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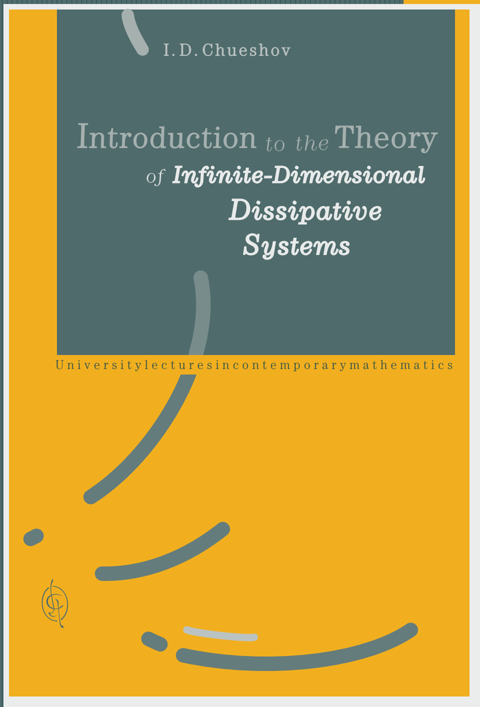

This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

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Chapter 6

Homoclinic Chaos in Infinite-Dimensional Systems

Contents

In this chapter we consider some questions on the asymptotic behaviour of a discrete dynamical system. We remind (see Chapter 1) that a discrete dynamical system is defined as a pair (X, S) consisting of a metric space X and a continuous mapping of X into itself. Most assertions on the existence and properties of attractors given in Chapter 1 remain true for these systems. It should be noted that the following examples of discrete dynamical systems are the most interesting from the point of view of applications: a) systems generated by monodromy operators (period mappings) of evolutionary equations, with coefficients being periodic in time; b) systems generated by difference schemes of the type $\tau^{-1}(u_{n+1}-u_n) = F(u_n)$, $n = 0, 1, 2, \ldots$ in a Banach space X (see Examples 1.5 and 1.6 of Chapter 1).

The main goal of this chapter is to give a strict mathematical description of one of the mechanisms of a complicated (irregular, chaotic) behaviour of trajectories. We deal with the phenomenon of the so-called homoclinic chaos. This phenomenon is well-known and is described by the famous Smale theorem (see, e.g., [1–3]) for finite-dimensional systems. This theorem is of general nature and can be proved for infinite-dimensional systems. Its proof given in Section 5 is based on an infinite-dimensional variant of Anosov's lemma on ε-trajectories (see Section 4). The considerations of this Chapter are based on the paper [4] devoted to the finite-dimensional case as well as on the results concerning exponential dichotomies of infinite-dimensional systems given in Chapter 7 of the book [5]. We follow the arguments given in [6] while proving Anosov's lemma.

§ 1 *Bernoulli Shift as a Model of Chaos*

Mathematical simulation of complicated dynamical processes which take place in real systems requires that the notion of a state of chaos be formalized. One of the possible approaches to the introduction of this notion relies on a selection of a class of explicitly solvable models with complicated (in some sense) behaviour of trajectories. Then we can associate every model of the class with a definite type of chaotic behaviour and use these models as standard ones comparing their dynamical structure with a qualitative behaviour of the dynamical system considered. A discrete dynamical system known as *the Bernoulli shift* is one of these explicitly solvable models.

Let $m \ge 2$ and let

$$\Sigma_m = \left\{ x = (\ldots, x_{-1}, x_0, x_1, \ldots): \;\; x_j \in \{1, 2, \ldots, m\},\; j \in \mathbb{Z} \right\},$$

i.e. Σ_m is a set of two-sided infinite sequences the elements of which are the integers $1, 2, \ldots, m$. Let us equip the set Σ_m with a metric

$$d(x, y) = \sum_{i=-\infty}^{\infty} 2^{-|i|} \frac{|x_i - y_i|}{1 + |x_i - y_i|} \,. \tag{1.1}$$

Here $x = \{x_i\colon\ i \in \mathbb{Z}\}$ and $y = \{y_i\colon\ i \in \mathbb{Z}\}$ are elements of Σ_m. Other methods of introduction of a metric in Σ_m are given in Example 1.1.7 and Exercise 1.1.5.

Exercise *1.1* Show that the function $d(x, y)$ satisfies all the axioms of a metric.

Exercise *1.2* Let $x = \{x_i\}$ and $y = \{y_i\}$ be elements of the set Σ_m. Assume that $x_i = y_i$ for $|i| \le N$ and for some integer N. Prove that $d(x, y) \le 2^{-N+1}$.

Exercise *1.3* Assume that equation $d(x, y) < 2^{-N}$ holds for $x, y \in \Sigma_m$, where N is a natural number. Show that $x_i = y_i$ for all $|i| \le N-1$ (*Hint*: $d(x, y) \ge 2^{-|i|-1}$ if $x_i \ne y_i$).

Exercise *1.4* Let $x \in \Sigma_m$ and let

$$\mathcal{U}^N(x) = \{y \in \Sigma_m\colon\ x_i = y_i \quad \text{for } |i| \le N\}. \tag{1.2}$$

Prove that for any $0 < \varepsilon < 1$ the relation

$$\mathcal{U}^{N(\varepsilon)+2}(x) \subset \{y\colon\ d(x, y) < \varepsilon\} \subset \mathcal{U}^{N(\varepsilon)-1}(x)$$

holds, where $N(\varepsilon)$ is an integer with the property

$$N(\varepsilon) < \frac{\ln 1/\varepsilon}{\ln 2} \le N(\varepsilon) + 1 \,.$$

Exercise *1.5* Show that the space Σ_m with metric (1.1) is a compact metric space.

In the space Σ_m we define a mapping S which shifts every sequence one symbol left, i.e.

$$[Sx]_i = x_{i+1}\,, \qquad i \in \mathbb{Z}\,, \qquad x = \{x_i\} \in \Sigma_m\,.$$

Evidently, S is invertible and the relations

$$d(Sx, Sy) \le 2\,d(x, y)\,, \qquad d(S^{-1}x, S^{-1}y) \le 2\,d(x, y)$$

hold for all $x, y \in \Sigma_m$. Therefore, the mapping S is a homeomorphism.

The discrete dynamical system (Σ_m, S) is called ***the Bernoulli shift*** of the space of sequences of m symbols. Let us study the dynamical properties of the system (Σ_m, S).

Exercise *1.6* Prove that (Σ_m, S) has m fixed points exactly. What structure do they have?

We call an arbitrary ordered collection $a = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_j \in \{1, \ldots, m\}$ a segment (of the length N). Each element $x \in \Sigma_m$ can be considered as an ordered infinite family of finite segments while the elements of the set Σ_m can be constructed from segments. In particular, using the segment $a = (\alpha_1, \ldots, \alpha_N)$ we can construct a periodic element $\bar{a} \in \Sigma_m$ by the formula

$$\bar{a}_{Nk+j} = \alpha_j, \quad j \in \{1, \ldots, N\}, \quad k \in \mathbb{Z}. \tag{1.3}$$

— Exercise *1.7* Let $a = (\alpha_1, \ldots, \alpha_N)$ be a segment of the length N and let $\bar{a} \in \Sigma_m$ be an element defined by (1.3). Prove that $\bar{a}$ is a periodic point of the period N of the dynamical system (Σ_m, S), i.e. $S^N \bar{a} = \bar{a}$.

— Exercise *1.8* Prove that for any natural N there exists a periodic point of the minimal period equal to N.

— Exercise *1.9* Prove that the set of all periodic points is dense in Σ_m, i.e. for every $x \in \Sigma_m$ and $\varepsilon > 0$ there exists a periodic point a with the property $d(x, a) < \varepsilon$ (*Hint*: use the result of Exercise 1.4).

— Exercise *1.10* Prove that the set of nonperiodic points is not countable.

— Exercise *1.11* Let $a = (\ldots, \alpha, \alpha, \alpha, \ldots)$ and $b = (\ldots, \beta, \beta, \beta, \ldots)$ be fixed points of the system (Σ_m, S). Let $C = \{c_i\}$ be an element of Σ_m such that $c_i = \alpha$ for $i \le -M_1$ and $c_i = \beta$ for $i \ge M_2$, where M_1 and M_2 are natural numbers. Prove that

$$\lim_{n \to -\infty} S^n c = a, \qquad \lim_{n \to \infty} S^n c = b. \tag{1.4}$$

Assume that an element $c \in \Sigma_m$ possesses property (1.4) with $c \ne a$ and $c \ne b$. If $a \ne b$, then the set

$$\gamma_{a,b} = \{S^n c : \ n \in \mathbb{Z}\}$$

is called a ***heteroclinic trajectory*** that connects the fixed points a and b. If $a = b$, then $\gamma_a = \gamma_{a,a}$ is called a ***homoclinic trajectory*** of the point a. The elements of a heteroclinic (homoclinic, respectively) trajectory are called heteroclinic (homoclinic, respectively) points.

— Exercise *1.12* Prove that for any pair of fixed points there exists an infinite number of heteroclinic trajectories connecting them whereas the corresponding set of heteroclinic points is dense in Σ_m.

— Exercise *1.13* Let

$$\gamma_1 = \{S^n a : \ n \in \mathbb{Z}\} = \{S^n a : \ n = 0, 1, \ldots, N_1 - 1\}$$

and

$$\gamma_2 = \{S^n b : \ n \in \mathbb{Z}\} = \{S^n b : \ n = 0, 1, \ldots, N_2 - 1\}$$

be cycles (periodic trajectories). Prove that there exists a heteroclinic trajectory $\gamma_{1,2} = \{S^n c: \; n \in \mathbb{Z}\}$ that connects the cycles γ_1 and γ_2, i.e. such that

$$\operatorname{dist}(S^n c, \gamma_1) \equiv \inf_{x \in \gamma_1} d(S^n c, x) \to 0, \qquad n \to -\infty$$

and

$$\operatorname{dist}(S^n c, \gamma_2) = \inf_{x \in \gamma_2} d(S^n c, x) \to 0, \qquad n \to +\infty.$$

For every N there exists only a finite number of segments of the length N. Therefore, the set $\mathcal{L}$ of all segments is countable, i.e. we can assume that $\mathcal{L} = \{a_k: k = 1, 2, \dots\}$, therewith the length of the segment a_{k+1} is not less than the length of a_k. Let us construct an element $b = \{b_i: \; i \in \mathbb{Z}\}$ from Σ_m taking $b_i = 1$ for $i \le 0$ and sequentially putting all the segments a_k to the right of the zeroth position. As a result, we obtain an element of the form

$$b = (\dots, 1, 1, 1, a_1, a_2, a_3, \dots), \qquad a_j \in \mathcal{L}. \tag{1.5}$$

— Exercise *1.14* Prove that a positive semitrajectory $\gamma_+ = \{S^n b, \; n \ge 0\}$ with b having the form (1.5) is dense in Σ_m, i.e. for every $x \in \Sigma_m$ and $\varepsilon > 0$ there exists $n = n(x, \varepsilon)$ such that $d(x, S^n b) < \varepsilon$.

— Exercise *1.15* Prove that the semitrajectory γ_+ constructed in Exercise 1.14 returns to an ε-vicinity of every point $x \in \Sigma_m$ infinite number of times (*Hint*: see Exercises 1.4 and 1.9).

— Exercise *1.16* Construct a negative semitrajectory $\gamma_- = \{S^n c: \; n \le 0\}$ which is dense in Σ_m.

Thus, summing up the results of the exercises given above, we obtain the following assertion.

Theorem 1.1.

The dynamical system (Σ_m, S) of the Bernoulli shift of sequences of m symbols possesses the properties:

1) *there exists a finite number of fixed points;*
2) *there exist periodic orbits of any minimal period and the set of these orbits is dense in the phase space Σ_m;*
3) *the set of nonperiodic points is uncountable;*
4) *heteroclinic and homoclinic points are dense in the phase space;*
5) *there exist everywhere dense trajectories.*

All these properties clearly imply the extraordinarity and complexity of the dynamics in the system (Σ_m, S). They also give a motivation for the following definitions.

Let (X, f) be a discrete dynamical system. The dynamics of the system (X, f) is called ***chaotic*** if there exists a natural number k such that the mapping f^k is topologically conjugate to the Bernoulli shift for some m, i.e. there exists a homeomorphism $h: X \to \Sigma_m$ such that $h(f^k(x)) = S(h(x))$ for all $x \in X$. We also say that chaotic dynamics is observed in the system (X, f) if there exist a number k and a set Y in X invariant with respect to f^k $(f^k Y \subset Y)$ such that the restriction of f^k to Y is topologically conjugate to the Bernoulli shift.

It turns out that if a dynamical system (X, f) has a fixed point and a corresponding homoclinic trajectory, then chaotic dynamics can be observed in this system under some additional conditions (this assertion is the core of the Smale theorem). Therefore, we often speak about homoclinic chaos in this situation. It should also be noted that the approach presented here is just one of the possible methods used to describe chaotic behaviour (for example, other approaches can be found in [1] as well as in book [7], the latter contains a survey of methods used to study the dynamics of complicated systems and processes).

