \gamma|[t_i, t_{i+1}]$ is smooth. Then γ can be viewed as the (smooth) 1-chain $\gamma = \sum \gamma_i$ or a continuous 1-simplex. Clearly, in both cases, $\partial\gamma = \gamma(b) - \gamma(a)$.

Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow U$ be a continuous closed curve, i.e. $\gamma(a) = \gamma(b)$. Consider the map $\pi : [a,b] \longrightarrow S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$, $\pi((1-t)a + tb)) = (\cos 2\pi t, \sin 2\pi t)$. Since γ is closed, $\tilde{\gamma} = \gamma \circ \pi^{-1}$ is a well defined continuous map of S^1 into U. Conversely, any such a map defines a continuous closed curve. From this point of view, continuous closed curves and continuous maps of the circle in U look like to be the same thing. However, there are some difference:

• If γ is a smooth curve $\tilde{\gamma}$ will be just piecewise smooth. It will be smooth if and only if the derivatives of all orders of γ at a, coincide with the derivatives of the corresponding order of γ at b.

• Any curve $\gamma : [a,b] \longrightarrow U$ is homotopic to a constant (see Exercise 5.3). This is not the case for maps of S^1 into U. The following result,whose proof is quite obvious, relates the two situations:

4.1. Lemma. *Let $\tilde{\gamma}_i : S^1 \longrightarrow U,\ i = 0, 1$ be continuous maps and γ_i be the corresponding closed curves. Then $\tilde{\gamma}_0 \sim \tilde{\gamma}_1$ if and only if there is a homotopy $H : [a,b] \times [0,1] \longrightarrow U$ between γ_0 and γ_1 such that $H(a,s) = H(b,s)\ \ \forall s \in [0,1]$.*

---

[5] Here U can be any topological space.

4.2. REMARK. A homotopy like the one in Lemma 4.1 is called a *free homotopy* and the maps $\tilde{\gamma}_i$ are said to be *freely homotopic* The word "free" is to distinguish this concept from the one of *based homotopy*, frequently used in homotopy theory, for example in the definition of the fundamental group.

4.3. REMARK. There is one more way to look at closed curves particularly convenient when we talk about differentiability. In fact a continuous closed curve $\gamma : [0,1] \longrightarrow U$ is the restriction to $[0,1]$ of a continuous function $\overline{\gamma} : \mathbb{R} \longrightarrow U$, $\overline{\gamma}(t) := \gamma(t - [t])$, where $[t]$ is the biggest integer less or equal to t. If γ is piecewise smooth, so is $\overline{\gamma}$. Also, if γ is smooth, $\overline{\gamma}$ is piecewise smooth, and smooth if the derivatives of all orders of γ at 0 coincide with the derivatives of the corresponding order of γ at 1, i.e. if γ is a *smooth* closed curve.

When clear from the context we will make no difference between the three points of view.

Let $\gamma : [a,b] \longrightarrow U$ be a closed piecewise smooth curve. Then, if we think of γ as a smooth 1-chain, $\partial\gamma = 0$, and it determines an element $[\gamma] \in H_1(U)$.

4.4. LEMMA. *If γ_0 and γ_1 are freely homotopic piecewise smooth closed curves, then $[\gamma_0] = [\gamma_1]$ in $H_1(U)$.*

PROOF. Let $H : [a,b] \times [0,1] \longrightarrow U$ be a free homotopy between the two curves. Subdividing $[a,b] \times [0,1]$ into triangles and using linear simplexes as in the proof of homotopy invariance for singular homology (see Theorem 2.9), we get a chain $\tilde{H}$ with $\partial\tilde{H} = \gamma_1 - \gamma_0$. ☐

An other important variant of the concept of homotopy of curves is the following:

4.5. DEFINITION. Let $\gamma_i : [a,b] \longrightarrow U$, $i = 0,1$ be curves such that $\gamma_0(a) = \gamma_1(a)$, $\gamma_0(b) = \gamma_1(b)$. An *endpoints fixing homotopy* between the two curves is an homotopy $H : [a,b] \times [0,1] \longrightarrow U$ such that $H(a,s) = \gamma_0(a)$, $H(b,s) = \gamma_0(b)$, $\forall\ s \in [0,1]$.

If such homotopy exists, we will say that the curves are *homotopic relative to the endpoints.*

The following Proposition follows easily from Stokes Theorem (Exercice 5.11).

4.6. PROPOSITION. *Let $\omega \in \Omega^1(U)$ be a closed 1-form.*

(1) *If γ_i, $i = 0,1$ are freely homotopic piecewise smooth closed curves (resp. curves homotopic relative to the endpoints) then:*
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$
(2) *ω is exact if and only if for all closed curves γ*
$$\int_{\gamma} \omega = 0.$$

4.7. DEFINITION. A connected open set $U \subseteq \mathbb{R}^n$ is *simply connected* if every closed curve is freely homotopic to a constant curve [6].

From Proposition 4.6 we have:

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[6] The concept of simply connectedness is usually defined in terms of vanishing of the fundamental group. In this group, two freely homotopic closed curves are in the same conjugacy class (and conversely), but they may not be the same element of the group. However, the vanishing of the fundamental group is equivalent to the fact that every two closed curves are freely homotopic.

4.8. Corollary. *If U is simply connected, then $H^1(U) = \{0\}$.*

4.9. Remark. A natural question is if $H^1(U) = \{0\}$ implies that U is simply connected. The answer to this question is affirmative for $n = 2$ (see Exercise 5.23) and negative if $n \geq 3$. For example there are, in $\mathbb{R}^3$, (complicated) closed sets, homeomorphic to the 3-dimensional closed disk, whose complement are not simply connected (for example the so called “horned sphere”). The complement of such a disks has, by the Jordan-Alexander duality (see Theorem 6.4 of Chapter 1), the same cohomology of the complement of the standard 3-dimensional disk, hence vanishing first cohomology group (see Example 5.13 of Chapter 1). We do not know of any simpler example in dimension 3. For $n \geq 4$ there are simpler examples.

We will focus now on closed curves in $U = \mathbb{R}^2 \setminus \{0\}$. In U there is a very important 1-form, the *angle form*:

$$\omega = \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y.$$

It is easily seen that $\mathrm{d}\omega = 0$, in fact, locally, $\omega = \mathrm{d}\arctan(\frac{y}{x})$. ω is not exact since, if $\gamma : [0,1] \longrightarrow U$ is the closed curve $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$, we have:

$$\int_\gamma \omega = \int_0^1 2\pi[\sin^2(2\pi t) + \cos^2(2\pi t)]\mathrm{d}t = 2\pi \neq 0.$$

In particular, $dR([\omega])([\gamma]) = 2\pi$. Since $H^1(U) \cong \mathbb{R}$, by Examples 5.13 of Chapter 1, $[\omega]$ spans $H^1(U)$. Also, $[\gamma]$ spans $H_1(U) \cong \mathbb{R}$.

4.10. Definition. Let $\gamma : [0,1] \longrightarrow U$ be a piecewise smooth curve. An *angular function* for γ is a piecewise smooth function $\theta : [0,1] \longrightarrow \mathbb{R}$ such that $\theta(t)$ is one of the determinations, in radians, of the (oriented) angle between e_1 and $\gamma(t)$.

4.11. Lemma. *Any piecewise smooth curve $\gamma : [0,1] \longrightarrow U$ admits angular functions and two angular functions for γ differ by an entire multiple of 2π.*

Proof. Let $\theta_0 \in [0, 2\pi)$ be the angle between e_1 and $\gamma(0)$, and ω the angle form. Define

$$\theta(t) = \int_{\gamma|[0,t]} \omega \ + \ \theta_0.$$

Since, locally, $\omega = \mathrm{d}\arctan(\frac{y}{x})$, θ is an angular function for γ. Finally we observe that two angular functions, at a given time, are determinations of the same angle, so they differ, at that time, by an entire multiple of 2π. This multiple does not depend on the time since the difference of the two angular functions is an integer valued continuous function defined on a connected set, hence constant. □

4.12. Remark. The advantage of having angular functions is that we can write γ in *polar coordinates* :

$$\gamma(t) = \|\gamma(t)\| \ e^{i\theta(t)} = \|\gamma(t)\| \ (\cos\theta(t), \sin\theta(t)).$$

Let $\gamma : [0,1] \longrightarrow U$ be a closed curve and θ an angular function. Since $\gamma(0) = \gamma(1)$, $\theta(1) - \theta(0)$ is an entire multiple of 2π.

4.13. Definition. The *winding number* of γ is the integer:

$$w(\gamma) = \frac{\theta(1) - \theta(0)}{2\pi} \in \mathbb{Z}.$$

4.14. Remark. Since two angular functions differ by a multiple of 2π, the winding number does not depend on the particular angular function. Moreover:

$$w(\gamma) = \frac{1}{2\pi}\int_\gamma \omega.$$

where ω is the angular form.

4.15. Example. Consider the curve $\xi_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$, $t \in [0,1]$, n a given integer. Then $\theta(t) = 2\pi nt$ is an angular function and $w(\xi_n) = n$.

The main fact about winding numbers is the following:

4.16. Theorem. *Two piecewise smooth closed curves* $\gamma_i : [0,1] \longrightarrow U, \quad i = 0,1$, *are freely homotopic if and only if they have the same winding number.*

Proof. If the two curves are freely homotopic, by Proposition 4.6 and Remark 4.14, they have the same winding number. Suppose now that γ is a piecewise smooth closed curve with angular function θ and winding number $w(\gamma) = n \in \mathbb{Z}$. Let ξ_n be as in Example 4.15. Define:

$$H : [0,1]\times[0,1] \longrightarrow U, \quad H(t,s) = [s\|\gamma(t)\| + (1-s)](\cos(s\theta(t) + (1-s)2\pi nt), \sin(s\theta(t) + (1-s)2\pi nt)).$$

Then $H(t,0) = \xi_n(t)$, $H(t,1) = \gamma(t)$ and the condition $w(\gamma) = n$ implies $H(0,s) = H(1,s)$. Hence H is a free homotopy between ξ_n and γ. This concludes the proof since the relation of being freely homotopic is an equivalence relation. ☐

4.17. Remark. By a different argument we could show that any *continuous* curve in U admits *continuous* angular functions. Once we have angular functions, we can define the winding number for a continuous closed curve. Theorem 4.16 holds true in this more general situation (see Exercise 5.14).