§ 2 Exponential Dichotomy and Difference Equations

This is an auxiliary section. Nonautonomous linear difference equations of the form

$$x_{n+1} = A_n x_n + h_n, \qquad n \in \mathbb{Z}, \tag{2.1}$$

in a Banach space X are considered here. We assume that $\{A_n\}$ is a family of linear bounded operators in X, h_n is a sequence of vectors from X. Some results both on the dichotomy (splitting) of solutions to homogeneous $(h_n \equiv 0)$ equation (2.1) and on the existence and properties of bounded solutions to nonhomogeneous equation are given here. We mostly follow the arguments given in book [5] as well as in paper [4] devoted to the finite-dimensional case.

Thus, let $\{A_n: \ n \in \mathbb{Z}\}$ be a sequence of linear bounded operators in a Banach space X. Let us consider a homogeneous difference equation

$$x_{n+1} = A_n x_n, \qquad n \in J, \tag{2.2}$$

where J is an interval in $\mathbb{Z}$, i.e. a set of integers of the form

$$J = \{n \in \mathbb{Z}: \ m_1 < n < m_2\},$$

where m_1 and m_2 are given numbers, we allow the cases $m_1 = -\infty$ and $m_2 = +\infty$. Evidently, any solution $\{x_n: \ n \in J\}$ to difference equation (2.2) possesses the property

$$x_m = \Phi(m, n) x_n, \qquad m \geq n, \qquad m, n \in J,$$

where $\Phi(m, n) = A_{m-1} \cdot \ldots \cdot A_n$ for $m > n$ and $\Phi(m, m) = I$. The mapping $\Phi(m, n)$ is called an ***evolutionary operator*** of problem (2.2).

— Exercise *2.1* Prove that for all $m \ge n \ge k$ we have

$$\Phi(m, k) = \Phi(m, n)\Phi(n, k).$$

— Exercise *2.2* Let $\{P_n: n \in J\}$ be a family of projectors (i.e. $P_n^2 = P_n$) in X such that $P_{n+1}A_n = A_n P_n$. Show that

$$P_m \Phi(m, n) = \Phi(m, n)P_n, \quad m \ge n, \quad m, n \in J,$$

i.e. the evolutionary operator $\Phi(m, n)$ maps $P_n X$ into $P_m X$.

— Exercise *2.3* Prove that solutions $\{x_n\}$ to nonhomogeneous difference equation (2.1) possess the property

$$x_m = \Phi(m, n)x_n + \sum_{k=n}^{m-1} \Phi(m, k+1)h_k, \qquad m > n.$$

Let us give the following definition. A family of linear bounded operators $\{A_n\}$ is said to possess an ***exponential dichotomy*** over an interval J with constants $K > 0$ and $0 < q < 1$ if there exists a family of projectors $\{P_n: n \in J\}$ such that

a) $$P_{n+1}A_n = A_n P_n, \quad n, n+1 \in J;$$

b) $$\|\Phi(m, n)P_n\| \le Kq^{m-n}, \quad m \ge n, \quad m, n \in J;$$

c) for $n \ge m$ the evolutionary operator $\Phi(n, m)$ is a one-to-one mapping of the subspace $(1-P_m)X$ onto $(1-P_n)X$ and the following estimate holds:

$$\|\Phi(n, m)^{-1}(1-P_n)\| \le Kq^{n-m}, \quad m \le n, \quad m, n \in J.$$

If these conditions are fulfilled, then it is also said that difference equation (2.2) admits an exponential dichotomy over J. It should be noted that the cases $J = \mathbb{Z}$ and $J = \mathbb{Z}_\pm$ are the most interesting for further considerations, where $\mathbb{Z}_+$ ($\mathbb{Z}_-$) is the set of all nonnegative (nonpositive) integers.

The simplest case when difference equation (2.2) admits an exponential dichotomy is described in the following example.

— Example *2.1* (autonomous case)

Assume that equation (2.2) is autonomous, i.e. $A_n \equiv A$ for all n, and the spectrum $\sigma(A)$ does not intersect the unit circumference $\{z \in \mathbb{C}: |z|=1\}$. Linear operators possessing this property are often called hyperbolic (with respect to the fixed point $x = 0$). It is well-known (see, e.g., [8]) that in this case there exists a projector P with the properties:

a) $AP = PA$, i.e. the subspaces PX and $(1-P)X$ are invariant with respect to A;
b) the spectrum $\sigma(A|_{PX})$ of the restriction of the operator A to PX lies strictly inside of the unit disc;
c) the spectrum $\sigma(A|_{(1-P)X})$ of the restriction of A to the subspace $(1-P)X$ lies outside the unit disc.

— Exercise *2.4* Let C be a linear bounded operator in a Banach space X and let $\rho \equiv \max\{|z|:\ z \in \sigma(C)\}$ be its spectral radius. Show that for any $q > \rho$ there exists a constant $M_q \geq 1$ such that

$$\|C^n\| \leq M_q q^n, \qquad n = 0,\ 1,\ 2,\ \dots$$

(*Hint*: use the formula $\rho = \lim_{n\to\infty} \|C^n\|^{1/n}$ the proof of which can be found in [9], for example).

Applying the result of Exercise 2.4 to the restriction of the operator A to PX, we obtain that there exist $K > 0$ and $0 < q < 1$ such that

$$\|A^n P\| \leq K q^n, \qquad n \geq 0. \tag{2.3}$$

It is also evident that the restriction of the operator A to $(1-P)X$ is invertible and the spectrum of the inverse operator lies inside the unit disc. Therefore,

$$\|A^{-n}(1-P)\| \leq K q^n, \qquad n \geq 0, \tag{2.4}$$

where the constants $K > 0$ and $0 < q < 1$ can be chosen the same as in (2.3). The evolutionary operator $\Phi(m, n)$ of the difference equation $x_{n+1} = Ax_n$ has the form $\Phi(m, n) = A^{m-n}$, $m \geq n$. Therefore, the equality $AP = PA$ and estimates (2.3) and (2.4) imply that the equation $x_{n+1} = Ax_n$ admits an exponential dichotomy over $\mathbb{Z}$, provided the spectrum of the operator A does not intersect the unit circumference.

— Exercise *2.5* Assume that for the operator A there exists a projector P such that $AP = PA$ and estimates (2.3) and (2.4) hold with $0 < q < 1$. Show that the spectrum of the operator A does not intersect the unit circumference, i.e. A is hyperbolic.

Thus, the hyperbolicity of the linear operator A is equivalent to the exponential dichotomy over $\mathbb{Z}$ of the difference equation $x_{n+1} = Ax_n$ with the projectors P_n independent of n. Therefore, the dichotomy property of difference equation (2.2) should be considered as an extension of the notion of hyperbolicity to the nonautonomous case. The meaning of this notion is explained in the following two exercises.

— Exercise *2.6* Let A be a hyperbolic operator. Show that the space X can be decomposed into a direct sum of stable X^s and unstable X^u subspaces, i.e. $X = X^s + X^u$ therewith

$$\|A^n x\| \le K q^n \|x\|, \qquad x \in X^s, \qquad n \ge 0,$$

$$\|A^n x\| \ge K^{-1} q^{-n} \|x\|, \qquad x \in X^u, \quad n \ge 0,$$

with some constants $K > 0$ and $0 < q < 1$.

Exercise *2.7* Let $X = \mathbb{R}^2$ be a plane and let A be an operator defined by the formula

$$A(x_1, x_2) = (2x_1 + x_2;\ x_1 + x_2), \qquad x = (x_1;\ x_2) \in \mathbb{R}^2.$$

Show that the operator A is hyperbolic. Evaluate and display graphically stable X^s and unstable X^u subspaces on the plane. Display graphically the trajectory $\{A^n x:\ n \in \mathbb{Z}\}$ of some point x that lies neither in X^s, nor in X^u.

The next assertion (its proof can be found in the book [5]) plays an important role in the study of existence conditions of exponential dichotomy of a family of operators $\{A_n:\ n \in \mathbb{Z}\}$.

Theorem 2.1.

Let $\{A_n:\ n \in \mathbb{Z}\}$ be a sequence of linear bounded operators in a Banach space X. Then the following assertions are equivalent:

(i) ***the sequence $\{A_n:\ n \in \mathbb{Z}\}$ possesses an exponential dichotomy over $\mathbb{Z}$,***

(ii) ***for any bounded sequence $\{h_n:\ n \in \mathbb{Z}\}$ from X there exists a unique bounded solution $\{x_n:\ n \in \mathbb{Z}\}$ to the nonhomogeneous difference equation***

$$x_{n+1} = A_n x_n + h_n, \qquad n \in \mathbb{Z}. \tag{2.5}$$

In the case when the sequence $\{A_n\}$ possesses an exponential dichotomy, solutions to difference equation (2.5) can be constructed using ***the Green function***:

$$G(n, m) = \begin{cases} \Phi(n, m) P_m, & n \ge m, \\ -[\Phi(m, n)]^{-1}(1 - P_m), & n < m. \end{cases}$$

Exercise *2.8* Prove that $\|G(n, m)\| \le K q^{|n-m|}$.

Exercise *2.9* Prove that for any bounded sequence $\{h_n:\ n \in \mathbb{Z}\}$ from X a solution to equation (2.5) has the form

$$x_n = \sum_{m \in \mathbb{Z}} G(n, m+1) h_m, \qquad n \in \mathbb{Z}.$$

Moreover, the following estimate is valid:

$$\sup_n \|x_n\| \le K \frac{1+q}{1-q} \sup_n \|h_n\| .$$

The properties of the Green function enable us to prove the following assertion on the uniqueness of the family of projectors $\{P_n\}$.

Lemma 2.1.

Let a sequence $\{A_n\}$ possess an exponential dichotomy over $\mathbb{Z}$. Then the projectors $\{P_n :\ n \in \mathbb{Z}\}$ are uniquely defined.

Proof.

Assume that there exist two collections of projectors $\{P_n\}$ and $\{Q_n\}$ for which the sequence $\{A_n\}$ possesses an exponential dichotomy. Let $G_P(n,\ m)$ and $G_Q(n,\ m)$ be Green functions constructed with the help of these collections. Then Theorem 2.1 enables us to state (see Exercise 2.9) that

$$\sum_{m \in \mathbb{Z}} G_P(n,\ m+1)h_m = \sum_{m \in \mathbb{Z}} G_Q(n,\ m+1)h_m$$

for all $n \in \mathbb{Z}$ and for any bounded sequence $\{h_n\} \subset X$. Assuming that $h_m = 0$ for $m \ne k-1$ and $h_m = h$ for $m = k-1$, we find that

$$G_P(n,\ k)h = G_Q(n,\ k)h\,, \qquad h \in X\,, \quad n,\ k \in \mathbb{Z}\,, \quad n \ge k\,.$$

This equality with $n = k$ gives us that $P_n h = Q_n h$. Thus, the lemma is proved.

In particular, Theorem 2.1 implies that in order to prove the existence of an exponential dichotomy it is sufficient to make sure that equation (2.5) is uniquely solvable for any bounded right-hand side. It is convenient to consider this difference equation in the space $l_X^\infty \equiv l^\infty(\mathbb{Z},\ X)$ of sequences $\boldsymbol{x} = \{x_n :\ n \in \mathbb{Z}\}$ of elements of X for which the norm

$$|\boldsymbol{x}|_{l^\infty} = |\{x_n\}|_{l^\infty} = \sup\left\{\|x_n\| :\ n \in \mathbb{Z}\right\} \tag{2.6}$$

is finite. Assume that the condition

$$\sup\left\{\|A_n\| :\ n \in \mathbb{Z}\right\} < \infty \tag{2.7}$$

is valid. Then for any $\boldsymbol{x} = \{x_n\} \in l_X^\infty$ the sequence $\{y_n = x_n - A_{n-1}x_{n-1}\}$ lies in l_X^∞. Consequently, equation

$$(L\boldsymbol{x})_n = x_n - A_{n-1}x_{n-1}\,, \qquad \boldsymbol{x} = \{x_n\} \in l_X^\infty \tag{2.8}$$

defines a linear bounded operator acting in the space $l_X^\infty = l^\infty(\mathbb{Z},\ X)$. Therewith assertion (ii) of Theorem 2.1 is equivalent to the assertion on the invertibility of the operator L given by equation (2.8).

The assertion given below provides a sufficient condition of invertibility of the operator L. Due to Theorem 2.1 this condition guarantees the existence of an exponential dichotomy for the corresponding difference equation. This assertion will be used in Section 4 in the proof of Anosov's lemma. It is a slightly weakened variant of a lemma proved in [6].

Theorem 2.2.

Assume that a sequence of operators $\{A_n\colon\ n \in \mathbb{Z}\}$ ***satisfies condition*** (2.7)***. Let there exist a family of projectors*** $\{Q_n\colon\ n \in \mathbb{Z}\}$ ***such that***

$$\|Q_n\| \le K, \qquad \|1-Q_n\| \le K, \tag{2.9}$$

$$\|Q_{n+1}A_n(1-Q_n)\| \le \delta, \qquad \|(1-Q_{n+1})A_n Q_n\| \le \delta, \tag{2.10}$$

for all $n \in \mathbb{Z}$***. We also assume that the operator*** $(1-Q_{n+1})A_n$ ***is invertible as a mapping from*** $(1-Q_n)X$ ***into*** $(1-Q_{n+1})X$ ***and the estimates***

$$\|A_n Q_n\| \le \lambda, \qquad \|[(1-Q_{n+1})A_n]^{-1}(1-Q_{n+1})\| \le \lambda \tag{2.11}$$

are valid for every $n \in \mathbb{Z}$***. If***

$$K\lambda \le \frac{1}{8}, \qquad \delta \le \frac{1}{8}, \tag{2.12}$$

then the operator L ***acting in*** l_X^∞ ***according to formula*** (2.8) ***is invertible and*** $\|L^{-1}\| \le 2K+1$.