4.18. Remark. Geometrically, the winding number of a closed curve in $\mathbb{R}^2 \setminus \{0\}$ is the (algebraic) number of times that the curve goes around zero. We will make this statement more precise. Suppose, for simplicity, that γ is a regular smooth curve, i.e. $\dot\gamma(t) \neq 0, \quad \forall t$. Suppose also that there is a half line $\underline{a} = \{sv : v \in \mathbb{R}^2, \|v\| = 1, s \geq 0\}$, that intersects the curve γ transversally, i.e., if $p = \gamma(t_0) \in \underline{a}$, $\dot\gamma(t_0)$ and v are linearly independent [7]. For an intersection point $p = \gamma(t_0)$, we define $\epsilon(p) = \pm 1$ depending if $\{v, \dot\gamma(t_0)\}$ is a positive or negative basis for $\mathbb{R}^2$. It is known that, in this situation, the number of intersection points is finite. Then the winding number is given by

$$w(\gamma) = \sum_p \epsilon(p),$$

where p runs in the set of intersection points. We will leave the proof of this fact as an exercise (Exercise 5.15).

More generally, given a curve $\gamma : [0,1] \longrightarrow \mathbb{R}^2$, a point $p \in \mathbb{R}^2 \setminus \gamma([0,1])$ and an half line $\underline{\mathbf{a}} = \{p + sv : v \in S^1, \ s \geq 0\}$ we can define angular functions for γ with respect to the pair $(p, \underline{\mathbf{a}})$, and, if γ is closed, the winding number $w(\gamma, p, \underline{\mathbf{a}})$. It is easily seen that this number does not depend on $\underline{\mathbf{a}}$ but it *depends* on p. So we will use the notation $w(\gamma, p)$. The geometric interpretation, for the case of regular closed curves is like in Remark 4.18. One of the main features of this number is the following:

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[7] It is a consequence of Sard Theorem that such v's are dense in S^1.

4.19. Proposition. *If $\gamma : S^1 \longrightarrow \mathbb{R}^2$ is a closed curve and $p, p' \in \mathbb{R}^2$ belongs to the same connected component of $\mathbb{R}^2 \setminus \gamma(S^1)$, then $w(\gamma, p) = w(\gamma, p')$.*

Proof. Suppose first that the segment joining p and p' does not intersect $\gamma(S^1)$ Let $\underline{\mathbf{a}}$ be a half line starting at p, and $\underline{\mathbf{a}}'$ its translated by the vector $p' - p$. Let θ, θ' be angular function for γ in relation to $(p, \underline{\mathbf{a}})$, $(p', \underline{\mathbf{a}}')$ respectively. Denote by w, w' the winding number of γ in relation to p and p' respectively, and set:

$$\Delta(t) := \frac{\theta'(t) - \theta(t)}{\pi}.$$

Now $\Delta(1) - \Delta(0) = 2(w' - w)$. Hence, if $w' \neq w$, $|\Delta(1) - \Delta(0)| \geq 2$ and there exist t^* such that $\Delta(t^*)$ is an odd integer. Then $\gamma(t^*)$ belongs to the segment joining p and p', a contradiction.

For the general case, consider a polygonal in $\mathbb{R}^2 \setminus \gamma(S^1)$, joining p and p', such that the segments between two consecutive vertices do not intersect $\gamma(S^1)$. The the result follows applying the argument above to pairs of consecutive vertices of the polygonal. ☐

Proposition 4.19 suggest the following

4.20. Definition. The *index* of a connected component $\mathcal{C}$ of $\mathbb{R}^2 \setminus \gamma(S^1)$ is the winding number $w(\gamma, p)$, $p \in \mathcal{C}$.

4.21. Remark. It is easily seen that $\mathbb{R}^2 \setminus \gamma(S^1)$ has exactly one unbounded component and the index of such component is zero.

We will give now an alternative proof of the Jordan curve Theorem (see Theorem 6.8 of Chapter 1) in the case of regular curves, to better illustrate the concepts and facts discussed sofar. We will start with some preliminaries.

4.22. Definition. A regular curve $\gamma : [0,1] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ is a smooth curve such that $\underline{\mathbf{t}}(t) := \dot{\gamma}(t) \neq 0 \ \ \forall\, t \in [a, b]$.

Naturally, for a regular closed curve we will mean a smooth periodic curve with non vanishing tangent vector (see Remark 4.3).

We will be interested in the case $n = 2$. In this case there is, $\forall\ t \in [0,1]$, a (unique) unit vector $\underline{\mathbf{n}}(t)$, the *unit normal vector*, orthogonal to $\underline{\mathbf{t}}(t)$ and such that $\{\underline{\mathbf{t}}(t), \underline{\mathbf{n}}(t)\}$ is a positive bases for $\mathbb{R}^2$.

4.23. Theorem. [Tubular neighborhood Theorem] *Let $\gamma : S^1 \longrightarrow \mathbb{R}^2$ be a regular Jordan curve, i.e. γ is smooth, regular and injective. Then there exists $\epsilon > 0$ and a map:*

$$Tub : S^1 \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^2, \qquad Tub(t, 0) = \gamma(t),$$

which is a diffeomorphism onto an open neighborhood U of $\gamma(S^1)$.