Proof.

Let us first prove the injectivity of the mapping L. Assume that there exists a nonzero element $\boldsymbol{x} = \{x_n\} \in l_X^\infty$ such that $L\boldsymbol{x} = 0$, i.e. $x_n = A_{n-1}x_{n-1}$ for all $n \in \mathbb{Z}$. Let us prove that the sequence $\{x_n\}$ possesses the property

$$\|(1-Q_n)x_n\| \le \|Q_n x_n\| \tag{2.13}$$

for all $n \in \mathbb{Z}$. Indeed, let there exist $m \in \mathbb{Z}$ such that

$$\|(1-Q_m)x_m\| > \|Q_m x_m\|. \tag{2.14}$$

It is evident that this equation is only possible when $\|(1-Q_m)x_m\| > 0$. Let us consider the value

$$\begin{aligned} N_{m+1} &\equiv \|(1-Q_{m+1})x_{m+1}\| - \|Q_{m+1}x_{m+1}\| = \\ &= \|(1-Q_{m+1})A_m x_m\| - \|Q_{m+1}A_m x_m\|. \end{aligned} \tag{2.15}$$

It is clear that

$$\|(1-Q_{n+1})A_n x_n\| \ge \|(1-Q_{n+1})A_n(1-Q_n)x_n\| - \|(1-Q_{n+1})A_n Q_n x_n\|.$$

Since

$$[(1-Q_{n+1})A_n]^{-1}(1-Q_{n+1})A_n(1-Q_n) = (1-Q_n),$$

it follows from (2.11) that

$$\|(1-Q_n)x\| \le \lambda \|(1-Q_{n+1})A_n(1-Q_n)x\|$$

for every $x \in X$ and for all $n \in \mathbb{Z}$. Therefore, we use estimates (2.10) to find that

$$\|(1-Q_{n+1})A_n x_n\| \ge \lambda^{-1}\|(1-Q_n)x_n\| - \delta\|x_n\| . \tag{2.16}$$

Then it is evident that

$$\|Q_{n+1}A_n x_n\| \le \|Q_{n+1}A_n Q_n x_n\| + \|Q_{n+1}A_n(1-Q_n)x_n\| \le$$

$$\le \left(\|Q_{n+1}\| \cdot \|A_n Q_n\| + \|Q_{n+1}A_n(1-Q_n)\|\right)\|x_n\| .$$

Therefore, estimates (2.9)–(2.11) imply that

$$\|Q_{n+1}A_n x_n\| \le (K\lambda + \delta)\|x_n\| , \qquad n \in \mathbb{Z} . \tag{2.17}$$

Thus, equations (2.15)–(2.17) lead us to the estimate

$$N_{m+1} \ge \lambda^{-1}\|(1-Q_m)x_m\| - (2\delta + K\lambda)\|x_m\| .$$

It follows from (2.14) that

$$\|x_m\| \le \|Q_m x_m\| + \|(1-Q_m)x_m\| < 2\|(1-Q_m)x_m\| .$$

Therefore,

$$N_{m+1} > (\lambda^{-1} - 2K\lambda - 4\delta)\,\|(1-Q_m)x_m\| .$$

Hence, if conditions (2.12) hold, then

$$\|(1-Q_{m+1})x_{m+1}\| - \|Q_{m+1}x_{m+1}\| > 7\|(1-Q_m)x_m\| > 0 . \tag{2.18}$$

When proving (2.18) we use the fact that

$$\lambda^{-1} \ge 8K \ge 8\|Q_n\| \ge 8 .$$

Thus, equation (2.18) follows from (2.14), i.e. $N_m > 0$ implies $N_{m+1} > 0$. Hence,

$$\|(1-Q_n)x_n\| > \|Q_n x_n\| \quad \text{for all} \quad n \ge m .$$

Moreover, (2.18) gives us that

$$K \cdot \|x_n\| \ge \|(1-Q_n)x_n\| \ge 7^{n-m}\|(1-Q_m)x_m\| , \qquad n \ge m .$$

Therefore, $\|x_n\| \to +\infty$ as $n \to +\infty$. This contradicts the assumption $\boldsymbol{x} = \{x_n\} \in l_X^\infty$. Thus, for all $n \in \mathbb{Z}$ estimate (2.13) is valid. In particular it leads us to the inequality

$$\|x_n\| \le \|(1-Q_n)x_n\| + \|Q_n x_n\| \le 2\|Q_n x_n\| . \tag{2.19}$$

Therefore, it follows from (2.17) that

$$\|Q_{n+1}x_{n+1}\| = \|Q_{n+1}A_n x_n\| \le 2(K\lambda + \delta)\|Q_n x_n\|$$

for all $n \in \mathbb{Z}$. We use conditions (2.12) to find that

$$\|Q_{n+1}x_{n+1}\| \le \frac{1}{2}\|Q_n x_n\| , \qquad n \in \mathbb{Z} . \tag{2.20}$$

If $\boldsymbol{x} = \{x_n\} \neq 0$, then inequality (2.19) gives us that there exists $m \in \mathbb{Z}$ such that $\|Q_m x_m\| \neq 0$. Therefore, it follows from (2.20) that

$$\|Q_n x_n\| \geq 2^{m-n} \|Q_m x_m\| > 0$$

for all $n \leq m$. We tend $n \to -\infty$ to obtain that $\|Q_n x_n\| \to +\infty$ which is impossible due to (2.9) and the boundedness of the sequence $\{x_n\}$. Therefore, there does not exist a nonzero $\boldsymbol{x} \in l_X^\infty$ such that $L\boldsymbol{x} = 0$. Thus, the mapping L is injective.

Let us now prove the surjectivity of L. Let us consider an operator R in the space l_X^∞ acting according to the formula

$$(R\boldsymbol{y})_n = Q_n y_n - B_n(1-Q_{n+1})y_{n+1}, \qquad \boldsymbol{y} = \{y_n\} \in l_X^\infty,$$

where the operator $B_n = [(1-Q_{n+1})A_n]^{-1}$ acts from $(1-Q_{n+1})X$ into $(1-Q_n)X$ and is inverse to $(1-Q_{n+1})A_n\big|_{(1-Q_n)X}$. It follows from (2.9) and (2.11) that

$$\|R\boldsymbol{y}\|_{l^\infty} \leq (K+\lambda)\|\boldsymbol{y}\|_{l^\infty}, \qquad \boldsymbol{y} \in l_X^\infty. \tag{2.21}$$

It is evident that

$$(LR\boldsymbol{y})_n - y_n = -(1-Q_n)y_n - B_n(1-Q_{n+1})y_{n+1} -$$
$$- A_{n-1}Q_{n-1}y_{n-1} + A_{n-1}B_{n-1}(1-Q_n)y_n .$$

Since

$$(1-Q_n)A_{n-1}B_{n-1}(1-Q_n) = 1-Q_n ,$$

we have that

$$(LR\boldsymbol{y})_n - y_n = -B_n(1-Q_{n+1})y_{n+1} - A_{n-1}Q_{n-1}y_{n-1} +$$
$$+ Q_n A_{n-1}(1-Q_{n-1})B_{n-1}(1-Q_n)y_n .$$

Consequently,

$$\|(LR\boldsymbol{y})_n - y_n\| \leq \|B_n(1-Q_{n+1})\| \cdot \|y_{n+1}\| + \|A_{n-1}Q_{n-1}\| \cdot \|y_{n-1}\| +$$
$$+ \|Q_n A_{n-1}(1-Q_{n-1})\| \cdot \|B_{n-1}(1-Q_n)\| \cdot \|y_n\| .$$

Therefore, inequalities (2.10), (2.11), and (2.12) give us that

$$\|LR\boldsymbol{y} - \boldsymbol{y}\|_{l^\infty} \leq \lambda(2+\delta)\|\boldsymbol{y}\|_{l^\infty} \leq \frac{1}{2}\|\boldsymbol{y}\|_{l^\infty},$$

i.e. $\|LR - 1\| \leq 1/2$. That means that the operator LR is invertible and

$$\|(LR)^{-1}\| \leq (1-\|LR-1\|)^{-1} \leq 2 . \tag{2.22}$$

Let $\boldsymbol{h} = \{h_n\}$ be an arbitrary element of l_X^∞. Then it is evident that the element $\boldsymbol{y} = R(LR)^{-1}\boldsymbol{h}$ is a solution to equation $L\boldsymbol{y} = \boldsymbol{h}$. Moreover, it follows from (2.21) and (2.22) that

$$\|\boldsymbol{y}\|_{l^\infty} \leq 2(K+\lambda)\|\boldsymbol{h}\|_{l^\infty} .$$

Hence, L is surjective and $\|L^{-1}\| \leq 2K+1$. **Theorem 2.2 is proved**.

§ 3 Hyperbolicity of Invariant Sets for Differentiable Mappings

Let us remind the definition of the differentiable mapping. Let X and Y be Banach spaces and let $\mathcal{U}$ be an open set in X. The mapping f from $\mathcal{U}$ into Y is called (Frechét) ***differentiable*** at the point $x \in \mathcal{U}$ if there exists a linear bounded operator $Df(x)$ from X into Y such that

$$\lim_{\|v\| \to 0} \frac{1}{\|v\|} \|f(x+v) - f(x) - Df(x)v\| = 0 .$$

If the mapping f is differentiable at every point $x \in \mathcal{U}$, then the mapping Df: $x \to Df(x)$ acts from $\mathcal{U}$ into the Banach space $\mathcal{L}(X, Y)$ of all linear bounded operators from X into Y. If $Df: \mathcal{U} \to \mathcal{L}(X, Y)$ is continuous, then the mapping f is said to be ***continuously differentiable*** (or C^1-mapping) on $\mathcal{U}$. The notion of the derivative of any order can also be introduced by means of induction. For example, $D^2 f(x)$ is the Frechét derivative of the mapping $Df: \mathcal{U} \to \mathcal{L}(X, Y)$.

— Exercise *3.1* Let g and f be continuously differentiable mappings from $\mathcal{U} \subset X$ into Y and from $\mathcal{W} \subset Y$ into Z, respectively. Moreover, let $\mathcal{U}$ and $\mathcal{W}$ be open sets such that $g(\mathcal{U}) \subset \mathcal{W}$. Prove that $(f \circ g)(x) = f(g(x))$ is a C^1-mapping on $\mathcal{U}$ and obtain a chain rule for the differentiation of a composed function

$$D(f \circ g)(x) = Df(g(x))Dg(x), \qquad x \in \mathcal{U}.$$

— Exercise *3.2* Let f be a continuously differentiable mapping from X into X and let f^n be the n-th degree of the mapping f, i.e. $f^n(x) = f(f^{n-1}(x))$, $n \ge 1$, $f^1(x) \equiv f(x)$. Prove that f^n is a C^1-mapping on X and

$$(Df^n)(x) = Df(f^{n-1}(x)) \cdot Df(f^{n-2}(x)) \cdot \ldots \cdot Df(x) . \tag{3.1}$$

Now we give the definition of a hyperbolic set. Assume that f is a continuously differentiable mapping from a Banach space X into itself and Λ is a subset in X which is invariant with respect to f $(f(\Lambda) \subset \Lambda)$. The set Λ is called ***hyperbolic*** (with respect to f) if there exists a collection of projectors $\{P(x)\colon x \in \Lambda\}$ such that

a) $P(x)$ continuously depends on $x \in \Lambda$ with respect to the operator norm;

b) for every $x \in \Lambda$
$$Df(x) \cdot P(x) = P(f(x)) \cdot Df(x); \tag{3.2}$$

c) the mappings $Df(x)$ are invertible for every $x \in \Lambda$ as linear operators from $(1-P(x))X$ into $(I-P(f(x)))X$;

d) for every $x \in \Lambda$ the following equations hold:

$$\|(Df^n)(x)P(x)\| \le Kq^n, \quad n \ge 0, \tag{3.3}$$

$$\|[(Df^n)(x)]^{-1}(1-P(f^n(x)))\| \le Kq^n, \quad n \ge 0, \tag{3.4}$$

with the constants $K > 0$ and $0 < q < 1$ independent of $x \in \Lambda$. Here f^n is the n-th degree of the mapping f ($f^n(x) = f(f^{n-1}(x))$ for $n \ge 1$ and $f^1(x) \equiv f(x)$).

It should be noted that properties (b) and (c) as well as formula (3.1) enable us to state that $(Df^n)(x)$ maps $(1-P(x))X$ into $(1-P(f^n(x)))$ and is an invertible operator. Therefore, the value in the left-hand side of inequality (3.4) exists.

Exercise *3.3* Let $\Lambda = \{x_0\}$, where x_0 is a fixed point of a C^1-mapping f, i.e. $f(x_0) = x_0$. Then for the set Λ to be hyperbolic it is necessary and sufficient that the spectrum of the linear operator $Df(x_0)$ does not intersect the unit circumference (*Hint*: see Example 2.1).