Proof. Define the map

$$Tub : S^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2, \ \ Tub(t, s) = \gamma(t) + s\underline{\mathbf{n}}(t).$$

By definition, $Tub(t, 0) = \gamma(t)$. Moreover at a point $(t_0, 0) \in S^1 \times \mathbb{R}$ we have:

$$\frac{\partial Tub}{\partial t}(t_0, 0) = \dot{\gamma}(t), \qquad \frac{\partial Tub}{\partial t}(t_0, 0) = \underline{\mathbf{n}}(t).$$

Therefore $\mathrm{d}Tub(t_0,0)$ is invertible and hence, by the inverse function Theorem, Tub maps a neighborhood $(t_0-\eta,t_0+\eta)\times(-\epsilon(t_0),\epsilon(t_0))$ of $(t_0,0)$ diffeomorfically onto an open neighborhood of $\gamma(t_0)$. Since S^1 is compact, we can cover it (or better $S^1\times\{0\}$) with a finite number of such neighborhoods, say $U_i=(t_i-\eta_i,t_i+\eta_i)\times(-\epsilon(t_i),\epsilon(t_i))$. We claim that there exists $\epsilon>0$ such that $Tub_{|S^1\times(-\epsilon,\epsilon)}$ is injective. Suppose not. Then for all $n\in\mathbb{N}$ there are distinct points $(t_n,s_n),(t'_n,s'_n)\in S^1\times(-\frac{1}{n},\frac{1}{n})$ such that $Tub(t_n,s_n)=Tub(t'_n,s'_n)$. Since those points are in a compact neighborhood of $S^1\times\{0\}$, the two sequences have converging subsequences. Without loss of generality we can suppose that the sequence $\{(t_n,s_n)\}$ converges to $(\bar{t},0)$. The second sequence has a subsequence $\{(t'_{n_k},s'_{n_k})\}$ converging to $(\bar{t}',0)$. So the two sequences $\{(t_{n_k},s_{n_k})\}$ and $\{(t'_{n_k},s'_{n_k})\}$ converge to $(\bar{t},0)$ and $(\bar{t}',0)$ respectively. By continuity, $Tub(\bar{t},0):=\gamma(\bar{t})=\gamma(\bar{t}'):=Tub(\bar{t}',0)$. Since γ is injective, $\bar{t}=\bar{t}'$ (in S^1). Therefore, for n_k sufficiently large, (t_{n_k},s_{n_k}) and (t'_{n_k},s'_{n_k}) are in the same U_i, for some i, and this gives a contradiction since $Tub_{|U_i}$ is injective.

Finally, again by the inverse function Theorem, $U:=Tub(S^1\times(-\epsilon,\epsilon))$ is open and $[Tub_{|U}]^{-1}$ is smooth. □

4.24. DEFINITION. The neighborhood U is called a *tubular neighborhood* of γ.

4.25. THEOREM. [Jordan curve Theorem] *Let $\gamma:S^1\longrightarrow\mathbb{R}^2$ be a regular Jordan curve. Then $\mathbb{R}^2\setminus\{0\}$ has exactly two connected components. Moreover, one of the components is bounded of index ±1 and the other one is unbounded of index zero.*

PROOF. We will start proving that $\mathbb{R}^2\setminus\gamma(S^1)$ has, at most, two connected components. Let U be a tubular neighborhood of γ. Then $U\setminus\gamma(S^1)$ has two connected components, $U_+=Tub(S^1\times(0,\epsilon))$ and $U_-=Tub(S^1\times(-\epsilon,0))$. Let us denote by $G_\pm$ the connected components of $\mathbb{R}^2\setminus\gamma(S^1)$ containing $U_\pm$. Let G be a connected component of $\mathbb{R}^2\setminus\gamma(S^1)$. Take $p\in G$. It will be sufficient to prove that $p\in G_\pm$. If $p\in U$, there is nothing to prove. Suppose $p\notin U$ and let $\sigma:[0,1]\longrightarrow\mathbb{R}^2$ be a curve joining p with a point in $\gamma(S^1)$. Let $t_0=inf\{t\in[0,1]:\gamma(t)\notin U\}$. Then, for η sufficiently small, $\sigma([0,t_0+\eta])\subseteq\mathbb{R}^2\setminus\gamma(S^1)$ and $\sigma(t_0+\eta)\in U$. Let say $\sigma(t_0+\eta)\in U_+$. Then p may be connected, in $\mathbb{R}^2\setminus\gamma(S^1)$, to a point of G_+, so $p\in G_+$ and $G=G_+$.

We will prove now that $\mathbb{R}^2\setminus\gamma(S^1)$ is disconnected. We will give two different arguments.

First argument It is enough to show that there exists a continuous function $\tilde{g}:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that:

- $\tilde{g}$ assumes positive and negative values,
- $\tilde{g}(x)=0$ if and only if $x\in\gamma(S^1)$.