Let Λ be an invariant hyperbolic set of a C^1-mapping f and let $\gamma = \{x_n : n \in \mathbb{Z}\}$ be a complete trajectory (in Λ) for f, i.e. $\gamma = \{x_n\}$ is a sequence of points from Λ such that $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Let us consider a difference equation obtained as a result of linearization of the mapping f along γ:

$$u_{n+1} = Df(x_n)u_n, \quad n \in \mathbb{Z}. \tag{3.5}$$

Exercise *3.4* Prove that the evolutionary operator $\Phi(m, n)$ of difference equation (3.5) has the form

$$\Phi(m, n) = (Df^{m-n})(x_n), \quad m > n, \quad m, n \in \mathbb{Z}.$$

Exercise *3.5* Prove that difference equation (3.5) admits an exponential dichotomy over $\mathbb{Z}$ with (i) the constants K and q given by equations (3.3) and (3.4) and (ii) the projectors $P_n = P(x_n)$ involved in the definition of the hyperbolicity.

It should be noted that property (a) of uniform continuity implies that the projectors $P(x)$ are similar to one another, provided the values of x are close enough. The proof of this fact is based on the following assertion.

Lemma 3.1.

Let P and Q be projectors in a Banach space X. Assume that

$$\|P\| \le K, \quad \|1-P\| \le K, \quad \|P-Q\| < \frac{1}{2K} \tag{3.6}$$

for some constant $K \ge 1$. Then the operator

$$J = PQ + (1-P)(1-Q) \tag{3.7}$$

possesses the property $PJ = JQ$ *and is invertible, therewith*

$$\|J^{-1}\| \le (1-2K\cdot\|P-Q\|)^{-1} . \tag{3.8}$$

Proof.

Since $P^2+(1-P)^2 = 1$, we have

$$J-1 = J-P^2-(1-P)^2 = P(Q-P)+(1-P)(P-Q) .$$

It follows from (3.6) that

$$\|J-1\| \le 2K\|P-Q\| < 1 .$$

Hence, the operator J^{-1} can be defined as the following absolutely convergent series

$$J^{-1} = \sum_{n=0}^{\infty}(1-J)^n .$$

This implies estimate (3.8). The permutability property $PJ = JQ$ is evident. Lemma 3.1 is proved.

Exercise *3.6* Let Λ be a connected compact set and let $\{P(x): x\in\Lambda\}$ be a family of projectors for which condition (a) of the hyperbolicity definition holds. Then all operators $P(x)$ are similar to one another, i.e. for any $x, y \in \Lambda$ there exists an invertible operator $J = J_{x,\, y}$ such that $P(x) = JP(y)J^{-1}$.

The following assertion contains a description of a situation when the hyperbolicity of the invariant set is equivalent to the existence of an exponential dichotomy for difference equation (3.5) (cf. Exercise 3.5).

Theorem 3.1.

Let $f(x)$ ***be a continuously differentiable mapping of the space*** X ***into itself. Let*** x_0 ***be a hyperbolic fixed point of*** f ***(***$f(x_0)=x_0$***) and let*** $\{y_n: n\in\mathbb{Z}\}$ ***be a homoclinic trajectory (not equal to*** x_0***) of the mapping*** f***, i.e.***

$$f(y_n) = y_{n+1}, \qquad n\in\mathbb{Z}, \qquad y_n\to x_0, \quad n\to\pm\infty . \tag{3.9}$$

Then the set $\Lambda = \{x_0\}\cup\{y_n: n\in\mathbb{Z}\}$ ***is hyperbolic if and only if the difference equation***

$$u_{n+1} = Df(y_n)u_n, \qquad n\in\mathbb{Z}, \tag{3.10}$$

possesses an exponential dichotomy over $\mathbb{Z}$***.***

Proof.

If Λ is hyperbolic, then (see Exercise 3.5) equation (3.10) possesses an exponential dichotomy over $\mathbb{Z}$. Let us prove the converse assertion. Assume that equation (3.10) possesses an exponential dichotomy over $\mathbb{Z}$ with projectors $\{P_n: n\in\mathbb{Z}\}$

and constants K and q. Let us denote the spectral projector of the operator $Df(x_0)$ corresponding to the part of the spectrum inside the unit disc by P. Without loss of generality we can assume that

$$\left\|[Df(x_0)]^n P\right\| \le Kq^n, \qquad n \ge 0,$$

$$\left\|[Df(x_0)]^{-n}(1-P)\right\| \le Kq^{-n}, \qquad n \ge 0.$$

Thus, for every $x \in \Lambda$ the projector $P(x)$ is defined: $P(x_0) = P$, $P(y_k) = P_k$. The structure of the evolutionary operator of difference equation (3.10) (see Exercise 3.4) enables us to verify properties (b)–(d) of the definition of a hyperbolic set. In order to prove property (a) it is sufficient to verify that

$$\left\|P_k - P\right\| \to 0 \quad \text{as} \quad k \to \pm\infty. \tag{3.11}$$

Since Λ is a compact set, then

$$M = \sup\{\|Df(x)\|:\ x \in \Lambda\} < \infty. \tag{3.12}$$

Let us consider the following difference equations

$$v_{n+1} = Df(x_0)v_n, \qquad n \in \mathbb{Z}, \tag{3.13}$$

and

$$w_{n+1}^{(k)} = Df(y_{n+k})w_n^{(k)}, \qquad n \in \mathbb{Z}, \tag{3.14}$$

where k is an integer. It is evident that equation (3.14) admits an exponential dichotomy over $\mathbb{Z}$ with constants K and q and projectors $P_n^{(k)} = P_{n+k}$. Let $G(n, m)$ and $G^{(k)}(n, m)$ be the Green functions (see Section 2) of difference equations (3.13) and (3.14). We consider the sequence

$$x_n = G(n, 0)z - G^{(k)}(n, 0)z, \qquad z \in X.$$

Since (see Exercise 2.8)

$$\|G(n, 0)\| \le Kq^{|n|}, \qquad \|G^{(k)}(n, 0)\| \le Kq^{|n|}, \tag{3.15}$$

we have that the sequence $\{x_n\}$ is bounded. Moreover, it is easy to prove (see Exercise 2.9) that $\{x_n\}$ is a solution to the difference equation

$$x_{n+1} - Df(x_0)x_n = h_n \equiv [Df(x_0) - Df(y_{n+k})]\, G^{(k)}(n, 0)z.$$

It follows from (3.12) and (3.15) that the sequence $\{h_n\}$ is bounded. Therefore, (see Exercise 2.9),

$$G(n, 0)z - G^{(k)}(n, 0)z \equiv x_n = \sum_{m \in \mathbb{Z}} G(n, m+1)h_m.$$

If we take $n = 0$ in this formula, then from the definition of the Green function we obtain that

$$(P - P_k)z = \sum_{m \in \mathbb{Z}} G(0, m+1)(Df(x_0) - Df(y_{m+k}))\, G^{(k)}(m, 0)z.$$

Therefore, equation (3.15) implies that

$$\|(P-P_k)z\| \le K^2 \sum_{m \in \mathbb{Z}} \|Df(x_0) - Df(y_{m+k})\| \cdot q^{|m|+|m+1|} \cdot \|z\| .$$

Consequently,

$$\|P-P_k\| \le K^2 \sum_{m \in \mathbb{Z}} \|Df(x_0) - Df(y_{m+k})\| \cdot q^{2|m|-1} \le$$

$$\le K^2 \left\{ \max_{|m| \le N} \|Df(x_0) - Df(y_{m+k})\| \cdot \sum_{|m| \le N} q^{2|m|-1} + 2M \sum_{|m| > N} q^{2|m|-1} \right\},$$

where N is an arbitrary natural number. Upon simple calculations we find that

$$\|P-P_k\| \le \frac{2K^2}{q(1-q^2)} \left(\max_{|m| \le N} \|Df(x_0) - Df(y_{m+k})\| + 2Mq^{2N+2} \right)$$

for every $N \ge 1$. It follows that

$$\overline{\lim_{k \to \pm\infty}} \|P-P_k\| \le \frac{4K^2M}{1-q^2} q^{2N+1}, \qquad N = 1,\ 2,\ \ldots .$$

We assume that $N \to +\infty$ to obtain that

$$\overline{\lim_{k \to \pm\infty}} \|P-P_k\| \le 0 .$$

This implies equation (3.11). Therefore, **Theorem 3.1 is proved**.

It should be noted that in the case when the set $\Lambda = \{x_0\} \cup \{y_n\colon\ n \in \mathbb{Z}\}$ from Theorem 3.1 is hyperbolic the elements y_n of the homoclinic trajectory $\gamma = \{y_n\colon\ n \in \mathbb{Z}\}$ are called ***transversal homoclinic points***. The point is that in some cases (see, e.g., [4]) it can be proved that the hyperbolicity of Λ is equivalent to the transversality property at every point y_n of the stable $W^s(x_0)$ and unstable $W^u(x_0)$ manifolds of a fixed point x_0 (roughly speaking, transversality means that the surfaces $W^s(x_0)$ and $W^u(x_0)$ intersect at the point y_n at a nonzero angle). In this case the trajectory γ is often called a ***transversal homoclinic trajectory***.

§ 4 Anosov's Lemma on ε-trajectories

Let f be a C^1-mapping of a Banach space X into itself. A sequence $\{y_n\colon\ n \in \mathbb{Z}\}$ in X is called a δ***-pseudotrajectory*** (or δ-pseudoorbit) of the mapping f if for all $n \in \mathbb{Z}$ the equation

$$\|y_{n+1} - f(y_n)\| \le \delta$$

is valid. A sequence $\{x_n: \ n \in \mathbb{Z}\}$ is called an ε***-trajectory*** of the mapping f corresponding to a δ-pseudotrajectory $\{y_n: \ n \in \mathbb{Z}\}$ if

(a) $f(x_n) = x_{n+1}$ for any $n \in \mathbb{Z}$;

(b) $\|x_n - y_n\| \le \varepsilon$ for all $n \in \mathbb{Z}$.

It should be noted that condition (a) means that $\{x_n\}$ is an orbit (complete trajectory) of the mapping f. Moreover, if a pair of C^1-mappings f and g is given, then the notion of the ε-trajectory of the mapping g corresponding to a δ-pseudoorbit of the mapping f can be introduced.

The following assertion is the main result of this section.

Theorem 4.1.

Let f be a C^1-mapping of a Banach space X into itself and let Λ be a hyperbolic invariant $(f(\Lambda) \subset \Lambda)$ set. Assume that there exists a Δ-vicinity $\mathcal{O}$ of the set Λ such that $f(x)$ and $Df(x)$ are bounded and uniformly continuous on the closure $\overline{\mathcal{O}}$ of the set $\mathcal{O}$. Then there exists $\varepsilon_0 > 0$ possessing the property that for every $0 < \varepsilon \le \varepsilon_0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any δ-pseudoorbit $\{y_n: \ n \in \mathbb{Z}\}$ lying in Λ has a unique ε-trajectory $\{x_n: \ n \in \mathbb{Z}\}$ corresponding to $\{y_n\}$.

As the following theorem shows, the property of the mapping f to have an ε-trajectory is rough, i.e. this property also remains true for mappings that are close to f.

Theorem 4.2.

Assume that the hypotheses of Theorem 4.1 hold for the mapping f. Let $\mathcal{W}_\eta(f)$ be a set of continuously differentiable mappings g of the space X into itself such that the following estimates hold on the closure $\overline{\mathcal{O}}$ of the Δ-vicinity $\mathcal{O}$ of the set Λ:

$$\|f(x) - g(x)\| < \eta, \qquad \|Df(x) - Dg(x)\| < \eta. \tag{4.1}$$

Then $\varepsilon_0 > 0$ can be chosen to possess the property that for every $\varepsilon \in (0, \varepsilon_0]$ there exist $\delta = \delta(\varepsilon) > 0$ and $\eta = \eta(\varepsilon) > 0$ such that for any δ-pseudotrajectory $\{y_n: \ n \in \mathbb{Z}\}$ (lying in Λ) of the mapping f and for any $g \in \mathcal{W}_\eta(f)$ there exists a unique trajectory $\{x_n: \ n \in \mathbb{Z}\}$ of the mapping g with the property

$$\|y_n - x_n\| \le \varepsilon \quad \textit{for all} \quad n \in \mathbb{Z}.$$

It is clear that Theorem 4.1 is a corollary of Theorem 4.2 the proof of which is based on the lemmata below.

Lemma 4.1.

Let $\mathcal{U}$ be an open set in a Banach space X and let $\mathcal{F}\colon \mathcal{U} \to X$ be a continuously differentiable mapping. Assume that for some point $y \in \mathcal{U}$ there exist an operator $[D\mathcal{F}(y)]^{-1}$ and a number $\varepsilon_0 > 0$ such that

$$\|D\mathcal{F}(x) - D\mathcal{F}(y)\| \le \left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.2}$$

for all x with the property $\|x-y\| \le \varepsilon_0$. Assume that for some $\varepsilon \in (0, \varepsilon_0]$ the inequality

$$\|\mathcal{F}(y)\| \le \bar{q} \cdot \varepsilon \left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.3}$$

is valid with $0 < \bar{q} < 1$. Then for any C^1-mapping $\mathcal{G}\colon \mathcal{U} \to X$ such that

$$\|\mathcal{G}(x) - \mathcal{F}(x)\| \le \varepsilon(1-\bar{q})\left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.4}$$

and

$$\|D\mathcal{G}(x) - D\mathcal{F}(x)\| \le \frac{1}{2}\left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.5}$$

for $\|x-y\| \le \varepsilon_0$, the equation $\mathcal{G}(x) = 0$ has a unique solution x with the property $\|x-y\| \le \varepsilon$.

Proof.