Let U be a tubular neighborhood of γ. We will denote by $\pi:S^1\times(-\epsilon,\epsilon)\longrightarrow\mathbb{R}$ the projection on the second factor, $\pi(t,s)=s$. Then $\pi\circ[Tub]^{-1}:U\longrightarrow\mathbb{R}$ is a function with the two properties above. The problem is that it is not defined in the all $\mathbb{R}^2$, just in U. In order to obtain a function defined on the all $\mathbb{R}^2$ we first modify slightly the function near ∂U and then we will extend the modified function. Let $\lambda:(-\epsilon,\epsilon)\longrightarrow(-\epsilon,\epsilon)$ be a non decreasing smooth function such that $\lambda(s)=s$, if $|s|<\frac{\epsilon}{3}$, $\lambda(s)=\frac{\epsilon}{2}$ if $s>\frac{2}{3}\epsilon$ and $\lambda(s)=-\frac{\epsilon}{2}$ if $s<-\frac{2}{3}\epsilon$. Then the function $f=\lambda\circ\pi\circ[Tub]^{-1}:U\longrightarrow\mathbb{R}$ is again a function with the two properties above and it is locally constant near ∂U. In U we consider the 1-form $\omega=\mathrm{d}f$. Since $\omega=0$ near ∂U, we can extend it to a smooth 1-form on the all of $\mathbb{R}^2$, by setting it identically zero outside U. ω is a closed form, hence exact since $\mathbb{R}^2$ is contractible. Then $\omega=\mathrm{d}g$ where $g:\mathbb{R}^2\longrightarrow\mathbb{R}$ is a smooth function uniquely defined up to an additive constant. So, taking $g = 0$ on a point of $\gamma(S^1)$, we have $g = f$ in U. The function g assumes positive and negative values and, in U, it vanishes exactly on $\gamma(S^1)$. So, all we have to show is that $g(p) \neq 0$ if $p \notin U$. Let $p \notin U$ and $\sigma : [0,1] \longrightarrow \mathbb{R}^2$ be a smooth curve joining p with a point of $\gamma(S^1)$. Let $t_0 = sup\{t : \sigma(s) \notin U,\ \forall\ s < t\}$. Then $\sigma([0,t_0)) \subseteq \mathbb{R}^2 \setminus U,\ \ \sigma(t_0) \in \partial U$ and $g(\sigma(t_0)) \pm \frac{2}{3}\epsilon \neq 0$. But

$$g(\sigma(t_0)) - g(p) = \int_0^{t_0} g'(t)\mathrm{d}t = \int_{\sigma_{|[0,t_0]}} \omega = 0,$$

and hence $g(p) = g(\sigma(t_0)) \neq 0$.

Second argument: Consider the function $h(t) = \|\gamma(t)\|^2$. Let t_0 be a maximum of h. Then $h'(t_0) = 2\langle\gamma(t_0), \dot\gamma(t_0)\rangle = 0$. Then $\gamma(t_0)$ is parallel to $\underline{\mathbf{n}}(t_0)$ and the half line $s\gamma(t_0)$ meet γ transversally at $\gamma(t_0)$. Observe that if $s > 1,\ s\gamma(t_0) \notin \gamma(S^1)$. Let $p = (1-\epsilon)\gamma(t_0),\ q = (1+\epsilon)\gamma(t_0)$. If ϵ is sufficiently small it follows from Theorem 4.23, that the half line starting at p, parallel to $\gamma(t_0)$, meets $\gamma(S^1)$ only at $\gamma(t_0)$. Also the half line starting at q, parallel to $\gamma(t_0)$, does not meet $\gamma(S^1)$. Therefore by Remark 4.18,

$$w(\gamma, p) = \pm 1, \qquad w(\gamma, q) = 0.$$

By Proposition 4.19 p and q can not be in the same connected component of the complement of $\gamma(S^1)$ hence this complement has, at least, two distinct connected components.

The last claim follows from Remark 4.21 and the second argument above. □

4.26. Remark. The Jordan curve Theorem has the following refinement, due to Schoenflies, that we will not prove here.

4.27. Theorem. [Schoenflies Theorem] *A Jordan curve* $\gamma : S^1 \longrightarrow \mathbb{R}^2$ *extends to a homeomorphism* Γ *of the 2-disk onto the closure of the bounded component of* $\mathbb{R}^2 \setminus \gamma(S^1)$.

4.28. Remark. It is a natural question to ask if the Jordan curve Theorem holds for Jordan curves in general surfaces. The properties of $\mathbb{R}^2$ we have used in the proof above are:

- $\mathbb{R}^2$ is orientable. This allows to define the unit normal vector to a closed curve an to prove the tubular neighborhood Theorem.
- $H^1(\mathbb{R}^2) = \{0\}$. This allows to integrate the closed form ω.

Both conditions are essential for the proof and, in fact, for the validity of the Theorem. For example the real projective space has vanishing first (de Rham) cohomology group, but it is not orientable and the Theorem does not hold there. On the other side, the torus is orientable but the first cohomology group does not vanishes and, again, the Theorem does not hold for the torus.

We will see now some applications of the homotopy invariance of the winding number.

Let $D^2(r) := \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ be the disk of radius r and $S^1(r) := \{x \in \mathbb{R}^2 : \|x\| = r\}$ be its boundary. Consider a smooth function [8] $f : D^2(r) \longrightarrow \mathbb{R}^2$. A basic question is to find solutions of the equation $f(x) = 0$. In the case of a function $f : [-r, r] \longrightarrow \mathbb{R}$, the celebrated Theorem of Bolzano states that if $f(r)f(-r) < 0$ the equation has a solution. We will prove a similar result for our case, similar in the

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[8] By Remark 4.17 we will only need continuity of the function.

sense that we will give a condition on f, at the boundary of the disk, that will be sufficient for the existence of solutions of our equation.