Let $\Gamma = (2\|[D\mathcal{F}(y)]^{-1}\|)^{-1}$ and let

$$\eta(x) = \mathcal{F}(x) - \mathcal{F}(y) - D\mathcal{F}(y)(x-y)\,.$$

For $x_1, x_2 \in B_{\varepsilon_0} \equiv \{z\colon \|z-y\| \le \varepsilon_0\}$ we have that

$$\eta(x_1) - \eta(x_2) = \mathcal{F}(x_1) - \mathcal{F}(x_2) - D\mathcal{F}(y)(x_1 - x_2) =$$

$$= \int_0^1 (D\mathcal{F}(x_1 + \xi(x_2 - x_1)) - D\mathcal{F}(y))(x_1 - x_2)\,\mathrm{d}\xi\,.$$

Since

$$\|\eta(x_1) - \eta(x_2)\| \le \int_0^1 \|D\mathcal{F}(x_1 + \xi(x_2 - x_1)) - D\mathcal{F}(y)\|\,\mathrm{d}\xi\,\|x_1 - x_2\|\,,$$

it follows from (4.2) that

$$\|\eta(x_1) - \eta(x_2)\| \le \Gamma\|x_1 - x_2\| \tag{4.6}$$

for all x_1 and x_2 from B_{ε_0}. Now we rewrite the equation $\mathcal{G}(x) = 0$ in the form

$$x = T(x) \equiv y - [D\mathcal{F}(y)]^{-1}(\mathcal{G}(x) - \mathcal{F}(x) + \mathcal{F}(y) + \eta(x))\,.$$

Let us show that the mapping T has a unique fixed point in the ball $B_\varepsilon = \{x: \|x-y\| \le \varepsilon\}$. It is evident that

$$\|T(x)-y\| \le \|[D\mathscr{F}(y)]^{-1}\|\left(\|\mathscr{G}(x)-\mathscr{F}(x)\|+\|\mathscr{F}(y)\|+\|\eta(x)\|\right)$$

for any $x \in B_\varepsilon$. Since $\eta(y) = 0$, we obtain from (4.6) that

$$\|\eta(x)\| = \|\eta(x)-\eta(y)\| \le \Gamma\|x-y\| \le \Gamma\varepsilon .$$

Therefore, estimates (4.3) and (4.4) imply that

$$\|T(x)-y\| \le \varepsilon \quad \text{for } x \in B_\varepsilon ,$$

i.e. T maps the ball B_ε into itself. This mapping is contractive in B_ε. Indeed,

$$\|T(x_1)-T(x_2)\| \le \frac{1}{2\Gamma}\left(\|\mathscr{H}(x_1, x_2)\|+\|\eta(x_1)-\eta(x_2)\|\right),$$

where

$$\mathscr{H}(x_1, x_2) \equiv \mathscr{G}(x_1)-\mathscr{G}(x_2)-\mathscr{F}(x_1)+\mathscr{F}(x_2) =$$

$$= \int_0^1 [D\mathscr{G}(x_1+\xi(x_1-x_2))-D\mathscr{F}(x_1+\xi(x_1-x_2))]\,d\xi\,(x_1-x_2) .$$

It follows from (4.5) that

$$\|\mathscr{H}(x_1, x_2)\| \le \frac{1}{2}\Gamma\|x_1-x_2\| .$$

This equation and inequality (4.6) imply the estimate

$$\|T(x_1)-T(x_2)\| \le \frac{3}{4}\|x_1-x_2\| .$$

Therefore, the mapping T has a unique fixed point in the ball $B_\varepsilon = \{x: \|x-y\| \le \varepsilon\}$. The lemma is proved.

Let the hypotheses of Theorems 4.1 and 4.2 hold. We assume that $\eta < 1$ in (4.1). Then for any element $g \in \mathscr{W}_\eta(f)$ the following estimates hold:

$$\|g(x)\| \le M, \qquad \|Dg(x)\| \le M, \qquad x \in \overline{\mathscr{O}}, \tag{4.7}$$

where $M > 0$ is a constant. In particular, these estimates are valid for the mapping f.

Lemma 4.2.

Let $\{y_n: n \in \mathbb{Z}\}$ be a δ-pseudotrajectory of the mapping f lying in Λ. Then for any $k \ge 1$ the sequence $\{z_n \equiv y_{nk}: n \in \mathbb{Z}\}$ is a $\delta\cdot M_{k-1}$-pseudotrajectory of the mapping f^k. Here M_k has the form

$$M_k = 1+M+\ldots+M^k, \qquad k \ge 1, \quad M_0 = 1, \tag{4.8}$$

and M is a constant from (4.7).

Proof.

Let us use induction to prove that

$$\left\|y_{nk+i}-f^{i}(y_{nk})\right\| \le \delta\cdot M_{i-1}, \qquad 1\le i\le k. \tag{4.9}$$

Since $\{y_n\}$ is a δ-pseudotrajectory, then it is evident that for $i=1$ inequality (4.9) is valid. Assume that equation (4.9) is valid for some $i\ge 1$ and prove estimate (4.9) for $i+1$:

$$\left\|y_{nk+i+1}-f^{i+1}(y_{nk})\right\| \le \left\|y_{nk+i+1}-f(y_{nk+i})\right\|+\left\|f(y_{nk+i})-f(f^{i}(y_{nk}))\right\|.$$

With the help of (4.7) we obtain that

$$\left\|y_{nk+i+1}-f^{i+1}(y_{nk})\right\| \le \delta+M\left\|y_{nk+i}-f^{i}(y_{nk})\right\| \le \delta+M\cdot\delta M_{i-1}=\delta\cdot M_i.$$

Thus, Lemma 4.2 is proved.

Lemma 4.3.

Let $\{y_n\}$ be a δ-pseudoorbit of the mapping f lying in Λ. Let $\{x_n\}$ be a trajectory of the mapping $g\in\mathcal{W}_\eta(f)$ such that

$$\left\|y_{nk}-x_{nk}\right\|\le\varepsilon, \qquad n\in\mathbb{Z}, \tag{4.10}$$

for some $k\ge 1$. If

$$\max(\varepsilon,\ \delta+\eta)M_k\le\Delta, \tag{4.11}$$

then

$$\left\|y_n-x_n\right\|\le\max(\varepsilon,\ \delta+\eta)\cdot M_k, \tag{4.12}$$

where M_k has the form (4.8).

Proof.

We first note that

$$\left\|y_{nk+1}-x_{nk+1}\right\| = \left\|y_{nk+1}-g(x_{nk})\right\| \le$$

$$\le \left\|y_{nk+1}-f(y_{nk})\right\|+\left\|f(y_{nk})-g(y_{nk})\right\|+\left\|g(y_{nk})-g(x_{nk})\right\|.$$

Therefore, it is evident that

$$\left\|y_{nk+1}-x_{nk+1}\right\| \le \delta+\eta+M\left\|y_{nk}-x_{nk}\right\| \le \max(\varepsilon,\ \delta+\eta)(1+M). \tag{4.13}$$

Here we use the estimate

$$\|g(y)-g(x)\| \le \int_0^1\|Dg(y+\xi(x-y))\|\,\mathrm{d}\xi\cdot\|x-y\| \le M\|x-y\|$$

which follows from (4.7) and holds when the segment connecting the points x and y lies in $\overline{\mathcal{O}}$. Condition (4.11) guarantees the fulfillment of this property at each stage of reasoning. If we repeat the arguments from the proof of (4.13), then it is easy to complete the proof of (4.12) using induction as in Lemma 4.2. Lemma 4.3 is proved.

Lemma 4.4.

Let $y \in \Lambda$ *and* $\|x-y\| \le \varepsilon$. *Assume that*

$$\max(\varepsilon, \eta)(1+\ldots+M^{k-1}) < \Delta. \tag{4.14}$$

Then the estimates

$$\|f^j(y) - f^j(x)\| \le M^j\|x-y\|, \qquad \|Df^j(x)\| \le M^j, \tag{4.15}$$

$$\|f^j(y)-g^j(x)\| \le \max(\varepsilon, \eta)(1+\ldots+M^j), \tag{4.16}$$

$$\|f^j(x)-g^j(x)\| \le \eta(1+\ldots+M^{j-1}) \tag{4.17}$$

are valid for $j = 1, 2, \ldots, k$ *and for every mapping* $g \in \mathcal{W}_\eta(f)$.

Proof.

As above, let us use induction. If $j = 1$, it is evident that equations (4.15)–(4.17) hold. The transition from j to $j+1$ in (4.15) is evident. Let us consider estimate (4.16):

$$f^{j+1}(y) - g^{j+1}(x) = \Big(f(f^j(y)) - f(g^j(x))\Big) + f(g^j(x)) - g(g^j(x)). \tag{4.18}$$

Condition (4.14) and the induction assumption give us that $g^j(x)$ lies in the ball with the centre at the point $f^j(y) \in \Lambda$ lying in $\mathcal{O}$. Therefore, it follows from (4.18) that

$$\|f^{j+1}(y)-g^{j+1}(x)\| \le M\|f^j(y)-g^j(x)\| + \eta \le \max(\varepsilon, \eta)(1+\ldots+M^{j+1}).$$

The transition from j to $j+1$ in (4.17) can be made in a similar way. Lemma 4.4 is proved.

Lemma 4.5.

There exists $\Delta' \le \Delta$ *such that the equations*

$$\sup\left\{\|f^k(x)-g^k(x)\|:\ x \in \mathcal{O}'\right\} \le \rho_k(\eta) \tag{4.19}$$

and

$$\sup\left\{\|(Df^k)(x)-(Dg^k)(x)\|:\ x \in \mathcal{O}'\right\} \le \rho_k(\eta) \tag{4.20}$$

are valid in the Δ'*-vicinity* $\mathcal{O}'$ *of the set* Λ *for any function* $g \in \mathcal{W}_\eta(f)$. *Here* $\rho_k(\eta) \to 0$ *as* $\eta \to 0$.

The proof follows from the definition of the class of functions $\mathcal{W}_\eta(f)$ and estimates (4.7) and (4.17).

Let us also introduce the values

$$\omega_k(\varkappa) = \sup\left\{\|Df^k(y) - Df^k(x)\|:\ y \in \Lambda,\ \|x-y\| \le \varkappa\right\} \tag{4.21}$$

and

$$\omega(\varkappa) = \sup\left\{\|P(y)-P(x)\|,\quad x, y \in \Lambda,\ \|x-y\| \le \varkappa\right\}. \tag{4.22}$$

The requirement of the uniform continuity of the derivative $Df(x)$ (see the hypotheses of Theorem 4.1) and the projectors $P(x)$ (see the hyperbolicity definition) enables us to state that

$$\omega_k(\varkappa) \to 0\,, \qquad \omega(\varkappa) \to 0 \ \text{ as } \ \varkappa \to 0\,. \tag{4.23}$$

Let $\{y_n\}$ be a δ-pseudotrajectory of the mapping f lying in Λ. Then due to Lemma 4.2 the sequence $\{\bar{y}_n = y_{nk}:\ n \in \mathbb{Z}\}$ is a δM_{k-1}-pseudotrajectory of the mapping f^k. Let us consider the mappings $\mathcal{F}(\boldsymbol{x})$ and $\mathcal{G}(\boldsymbol{x})$ in the space $l_X^\infty \equiv l^\infty(\mathbb{Z}, X)$ (for the definition see Section 2) given by the equalities

$$[\mathcal{F}(\boldsymbol{x})]_n = \bar{y}_n + x_n - f^k(\bar{y}_{n-1} + x_{n-1})\,, \tag{4.24}$$

$$[\mathcal{G}(\boldsymbol{x})]_n = \bar{y}_n + x_n - g^k(\bar{y}_{n-1} + x_{n-1})\ , \tag{4.25}$$

where $\boldsymbol{x} = \{x_n:\ n \in \mathbb{Z}\}$ is an element from l_X^∞. Thus, the construction of ε-trajectories of the mapping f^k and g^k corresponding to the sequence $\{\bar{y}_n\}$ is reduced to solving of the equations

$$\mathcal{F}(\boldsymbol{x}) = 0 \ \text{ and } \ \mathcal{G}(\boldsymbol{x}) = 0$$

in the ball $\{\boldsymbol{x}:\ \|\boldsymbol{x}\|_{l_X^\infty} \le \varepsilon\}$. Let us show that for k large enough Lemma 4.1 can be applied to the mappings $\mathcal{F}$ and $\mathcal{G}$. Let us start with the mapping $\mathcal{F}$.

Lemma 4.6.

The function $\mathcal{F}$ is a C^1-smooth mapping in l_X^∞ with the properties

$$\|\mathcal{F}(0)\| \le \delta M_{k-1}\ , \tag{4.26}$$

$$\|D\mathcal{F}(\boldsymbol{x}) - D\mathcal{F}(0)\| \ \le\ \omega_k(\varepsilon)\,, \qquad \|\boldsymbol{x}\|_{l_X^\infty} \le \varepsilon\,. \tag{4.27}$$

Proof.

Estimate (4.26) follows from the fact that $\{\bar{y}_n\}$ is a δM_{k-1}-pseudotrajectory. Then it is evident that

$$[D\mathcal{F}(\boldsymbol{x})\boldsymbol{h}]_n \ =\ h_n - Df^k(\bar{y}_{n-1} + x_{n-1})h_{n-1}\,, \tag{4.28}$$

where $\boldsymbol{x} = \{x_n:\ n \in \mathbb{Z}\}$ and $\boldsymbol{h} = \{h_n:\ n \in \mathbb{Z}\}$ lie in l_X^∞. Therefore, simple calculations and equation (4.21) give us (4.27).

In order to deduce relations (4.2) and (4.3) from inequalities (4.26) and (4.27) for $\boldsymbol{y} = 0$, we use Theorem 2.2. Consider the operator $L = D\mathcal{F}(0)$. It is clear that

$$[L\boldsymbol{h}]_n = h_n - Df^k(\bar{y}_{n-1})h_{n-1}, \qquad \boldsymbol{h} = \{h_n\}.$$

Let us show that equations (2.9)–(2.11) are valid for $A_n = Df^k(\bar{y}_{n-1})$ and $Q_n = P(\bar{y}_{n-1})$ and then estimate the corresponding constants. Property (2.9) follows from the hyperbolicity definition. Equations (3.3) and (4.15) imply that

$$\|A_n Q_n\| \le Kq^k \quad \text{and} \quad \|A_n\| \le M^k.$$

Further, the permutability property (3.2) gives us that

$$Q_{n+1}A_n(1-Q_n) = [Q_{n+1}A_n - A_nQ_n](1-Q_n) =$$

$$= \Big[Q_{n+1} - P(f^k(\bar{y}_{n-1}))\Big]A_n(1-Q_n)\,.$$

Hence (see (4.22)),

$$\|Q_{n+1}A_n(1-Q_n)\| \le \omega(\delta M_{k-1})\cdot\|A_n\|\cdot\|1-Q_n\| \le \omega(\delta M_{k-1})M^k\cdot K\,.$$

Similarly, we find that

$$\|(1-Q_{n+1})A_nQ_n\| \le \omega(\delta M_{k-1})M^k\cdot K\,.$$

The operator

$$J_n = Q_{n+1}P(f^k(\bar{y}_{n-1})) - (1-Q_{n+1})(1-P(f^k(\bar{y}_{n-1})))$$

is invertible if (see Lemma 3.1)

$$\|Q_{n+1} - P(f^k(\bar{y}_{n-1}))\| \le \omega(\delta M_{k-1}) < \frac{1}{2K}\,.$$

Moreover,

$$\|J_n^{-1}\| \le (1 - 2K\omega(\delta M_{k-1}))^{-1} < 2\,,$$

provided $2K\omega(\delta M_{k-1}) < 1/2$. Due to the hyperbolicity of the set Λ, the operator A_n is an invertible mapping from $(1-Q_n)X$ into $(1-P(f^k(\bar{y}_{n-1})))X$. Therefore, since

$$(1-Q_{n+1})A_n(1-Q_n) = J_nA_n(1-Q_n)\,,$$

the operator $(1-Q_{n+1})A_n(1-Q_n)$ is invertible as a mapping from $(1-Q_n)X$ into $(1-Q_{n+1})X$. Moreover, by virtue of (3.4) we have that

$$\|[(1-Q_{n+1})A_n]^{-1}(1-Q_{n+1})\| = \|[A_n(1-Q_n)]^{-1}\cdot J_n^{-1}(1-Q_{n+1})\| \le 2K^2q^k\,.$$

Thus, under the conditions

$$4K\omega(\delta M_{k-1}) < 1\,, \quad 8KM^k\omega(\delta M_{k-1}) \le 1\,, \quad 16K^3q^k \le 1\,, \tag{4.29}$$

Theorem 2.2 implies that the operator $L = D\mathcal{F}(0)$ is invertible and $\|L^{-1}\| \le 2K+1$. Let us fix some $\bar{\varepsilon} > 0$. If

$$\delta M_{k-1} \le \frac{1}{2}(4K+2)^{-1}\cdot\bar{\varepsilon}\,, \qquad \omega_k(\bar{\varepsilon}) \le (4K+2)^{-1}\,, \tag{4.30}$$

then by Lemma 4.6 relations (4.2) and (4.3) hold with $y = 0$, $\varepsilon = \bar{\varepsilon}$, and $\bar{q} = 1/2$. If

$$\rho_k(\eta) \le \frac{1}{2}\min(\bar{\varepsilon},\ 1)(4K+2)^{-1}\,, \tag{4.31}$$

then equations (4.4) and (4.5) also hold with $\boldsymbol{y} = 0$, $\varepsilon = \bar{\varepsilon}$, and $\bar{q} = 1/2$. Hence, under conditions (4.29)–(4.31) there exists a unique solution to equation $\mathcal{G}(\boldsymbol{x}) = 0$ possessing the property $\|\boldsymbol{x}\|_{l^\infty} \le \bar{\varepsilon}$. This means that for any δ-pseudoorbit $\{y_n\colon n \in \mathbb{Z}\}$ (lying in Λ) of the mapping f there exists a unique trajectory $\{z_n\colon n \in \mathbb{Z}\}$ of the mapping $g \in \mathcal{W}_\eta(f)$ such that

$$\|z_{nk} - y_{nk}\| \le \bar{\varepsilon}\,, \qquad n \in \mathbb{Z}\,,$$

provided conditions (4.29)–(4.31) hold. Therefore, under the additional condition

$$(\bar{\varepsilon} + \delta + \eta)M_k \le \Delta$$

and due to Lemma 4.3 we get

$$\|y_n - z_n\| \le (\bar{\varepsilon} + \delta + \eta)M_k\,, \qquad n \in \mathbb{Z}\,.$$

These properties are sufficient for the completion of the proof of Theorem 4.2.

Let us fix k such that $16K^3q^k \le 1$. We choose $\varepsilon_0 \le \Delta' \le \Delta$ (Δ' is defined in Lemma 4.5) such that

$$\omega_k(\bar{\varepsilon}) \le (4K+2)^{-1} \quad \text{for all} \quad \bar{\varepsilon} \le \frac{\varepsilon_0}{2M_k}\,.$$

Let us fix an arbitrary $\varepsilon \in (0,\ \varepsilon_0]$ and take $\bar{\varepsilon} = \varepsilon[2M_k]^{-1}$. Now we choose $\delta = \delta(\varepsilon)$ and $\eta = \eta(\varepsilon)$ such that the following conditions hold:

$$4K\omega(\delta M_{k-1}) < 1\,, \qquad 8KM^k\omega(\delta M_{k-1}) \le 1\,,$$

$$\delta M_{k-1} \le \frac{1}{4}\bar{\varepsilon}(2K+1)^{-1}\,, \quad \rho_k(\eta) \le \frac{1}{4}\min(\bar{\varepsilon},\ 1)(2K+1)^{-1}\,, \quad 2(\delta+\eta)M_k \le \varepsilon\,.$$

It is clear that under such a choice of δ and η any δ-pseudoorbit (from Λ) of the mapping f has a unique ε-trajectory of the mapping g. Thus, **Theorem 4.2 is proved**.

Exercise *4.1* Let the hypotheses of Theorem 4.1 hold. Show that there exist $\Delta > 0$ and $\delta > 0$ such that for any two trajectories $\{x_n\colon n \in \mathbb{Z}\}$ and $\{y_n\colon n \in \mathbb{Z}\}$ of a dynamical system $(X,\ f)$ the conditions

$$\operatorname{dist}(x_n,\ \Lambda) < \Delta\,, \qquad \operatorname{dist}(y_n,\ \Lambda) < \Delta\,, \qquad \sup_n \|x_n - y_n\| \le \delta$$

imply that $x_n \equiv y_n$, $n \in \mathbb{Z}$. In other words, any two trajectories of the system $(X,\ f)$ that are close to a hyperbolic invariant set cannot remain arbitrarily close to each other all the time.

— Exercise *4.2* Show that Theorem 4.1 admits the following strengthening: if the hypotheses of Theorem 4.1 hold, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon)$ with the property that for any δ-pseudoorbit $\{y_n\}$ such that $\mathrm{dist}(y_n, \Lambda) < \delta$ there exists a unique ε-trajectory.

— Exercise *4.3* Prove the analogue of the assertion of Exercise 4.2 for Theorem 4.2.

— Exercise *4.4* Let $\Lambda = \{x_n\colon n \in \mathbb{Z}\}$ be a periodic orbit of the mapping f, i.e. $f^k(x_n) \equiv x_{n+k} = x_n$ for all $n \in \mathbb{Z}$ and for some $k \geq 1$. Assume that the hypotheses of Theorem 4.2 hold. Then for $\eta > 0$ small enough every mapping $g \in \mathcal{W}_\eta(f)$ possesses a periodic trajectory of the period k.

§ 5 *Birkhoff-Smale Theorem*

One of the most interesting corollaries of Anosov's lemma is the Birkhoff-Smale theorem that provides conditions under which the chaotic dynamics is observed in a discrete dynamical system (X, f). We remind (see Section 1) that by definition the possibility of chaotic dynamics means that there exists an invariant set Y in the space X such that the restriction of some degree f^k of the mapping f on Y is topologically equivalent to the Bernoulli shift S in the space Σ_m of two-sided infinite sequences of m symbols.

Theorem 5.1.

Let f be a continuously differentiable mapping of a Banach space X into itself. Let $x_0 \in X$ be a hyperbolic fixed point of f and let $\{y_n\colon n \in \mathbb{Z}\}$ be a homoclinic trajectory of the mapping f that does not coincide with x_0, i.e.

$$f(x_0) = x_0; \qquad f(y_n) = y_{n+1}, \quad y_n \neq x_0, \quad n \in \mathbb{Z}; \qquad y_n \to x_0, \ n \to \pm\infty.$$

Assume that the trajectory $\{y_n\colon n \in \mathbb{Z}\}$ is transversal, i.e. the set

$$\Lambda = \{x_0\} \cup \{y_n\colon n \in \mathbb{Z}\}$$

is hyperbolic with respect to f and there exists a vicinity $\mathcal{O}$ of the set Λ such that $f(x)$ and $Df(x)$ are bounded and uniformly continuous on the closure $\overline{\mathcal{O}}$. By $\mathcal{W}_\eta(f)$ we denote a set of continuously differentiable mappings g of the space X into itself such that

$$\|f(x) - g(x)\| \leq \eta, \qquad \|Df(x) - Dg(x)\| \leq \eta, \qquad x \in \overline{\mathcal{O}}.$$

Then there exists $\eta > 0$ ***such that for any mapping*** $g \in \mathcal{W}_\eta(f)$ ***and for any*** $m \ge 2$ ***there exist a natural number*** l ***and a continuous mapping*** φ ***of the space*** Σ_m ***into a compact subset*** $Y \equiv \varphi(\Sigma_m)$ ***in*** X ***such that***

a) $Y = \varphi(\Sigma_m)$ ***is strictly invariant with respect to*** g^l***, i.e.*** $g^l(Y) = Y$***;***

b) ***if*** $a = (\ldots a_{-1}, a_0, a_1, \ldots)$ ***and*** $a' = (\ldots a'_{-1}, a'_0, a'_1, \ldots)$ ***are elements of*** Σ_m ***such that*** $a_i \ne a'_i$ ***for some*** $i \ge 0$***, then*** $\varphi(a) \ne \varphi(a')$***;***

c) ***the restriction of*** g^l ***on*** Y ***is topologically conjugate to the Bernoulli shift*** S ***in*** Σ_m***, i.e.***

$$g^l(\varphi(a)) = \varphi(Sa), \qquad a \in \Sigma_m .$$

Moreover, if in addition we assume that for the mapping g ***there exists*** $\varepsilon_0 > 0$ ***such that for any two trajectories*** $\{x_n :\ n \in \mathbb{Z}\}$ ***and*** $\{\bar{x}_n :\ n \in \mathbb{Z}\}$ ***(of the mapping*** g***) lying in the*** ε_0***-vicinity of the set*** Λ ***the condition*** $x_{n_0} = \bar{x}_{n_0}$ ***for some*** $n_0 \in \mathbb{Z}$ ***implies that*** $x_n = \bar{x}_n$ ***for all*** $n \in \mathbb{Z}$***, then the mapping*** φ ***is a homeomorphism.***

The proof of this theorem is based on Anosov's lemma and mostly follows the standard scheme (see, e.g., [4]) used in the finite-dimensional case. The only difficulty arising in the infinite-dimensional case is the proof of the continuity of the mapping φ. It can be overcome with the help of the lemma presented below which is borrowed from the thesis by Jürgen Kalkbrenner (Augsburg, 1994) in fact.

It should also be noted that the condition under which φ is a homeomorphism holds if the mapping g does not "glue" the points in some vicinity of the set Λ, i.e. the equality $g(x) = g(\bar{x})$ implies $x = \bar{x}$.

Lemma 5.1.

Let the hypotheses of Theorem 5.1 *hold. Let us introduce the notation*

$$J_\nu \equiv J_\nu(k_0, \mu) = \{k \in \mathbb{Z}:\ |k - k_0| \le \mu\nu\},$$

where $k_0 \in \mathbb{Z}$ *and* $\mu, \nu \in \mathbb{N}$. *Let* $z = \{z_n :\ n \in J_\nu\}$ *be a segment (lying in* Λ*) of a* δ*-pseudoorbit of the mapping* f*:*

$$z_n \in \Lambda, \qquad \|z_{n+1} - f(z_n)\| \le \delta, \qquad n,\ n+1 \in J_\nu . \tag{5.1}$$

Assume that $x = \{x_n :\ n \in J_\nu\}$ *and* $\bar{x} = \{\bar{x}_n :\ n \in J_\nu\}$ *are segments of orbits of the mapping* $g \in \mathcal{W}_\eta(f)$*:*

$$g(x_n) = x_{n+1}, \qquad g(\bar{x}_n) = \bar{x}_{n+1}, \qquad n,\ n+1 \in J_\nu , \tag{5.2}$$

such that

$$\|z_n - x_n\| \le \varepsilon, \qquad \|z_n - \bar{x}_n\| \le \varepsilon . \tag{5.3}$$

Then there exist $\delta, \eta, \varepsilon > 0$, *and* $\mu \in \mathbb{N}$ *such that conditions* (5.1)–(5.3) *imply the inequality*

$$\|x_{k_0} - \bar{x}_{k_0}\| \le 2^{1-\nu}\varepsilon . \tag{5.4}$$

Proof.