4.29. Definition. Let $f : D^2(r) \longrightarrow \mathbb{R}^2$ be a smooth function. Suppose $f(x) \neq 0$ if $\|x\| = r$. The degree of f, $dg(f)$, is defined as the winding number of the closed curve:

$$\gamma_f : [0,1] \longrightarrow U := \mathbb{R}^2 \setminus \{0\}, \qquad \gamma_f(t) = f(r(\cos 2\pi t, \sin 2\pi t))$$

.

4.30. Example. Consider the complex plane $\mathbb{C} \cong \mathbb{R}^2$ with complex variable $z = x + iy$, and the map $g(z) = z^n$. Then $\gamma_g(t) = r(\cos 2\pi nt, \sin 2\pi nt)$. Hence $dg(g) = n$.

The announced result is the following:

4.31. Theorem. *If $dg(f) \neq 0$ then the equation $f(x) = 0$ has a solution.*

Proof. Suppose that $dg(f) \neq 0$ and that the equation has no solutions. Consider the map:

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad H(t,s) = f(sr(\cos 2\pi t, \sin 2\pi t)).$$

Since $f(x) \neq 0$, for $\|x\| \leq r$, H is a free homotopy, in $\mathbb{R}^2 \setminus \{0\}$, between γ_f and the constant curve $\alpha(t) = f(0)$. Therefore, by Theorem 4.16 $dg(f) := w(\gamma_f) = 0$, a contradiction. ☐

In order to compute degrees, the following fact is often useful:

4.32. Lemma. *Let $\gamma_i : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $i = 0, 1$ be two closed curves. If $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\|$ $\forall\, t \in [0,1]$, then the two curves are freely homotopic.*

Proof. Consider the map:

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2, \qquad H(t,s) = s\gamma_1(t) + (1-s)\gamma_0(t).$$

The condition $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\|$ implies that the segment joining $\gamma_0(t)$ and $\gamma_1(t)$ does not contain the origin. Then $H([0,1] \times [0,1]) \subseteq \mathbb{R}^2 \setminus \{0\}$ and H is a free homotopy between the two curves. ☐

As an application of Theorem 4.31, we will prove the *Fundamental Theorem of Algebra*:

4.33. Theorem. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a polynomial in the complex variable z. If $n \geq 1$, f has a complex root.*

Proof. Let $r > 1 + \sum_1^n |a_i|$. If $f(z) = 0$, for some $z \in S^1(r)$, there is nothing to prove. Suppose $f(z) \neq 0$ for $\|z\| = r$ and consider the function $g(z) = z^n$. For $\|z\| = r$ we have:

$$\|f(z) - g(z)\| \leq \sum_1^n |a_i| \|z\|^{n-i} < r^n = \|g(z)\|.$$

Hence, by Lemma 4.32, f and g have the same degree and $dg(g) = n \neq 0$, by Example 4.30. Hence, by Theorem 4.31, the polynomial has a root in $D^2(r)$. ☐

The arguments we presented may be generalized to higher dimensions and this will be sketched in the Exercises section (Exercises 5.21, 5.22).

5. Exercises

5.1. Let $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p \in \Omega^p(\mathbb{R}^n)$ and Δ^p be the standard p-simplex. Show that

$$\int_{\Delta^p} \omega = \frac{1}{p!} \qquad (= \text{volume of } \ \Delta^p).$$

5.2. Let $U, V \subseteq \mathbb{R}^n$ be connected open set and $F : U \longrightarrow V$ be a diffeomorphism. Let $D \subseteq U$ be the closure of a bounded open set and $f : V \longrightarrow \mathbb{R}$ a smooth function. The change of variables Theorem for multiple integrals states that:

$$\int_{F(D)} f(y_1, \ldots, y_n)\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_D f(F(x_1, \ldots, x_n))|\det(\mathrm{d}F)|\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Let $\omega \in \Omega^n(V)$. Prove that:

$$\int_{F(D)} \omega = \pm \int_D F^*\omega,$$

with the sign + (resp. −) if F preserves (resp. inverts) the orientation.

5.3. Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a continuous map. Prove that if U (resp. V) is contractible, then F is homotopic to a constant map.

5.4. Let $D^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $V \subseteq \mathbb{R}^m$. Show that a continuous map $F : S^n \longrightarrow V$ is continuously homotopic to a constant map if and only if it extends to a continuous map $\tilde{F} : D^{n+1} \longrightarrow V$ (observe that the concept of *continuous* homotopy may be defined even if the domains and codomains of the maps are not open).

5.5. Prove that an open set $U \subseteq \mathbb{R}^n$ is connected if and only if $H_0(U) \cong \mathbb{R}$ (see Example 2.8).

5.6. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open set and $F : U \longrightarrow V$ a smooth map. Prove that if U is connected, $F_* : H_0(U) \longrightarrow H_0(V)$ is injective. Study the case when U is not connected (see Example 2.8).

5.7. Consider the following generalization of the concept of homotopy:
Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F, G : U \longrightarrow V$ continuous maps. Let $C \subseteq U$ be a closed set such that $F|C = G|C$. A *homotopy between F and G, relative to C*, is a homotopy $H : U \times [0,1] \longrightarrow V$, between F and G, such that $H(x,t) = F(x) = G(x), \ \forall\, t \in [0,1], \ \forall\, x \in C$. If there exists such an homotopy, we will write $F \sim G \ \ (\text{rel } C)$.