It follows from (5.2) that

$$x_{k_0+\mu} - \bar{x}_{k_0+\mu} = (Df^{\mu})(z_{k_0})(x_{k_0} - \bar{x}_{k_0}) + R_{k_0}, \tag{5.5}$$

where

$$R_{k_0} = g^{\mu}(x_{k_0}) - g^{\mu}(\bar{x}_{k_0}) - (Df^{\mu})(z_{k_0})(x_{k_0} - \bar{x}_{k_0}).$$

Since the set Λ is hyperbolic with respect to f, there exists a family of projectors $\{P(x):\ x \in \Lambda\}$ for which equations (3.2)–(3.4) are valid. Therefore,

$$(1 - P(f^{\mu}(z_{k_0})))(x_{k_0+\mu} - \bar{x}_{k_0+\mu}) =$$

$$= (Df^{\mu})(z_{k_0})(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0}) + (1 - P(f^{\mu}(z_{k_0})))R_{k_0}.$$

It means that

$$(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0}) =$$

$$= [(Df^{\mu})(z_{k_0})]^{-1}[1 - P(f^{\mu}(z_{k_0}))][x_{k_0+\mu} - \bar{x}_{k_0+\mu} - R_{k_0}].$$

Consequently, equation (3.4) implies that

$$\|(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0})\| \le Kq^{\mu}(\|x_{k_0+\mu} - \bar{x}_{k_0+\mu}\| + \|R_{k_0}\|).$$

Let us estimate the value R_{k_0}. It can be rewritten in the form

$$R_{k_0} = \int_0^1 [(Dg^{\mu})(\xi x_{k_0} + (1-\xi)\bar{x}_{k_0}) - Df^{\mu}(z_{k_0})]\,\mathrm{d}\xi \cdot (x_{k_0} - \bar{x}_{k_0}).$$

It follows from (5.3) that $\|\xi x_{k_0} + (1-\xi)\bar{x}_{k_0} - z_{k_0}\| \le \varepsilon$. Hence, using (4.20) and (4.21), for $\varepsilon > 0$ small enough we obtain that

$$\|R_{k_0}\| \le (\rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon))\|x_{k_0} - \bar{x}_{k_0}\|, \tag{5.6}$$

where $\rho_{\mu}(\eta) \to 0$ as $\eta \to 0$ and $\omega_{\mu}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore,

$$\|(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0})\| \le Kq^{\mu}(\|x_{k_0+\mu} - \bar{x}_{k_0+\mu}\| + \|x_{k_0} - \bar{x}_{k_0}\|), \tag{5.7}$$

provided $\rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon) \le 1$. Further, we substitute the value $k_0 - \mu$ for k_0 in (5.5) to obtain that

$$x_{k_0} - \bar{x}_{k_0} = Df^{\mu}(z_{k_0-\mu})(x_{k_0-\mu} - \bar{x}_{k_0-\mu}) + R_{k_0-\mu}.$$

Therefore, using (3.2) we find that

$$P(f^{\mu}(z_{k_0-\mu}))(x_{k_0} - \bar{x}_{k_0}) =$$

$$= Df^{\mu}(z_{k_0-\mu})P(z_{k_0-\mu})(x_{k_0-\mu} - \bar{x}_{k_0-\mu}) + P(f^{\mu}(z_{k_0-\mu}))R_{k_0-\mu}.$$

Hence, equations (3.3) and (5.6) with $k_0-\mu$ instead of k_0 give us that

$$\left\|P(f^{\mu}(z_{k_0-\mu}))(x_{k_0}-\bar{x}_{k_0})\right\| \le$$
$$\le K\{q^{\mu}+(\rho_{\mu}(\eta)+\omega_{\mu}(\varepsilon))\}\left\|x_{k_0-\mu}-\bar{x}_{k_0-\mu}\right\| . \tag{5.8}$$

Since

$$z_{k_0}-f^{\mu}(z_{k_0-\mu}) = \sum_{j=0}^{\mu-1}\left\{f^j(z_{k_0-j})-f^j(f(z_{k_0-j-1}))\right\},$$

it follows from (4.15) and (5.1) that

$$\left\|z_{k_0}-f^{\mu}(z_{k_0-m})\right\| \le \delta\sum_{j=0}^{\mu-1}M^j \le \delta\cdot\mu(1+M^{\mu})$$

for δ small enough. Therefore,

$$\left\|P(z_{k_0})-P(f^{\mu}(z_{k_0-\mu}))\right\| \le \omega(\delta\cdot\mu(1+M^{\mu})) \equiv \omega(\delta,\ \mu),$$

where $\omega(\xi)\to 0$ as $\xi\to 0$ (cf. (4.22)). Consequently, estimate (5.8) implies that

$$\left\|P(z_{k_0})(x_{k_0}-\bar{x}_{k_0})\right\| \le$$
$$\le K\{q^{\mu}+\rho_{\mu}(\eta)+\omega_{\mu}(\varepsilon)+K^{-1}\cdot\omega(\delta,\ \mu)\}\left\|x_{k_0-\mu}-\bar{x}_{k_0-\mu}\right\| . \tag{5.9}$$

It is evident that estimates (5.7) and (5.9) enable us to choose the parameters $\mu,\ \eta,\ \varepsilon$, and δ such that

$$\left\|x_{k_0}-\bar{x}_{k_0}\right\| \le \frac{1}{2}\max\{\left\|x_k-\bar{x}_k\right\|:\ k\in J_1(k_0,\ \mu)\} .$$

Using this inequality with k instead of k_0 we obtain that

$$\left\|x_k-\bar{x}_k\right\| \le \frac{1}{2}\max\{\left\|x_n-\bar{x}_n\right\|:\ n\in J_2(k_0,\ \mu)\}$$

for all $k\in J_1(k_0,\ \mu)$. Therefore,

$$\left\|x_{k_0}-\bar{x}_{k_0}\right\| \le \frac{1}{4}\max\{\left\|x_n-\bar{x}_n\right\|:\ n\in J_2(k_0,\ \mu)\} .$$

If we continue to argue like that, then we find that

$$\left\|x_{k_0}-\bar{x}_{k_0}\right\| \le 2^{-\nu}\max\{\left\|x_n-\bar{x}_n\right\|:\ n\in J_{\nu}(k_0,\ \mu)\} .$$

Since $\left\|x_n-\bar{x}_n\right\| \le \left\|x_n-z_n\right\|+\left\|\bar{x}_n-z_n\right\|$, this and estimate (5.3) imply (5.4). Lemma 5.1 is proved.

Proof of Theorem 5.1.

Let $p_1,\ p_2,\ \dots,\ p_{m-1}$ be distinct integers. Let us choose and fix the parameters $\varepsilon,\ \eta,\ \delta>0$ and the integer $\mu>0$ such that (i) Theorem 4.2 and Lemma 5.1 can be applied to the hyperbolic set $\Lambda=\{x_0\}\cup\{y_n:\ n\in\mathbb{Z}\}$ and (ii)

$$\varepsilon < \frac{1}{2} \min\left\{ \left\| y_{p_i} - y_{p_j} \right\|, \left\| y_{p_i} - x_0 \right\| : \quad i, j = 1, \ldots, m-1, \quad i \neq j \right\}. \tag{5.10}$$

Assume that N is such that

$$\left\| y_n - x_0 \right\| < \frac{\delta}{2} \quad \text{for} \quad n = N + p_i + 1 \quad \text{and} \quad n = p_i - N \tag{5.11}$$

for all $i = 1, 2, \ldots, m-1$. Let us consider the segments C_i of the orbits of the mapping f of the form

$$C_i = (y_{p_i - N}, \ldots, y_{p_i}, \ldots, y_{p_i + N}), \qquad i = 1, 2, \ldots, m-1,$$

$$C_m = (x_0, x_0, \ldots, x_0).$$

The length of every such segment is $2N+1$. Let $a = (\ldots a_{-1} a_0 a_1 \ldots) \in \Sigma_m$. Let us consider a sequence of elements γ_a made up of the segments C_i by the formula

$$\gamma_a = (\ldots C_{a_{-1}} C_{a_0} C_{a_1} \ldots). \tag{5.12}$$

It is clear that $\gamma_a \in \Lambda$ and by virtue of (5.11) γ_a is a δ-pseudoorbit of the mapping f. Therefore, due to Theorem 4.2 there exists a unique trajectory $\{w_n \equiv w_n(a): n \in \mathbb{Z}\}$ of the mapping g such that

$$\left\| w_{n(N,\, i,\, j)} - z_{ij}(a) \right\| \leq \varepsilon, \tag{5.13}$$

where $n(N, i, j) = N + (i-1)(2N+1) + j$ and $z_{ij}(a)$ is the j-th element of the segment C_{a_i}, $i \in \mathbb{Z}$, $j = 1, 2, \ldots, 2N+1$. Let us define the mapping φ from Σ_m into X by the formula

$$\varphi(a) = w_0, \tag{5.14}$$

where w_0 is the zeroth element of the trajectory $\{w_n\}$. Since the trajectory $\{w_n\}$ possessing the property (5.13) is uniquely defined, equation (5.14) defines a mapping from Σ_m into X.

If we substitute $i+1$ for i in (5.13) and use the equations

$$w_{n(N,\, i+1,\, j)} = w_{n(N,\, i,\, j) + 2N+1} = g^{2N+1}\left(w_{n(N,\, i,\, j)}\right),$$

we obtain that

$$\left\| g^{2N+1}(w_{n(N,\, i,\, j)}) - z_{i+1,\, j}(a) \right\| \leq \varepsilon$$

for all $i \in \mathbb{Z}$ and $j = 1, 2, \ldots, 2N+1$. Therefore, the equality $z_{i+1,\, j}(a) = z_{ij}(Sa)$ with S being the Bernoulli shift in Σ_m leads us to the equation

$$\left\| g^{2N+1}(w_{n(N,\, i,\, j)}) - z_{i+1,\, j}(Sa) \right\| \leq \varepsilon.$$

Consequently, the uniqueness property of the ε-trajectory in Theorem 4.2 gives us the equation

$$w_n(Sa) = g^{2N+1}(w_n(a)), \qquad n \in \mathbb{Z}.$$

This implies that

$$\varphi(Sa) = g^{2N+1}(\varphi(a)), \qquad a \in \Sigma_m, \tag{5.15}$$

i.e. property (c) is valid for $l = 2N+1$. It follows from (5.15) that

$$g^{2N+1}(\varphi(S^{-1}a)) = \varphi(a).$$

Therefore, the set $Y = \varphi(\Sigma_m)$ is strictly invariant with respect to g^{2N+1}. Thus, assertion (a) is proved.

Let us prove the continuity of the mapping φ. Assume that the sequence of elements $a^{(s)} = (\dots, a_{-1}^{(s)}, a_0^{(s)}, a_1^{(s)}, \dots)$ of Σ_m tends to $a = (\dots, a_{-1}, a_0, a_1, \dots) \in \in \Sigma_m$ as $s \to +\infty$. This means (see Exercise 1.4) that for any $M \in \mathbb{N}$ there exists $s_0 = s_0(M)$ such that

$$a_i^{(s)} = a_i \quad \text{for} \quad |i| \le M, \; s \ge s_0. \tag{5.16}$$

Assume that $\gamma^{(s)} = \{z_k^{(s)}\}$ and $\gamma = \{z_k\}$ are δ-pseudoorbits in Λ constructed according to (5.12) for the symbols $a^{(s)}$ and a, respectively. Equation (5.16) implies that $z_k^{(s)} = z_k$ for $|k| \le M(2N+1)$. Let $\{w_n\}$ and $\{w_n^{(s)}\}$ be ε-trajectories corresponding to γ and $\gamma^{(s)}$, respectively. Lemma 5.1 gives us that

$$\left\| w_0^{(s)} - w_0 \right\| \le 2^{1-\nu}\varepsilon, \tag{5.17}$$

provided $M(2N+1) \ge \nu\mu$, i.e. for any $\nu \in \mathbb{N}$ equations (5.17) is valid for $s \ge s_0(\nu, M)$. This means that

$$\|\varphi(a^{(s)}) - \varphi(a)\| \equiv \left\| w_0^{(s)} - w_0 \right\| \to 0$$

as $s \to +\infty$. Thus, the mapping φ is continuous and $Y = \varphi(\Sigma_m)$ is a compact strictly invariant set with respect to g^{2N+1}.