(1) Prove that this relation is an equivalence relation.
(2) Reformulate Definition 4.5 in this context.

5.8. For an open set $U \subseteq \mathbb{R}^n$ define the reduced homology, $\tilde{H}_p(U)$, as the homology of the augmented chain complex

$$\cdots \longrightarrow C_p(U) \longrightarrow C_{p-1}(U) \longrightarrow \cdots \longrightarrow C_0(U) \longrightarrow \mathbb{R} \longrightarrow \{0\},$$

where the last map sends any singular 0-simplex to $1 \in \mathbb{R}$ and is extended by linearity (the other maps are the usual boundaries). Find the relation between $H_p(U)$ and $\tilde{H}_p(U)$ and prove the homotopy invariance and the exactness of Mayer-Vietoris sequence for reduced homology.

5.9. Prove the claim made in Remak 3.4 that the homology of the complex of continuous singular simplexes is isomorphic to the homology of the complex of the smooth singular simplexes (hint: use Lemma 3.1 and the Mayer-Vietoris exact sequences).

5.10. Compute the homology of $\mathbb{R}^n \setminus \{0\}$ without using the de Rham Theorem (hint: look at the Example 5.13 of Chapter 1).

5.11. Prove Proposition 4.6.

5.12. Given a close smooth curve $\gamma : [0,1] \longrightarrow U \subseteq \mathbb{R}^n$, we define the n-iterated, $\gamma_n : [0,n] \longrightarrow U,\ \gamma_n(t+m) = \gamma(t),\ m = 0,1,\dots n-1,\ t \in [0,1]$.

1. Prove that, if $\omega \in \Omega^1(U),\ \int_{\gamma_n} \omega = n \int_\gamma \omega$.
2. Prove that, if U has the property that for a given closed curve $\gamma : [0,1] \longrightarrow U$ there exist $n \in \mathbb{N}$ such that γ_n is homotopic to a constant, then $H^1(U) = \{0\}$.

5.13. Prove that an open set $U \subseteq \mathbb{R}^n$ is simply connected if and only if any two curves $\gamma_i : [0,1] \longrightarrow U,\ i = 0,1$ with with the same endpoints are homotopic relative to $\{0,1\}$.

5.14. Prove that any *continuous* curve $\gamma : [a,b] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ admits angular functions (hint: use polar coordinates to prove the claim when the image of γ is contained in a half plane. Then...). Extend Theorem 4.16, Definition 4.29 and Theorem 4.31 to the case of continuous functions.

5.15. Prove the formula in Remark 4.18.

5.16. Let $\gamma : S^1 \longrightarrow R^2 \setminus \{0\}$ be an *odd* closed curve, i.e. $\gamma(-t) = -\gamma(t),\quad t \in S^1$. Prove that $w(\gamma)$ is odd.

5.17. Prove the following Theorem of Borsuk: if $f, g : S^2 \longrightarrow \mathbb{R}$ are *odd* continuous functions, there exists $p \in S^2$ such that $f(p) = 0 = g(p)$ (hint: use the projection of the closed upper hemisphere onto the unit disk to define a function of the disk in $\mathbb{R}^2$).

5.18. Let $f, g : S^2 \longrightarrow \mathbb{R}$ be continuous functions. Prove that there exists $p \in S^2$ such that $f(p) = f(-p),\ g(p) = g(-p)$.

5.19. Prove that there are no injective continuous function $F : S^2 \longrightarrow \mathbb{R}^2$.

5.20. Let $\omega = a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y$ be a smooth closed 1-form in $\mathbb{R}^2 \setminus \{0\}$. Suppose that, for $0 < x^2 + y^2 \leq K$, the function a, b are bounded. Prove that ω is exact (hint: use homotopy invariance to show that for all closed curves $\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\},\ \int_\gamma \omega = 0$).

5.21. Let $F : S^n \longrightarrow S^n$ be a smooth function and $\tilde{F} : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\},\ \tilde{F}(tx) = tF(x)$. Then we have an induced linear map $\tilde{F}_* : H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R} \longrightarrow H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R}$. This map is multiplication by a real number $dg(F)$, called the *degree* of F. It is known that $dg(F) \in \mathbb{Z}$ [9] . Let D^{n+1} be the unit disk and $G : D^{n+1} \longrightarrow \mathbb{R}^{n+1}$ a smooth function not vanishing on the unit sphere $S^n = \partial D^{n+1}$. Then the degree of G, $dg(G)$, is defined as the degree of the map $\tilde{G}(x) = \frac{G(x)}{\|G(x)\|}$. Prove that, if $dg(G) \neq 0$, then the equation $G(x) = 0$ has a solution.

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[9] It follows, from homotopy invariance, that homotopic maps have the same degree. A basic fact in homotopy theory is the Theorem of Hopf: if tho maps from S^n to S^n have the same degree, then they are homotopic.

5.22. Prove that there are not smooth maps $F : D^{n+1} \longrightarrow S^n = \partial D^{n+1}$ such that $F(x) = x \ \ \forall x \in S^n$. Use this fact to prove the celebrated Brouwer fix point Theorem: any continuous map $G : D^{n+1} \longrightarrow D^{n+1}$ has a fixed point, i.e. a point $x \in D^{n+1}$ such that $G(x) = x$ (hint for the Brouwer fix point Theorem: if $G(x) \neq x \ \ \forall\ x \in D^{n+1}$, take the halph line starting at $G(x)$ containing x and define $F(x)$ to be the intersection of this halph line with S^n. Then ...).

5.23. Let $U \subseteq \mathbb{R}^2$ be an open set such that $H^1(U) = \{0\}$. Prove that any smooth Jordan curve $\gamma : S^1 \longrightarrow U$ is homotopic, in U, to a constant curve (hint: by Theorem 4.27, $\gamma(S^1)$ is the boundary of a disk in $\mathbb{R}^2$. If the disk is in U, the curve is contractible by Exercise 5.4. If not use the angle form to get a contradiction).
REMARK: This fact implies that U is simply connected (see Remark 4.9).

5.24. Let $U \subseteq \mathbb{R}^2$ be an open set and $X : U \longrightarrow \mathbb{R}^2$ a smooth vector field. Let $D_\epsilon \subseteq U$ be a disk of radius ϵ, with center $p \in U$, such that $X(q) \neq 0$, $\forall q \in D_\epsilon \setminus \{p\}$. The point p is called an (*isolated*) *singularity* of X. The *index* of X at p, $i(X, p)$, is defined as the degree of $X_{|D_\epsilon}$, i.e. the winding number of the curve $X(p + \epsilon \cos 2\pi t, p + \epsilon \sin 2\pi t)$, $t \in [0, 1]$.

(1) Let $\gamma : [0, 1] \longrightarrow U$ be a piecewise smooth, positively oriented closed Jordan curve bounding a disk in U, containing p in its interior. Prove that $i(X, p)$ is the winding number of $X \circ \gamma$.

(2) If $X(x, y) = (f(x, y), g(x, y))$, prove that

$$i(x, p) = \frac{1}{2\pi}\int_\gamma \theta,$$

where γ is as in the preceding item and

$$\theta = \frac{-g\mathrm{d}x}{f^2 + g^2} + \frac{f\mathrm{d}y}{f^2 + g^2} = X^*\omega,$$

where ω is the angle form.

(3) Prove that if $X(p) \neq 0$, then $i(X, p) = 0$.

(4) Let $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isomorphism. Prove that $i(x, 0) = 1$ if $\det X > 0$ and $i(x, 0) = -1$ if $\det X < 0$.

(5) Assume that $X(p) = 0$ and $\mathrm{d}X(p)$ is invertible. In this case we will say that p is a *simple* singularity of X, positive, if $\det \mathrm{d}X(p) > 0$, negative otherwise. Prove that a simple singularity is isolated and $i(X, p) = \pm 1$, depending if p is a positive or negative simple singularity (hint: by Taylor's formula $X(q) = \mathrm{d}X(0)(q) + R(q)\|q\|$, with $\lim_{q \to 0} R(q) = 0$. Prove that $H(q, s) = \mathrm{d}X(0)(q) + (1 - t)R(q)\|q\| \neq 0$, if $\|q\|$ is sufficiently small. Hence...).

(6) Prove the following formula, called the *Kronecker formula*.
Let $D \subseteq \mathbb{R}^2$ be a closed disk, with center q and radius r, and $X : D \longrightarrow \mathbb{R}^2$ be a vector field with only simple singularities, none of which is in ∂D. Then

$$\frac{1}{2\pi}\int_\gamma \theta = P - N,$$

where $\gamma(t) = p + r(\cos 2\pi t, \sin 2\pi t)$, P is the number of the positive singularities and N the number of the negative ones.

Remark: The condition $i(X,p)=0$ does not imply $X(p)\neq 0$ (find an example!). However, if $i(X,p)=0$, given $\epsilon>0$, we can find a vector field $\tilde{X}$ which coincide with X outside a disk of radius ϵ and center p, without zeros in that disk.

5.25. Let $f: U\subseteq \mathbb{C}=\mathbb{R}^2 \longrightarrow \mathbb{C}$ be a holomorphic function (see Exercise 9.28 of Chapter 1), $f=u+iv$.

(1) Prove the following Cauchy's Theorem:
Theorem: If U is simply connected and $\gamma: S^1 \longrightarrow U$ is a closed piecewise smooth curve then
$$\int_\gamma f(z)\mathrm{d}z := \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\gamma (u\mathrm{d}y + v\mathrm{d}x) = 0.$$

(2) Suppose that $f'(z)\neq 0$ for z in a disk $D\subseteq U$ and $f(z)\neq 0$ for $z\in\partial U$. Prove that the number of zeros in D is given by
$$\frac{1}{2\pi i}\int_{\partial D}\frac{\mathrm{d}f}{f}$$
(hint: prove that the singularities of the vector field $X(x,y)=(u(x,y),(v(x,y))$ are all simple and positive. Then....).

5.26. Use Exercise 5.21 to define the index of a vector field $X: U\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$ at a point $p\in U$ and try to extend, as much as you can, the facts claimed in Exercise 5.24 for this situation.

5.27. Make the following claim precise and prove it
Claim: the de Rham isomorphism is natural with respect to smooth maps.

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