Let us now prove nontriviality property (b) of the mapping φ. Let $a, a' \in \Sigma_m$ be such that $a_i \ne a_i'$ for some $i \ge 0$. Let $\{w_n\}$ and $\{w_n'\}$ be ε-trajectories corresponding to the symbols a and a', respectively. Then

$$\left\| w_{n(N,\, i,\, N+1)} - w'_{n(N,\, i,\, N+1)} \right\| \ge \left\| z_{i,\, N+1}(a) - z_{i,\, N+1}(a') \right\| -$$

$$- \left\| w_{n(N,\, i,\, N+1)} - z_{i,\, N+1}(a) \right\| - \left\| w'_{n(N,\, i,\, N+1)} - z_{i,\, N+1}(a') \right\| .$$

Therefore, it follows both from (5.13) and the definition of the elements $z_{ij}(a)$ that

$$\left\| w_{(2N+1)i} - w'_{(2N+1)i} \right\| \ge \left\| y_{q_i} - y_{q_i'} \right\| - 2\varepsilon,$$

where $q_i = p_{a_i}$ and $q_i' = p_{a_i'}$. We apply (5.10) to obtain that

$$\left\| w_{(2N+1)i} - w'_{(2N+1)i} \right\| > 0. \tag{5.18}$$

Therefore, if $i \ge 0$, then

$$g^{(2N+1)i}(w_0) \ne g^{(2N+1)i}(w_0').$$

Hence, $\varphi(a) \equiv w_0 \ne w_0' \equiv \varphi(a')$. This completes the proof of assertion (b).

If the trajectories of the mapping g cannot be "glued" (see the hypotheses of Theorem 5.1), then for some $i \in \mathbb{Z}$ equation (5.18) gives us that $w_0 \neq w_0'$, i.e. $\varphi(a) \neq \varphi(a')$ if $a \neq a'$. Thus, the mapping φ is injective in this case. Since Σ_m is a compact metric space, then the injectivity and continuity of φ imply that φ is a homeomorphism from Σ_m onto $\varphi(\Sigma_m)$. **Theorem 5.1 is proved**.

It should be noted that equations (5.13) and (5.14) imply that the set $Y = \varphi(\Sigma_m)$ lies in the ε-vicinity of the hyperbolic set Λ. Therewith, the values η and l involved in the statement of the theorem depend on ε and one can state that for any vicinity $\mathcal{U}$ of the set Λ there exist η and l such that the conclusions of Theorem 5.1 are valid and $\varphi(\Sigma_m) \subset \mathcal{U}$. It is also clear that the set $Y = \varphi(\Sigma_m)$ is not uniquely determined.

Exercise *5.1* Assume that $g = f$ in Theorem 5.1. Prove that the mapping φ can be constructed such that $\varphi(\Sigma_m) \supset \{x_0\} \cup \{y_{nl} \colon\ n \geq 0\}$, where $\{y_n \colon\ n \in \mathbb{Z}\}$ is a homoclinic orbit of the mapping f.

Exercise *5.2* Prove the Birkhoff theorem: if the hypotheses of Theorem 5.1 hold, then for any $\varepsilon > 0$ small enough there exist $\eta > 0$ and $l \in \mathbb{N}$ such that for every mapping $g \in \mathcal{W}_\eta(f)$ there exist periodic trajectories of the mapping g^l of any minimal period in the ε-vicinity of the set Λ.

Exercise *5.3* Use Theorem 1.1 to describe all the possible types of behaviour of the trajectories of the mapping g on a set

$$W = \bigcup_{k=1}^{l} g^k(\varphi(\Sigma_m)).$$

In conclusion, it should be noted that different infinite-dimensional versions of Anosov's lemma and the Birkhoff-Smale theorem have been considered by many authors (see, e.g., [6], [10], [11], [12], and the references therein).

§ 6 *Possibility of Chaos in the Problem of Nonlinear Oscillations of a Plate*

In this section the Birkhoff-Smale theorem is applied to prove the existence of chaotic regimes in the problem of nonlinear plate oscillations subjected to a periodic load. The results presented here are close to the assertions proved in [13]. However, the methods used differ from those in [13].

Let us remind the statement of the problem. We consider its abstract version as in Chapter 4. Let H be a separable Hilbert space and let A be a positive operator with discrete spectrum in H, i.e. there exists an orthonormalized basis $\{e_k\}$ in H such that

$$Ae_k = \lambda_k e_k\,, \qquad 0 < \lambda_1 \le \lambda_2 \le \ldots\,, \qquad \lim_{n\to\infty} \lambda_n = \infty\,.$$

The following problem is considered:

$$\begin{cases} \ddot{u} + \gamma\dot{u} + A^2 u + (\varkappa\|A^{1/2}u\|^2 - \Gamma)Au + Lu = h\cos\omega t\,, & (6.1)\\ u\big|_{t=0} = u_0 \in F_1 = D(A), \quad \dot{u}\big|_{t=0} = u_1 \in H\,. & (6.2)\end{cases}$$

Here γ, $\varkappa$, Γ, and ω are positive parameters, h is an element of the space H, L is a linear operator in H subordinate to A, i.e.

$$\|Lu\| \le K\|Au\|\,, \tag{6.3}$$

where K is a constant. The problem of the form (6.1) and (6.2) was studied in Chapter 4 in details (nonlinearity of a more general type was considered there). The results of Section 4.3 imply that problem (6.1) and (6.2) is uniquely solvable in the class of functions

$$\mathscr{W}_+ = C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H)\,. \tag{6.4}$$

Moreover, one can prove (cf. Exercise 4.3.9) that Cauchy problem (6.1) and (6.2) is uniquely solvable on the whole time axis, i.e. in the class

$$\mathscr{W} = C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, H)\,.$$

This fact as well as the continuous dependence of solutions on the initial conditions (see (4.3.20)) enables us to state that the monodromy operator G acting in $\mathscr{H} = D(A) \times H$ according to the formula

$$G(u_0;\ u_1) = \left(u\left(\frac{2\pi}{\omega}\right),\ \dot{u}\left(\frac{2\pi}{\omega}\right)\right) \tag{6.5}$$

is a homeomorphism of the space $\mathscr{H}$ (see Exercise 4.3.11). Here $u(t)$ is a solution to problem (6.1) and (6.2)

The aim of this section is to prove the fact that under some conditions on L and h chaotic dynamics is observed in the discrete dynamical system $(\mathscr{H}, G)$ for some set of parameters γ, $\varkappa$, Γ, and ω.

Lemma 6.1.

The mapping G defined by equality (6.5) *is a diffeomorphism of the space $\mathscr{H} = D(A) \times H$.*

Proof.

We use the method applied to prove Lemma 4.7.3. Let $u_1(t)$ be a solution to problem (6.1) and (6.2) with the initial conditions $\overline{y} = (u_0, u_1) \in \mathscr{H}$ and let

$u_2(t)$ be a solution to it with the initial condions $\bar{y}+z \equiv (u_0+z_0,\ u_1+z_1) \in \mathcal{H}$. Let us consider a linearization of problem (6.1) and (6.2) along the solution $u_1(t)$:

$$\begin{cases} \ddot{w}(t)+\gamma\dot{w}(t)+A^2w+(\varkappa\left\|A^{1/2}u_1(t)\right\|^2-\Gamma)Aw+ \\ \qquad +2\varkappa(A^{1/2}u_1(t),\ A^{1/2}w(t))\,Au_1(t)+Lw \;=\; 0\ , & (6.6) \\ w\big|_{t=0}=z_0, \quad \dot{w}\big|_{t=0}=z_1\ . & (6.7) \end{cases}$$

As in the proof of Theorem 4.2.1, it is easy to find that problem (6.6) and (6.7) is uniquely solvable in the class of functions (6.4). Let $v(t) = u_2(t)-u_1(t)-w(t)$. It is evident that $v(t)$ is a weak solution to problem

$$\begin{cases} \ddot{v}+\gamma\dot{v}(t)+A^2v(t) \;=\; F(t) \equiv F(u_1(t),\ u_2(t),\ w(t))\ , & (6.8) \\ v\big|_{t=0}=0\,, \quad \dot{v}\big|_{t=0}=0\ , & (6.9) \end{cases}$$

where

$$F(u_1,\ u_2,\ w) \;=\; -\Big(\varkappa\left\|A^{1/2}u_2\right\|^2-\Gamma\Big)Au_2+\Big(\varkappa\left\|A^{1/2}u_1\right\|^2-\Gamma\Big)Au_1+$$

$$+\Big(\varkappa\left\|A^{1/2}u_1\right\|^2-\Gamma\Big)\,Aw+2\varkappa(Au_1,\ w)\,Au_1-L(u_2-u_1-w)\ .$$

A simple calculation shows that

$$F(u_1,\ u_2,\ w) = F_1(u_1,\ u_2,\ w)+F_2(u_1,\ u_2,\ w) \equiv F_1+F_2\,,$$

where

$$F_1 = -\Big(\varkappa\left\|A^{1/2}u_2\right\|^2-\Gamma\Big)Av-Lv-2\varkappa(Au_1,\ v)\,A(u_1+w)\,,$$

$$F_2 \;=\; -\varkappa\left\|A^{1/2}(u_1-u_2)\right\|^2A(u_1+w)-2\varkappa(Au_1,\ w)Aw\,.$$

We assume that $\|\bar{y}\|_{\mathcal{H}} \le R$ and $\|z\|_{\mathcal{H}} \le 1$. In this case (see Section 4.3) the estimates

$$\begin{aligned} \left\|Au_j(t)\right\| &\le C_{R,\,T}\,, \\ \left\|A(u_1(t)-u_2(t))\right\| &\le C_{R,\,T}\|z\|_{\mathcal{H}}\,, \\ \|Aw(t)\| &\le C_{R,\,T}\|z\|_{\mathcal{H}}\,, \end{aligned} \qquad (6.10)$$

are valid on any segment $[0,\ T]$. Here $C_{R,\,T}$ is a constant. Therefore,

$$\left\|F_1\right\| \le C_1\|Av\|\,, \qquad \left\|F_2\right\| \le C_2\|z\|_{\mathcal{H}}^2\,, \qquad t\in[0,\ T]\,,$$

where C_1 and C_2 are constants depending on R and T. Hence,

$$\|F(t)\|^2 \le C_1\|Av\|^2+C_2\|z\|_{\mathcal{H}}^4\,, \qquad t\in[0,\ T]\,.$$

Therefore, the energy equation

$$\frac{1}{2}\left(\|\dot{v}(t)\|^2+\|Av(t)\|^2\right)+\gamma\int_0^t\|\dot{v}(\tau)\|^2\,\mathrm{d}\tau=\int_0^t(F(\tau),\,\dot{v}(\tau))\,\mathrm{d}\tau \tag{6.11}$$

for problem (6.8) and (6.9) leads us to the estimate

$$\|\dot{v}(t)\|^2+\|Av(t)\|^2\ \le\ \int_0^t\left(C_1\|Av(\tau)\|^2+C_2\|z\|^4\right)\mathrm{d}\tau\ ,\qquad t\in[0,\,T].$$

Using Gronwall's lemma we find that

$$\|\dot{v}(t)\|^2+\|Av(t)\|^2\ \le\ C\|z\|^4,\qquad t\in[0,\,T],$$

where the constant C depends on T and R. This estimate implies that the mapping

$$G':\ z\equiv(z_0;\ z_1)\to\left(w\left(\frac{2\pi}{\omega}\right);\ \dot{w}\left(\frac{2\pi}{\omega}\right)\right) \tag{6.12}$$

is a Frechét derivative of the mapping G defined by equality (6.5). Here $w(t)$ is a solution to problem (6.6) and (6.7). It follows from (6.10) and (6.6) that G' is a continuous linear mapping of $\mathcal{H}$ into itself. Using (6.10) it is also easy to see that $G'=G'[u_0,\,u_1]$ continuously depends on $\bar{y}=(u_0;\,u_1)$ with respect to the operator norm. Lemma 6.1 is proved.

Further we will also need the following assertion.

Lemma 6.2.

Let G_j be the monodromy operator of problem (6.1) *and* (6.2) *with $L=L_j$ and $h=h_j$, $j=1,2$. Assume that for $L=L_j$ equation* (6.3) *is valid and $h_j\in H$, $j=1,\,2$. Moreover, assume that*

$$\left\|L_jA^{-1}\right\|\ \le\ \rho,\qquad \left\|h_j\right\|\le\rho,\qquad j=1,\ 2.$$

Then the estimates

$$\sup_{y\ \in\ B_R}\|G_1(y)-G_2(y)\|\ \le\ C\left(\left\|(L_1-L_2)A^{-1}\right\|+\|h_1-h_2\|\right) \tag{6.13}$$

and

$$\sup_{y\ \in\ B_R}\|G_1'(y)-G_2'(y)\|\ \le\ C\left(\left\|(L_1-L_2)A^{-1}\right\|+\|h_1-h_2\|\right) \tag{6.14}$$

are valid. Here B_R is a ball of the radius R in $\mathcal{H}=D(A)\times H$, $R>0$ is an arbitrary number while the constant C depends on R and ρ but does not depend on the parameters $\omega,\gamma,\varkappa\ge 0$, and Γ provided they vary in bounded sets.

Proof.

Let $u_j(t)$ be a solution to problem (6.1) and (6.2) with $L = L_j$ and $h = h_j$, $j = 1, 2$. It is evident (see Section 4.3) that

$$\|Au_j(t)\| \le C \equiv C(R, \rho, T), \qquad t \in [0, T]. \tag{6.15}$$

Therefore, it is easy to find that the difference $u(t) = u_1(t) - u_2(t)$ satisfies the equation

$$\begin{cases} \ddot{u} + \gamma u + A^2 u = F(t, u_1, u_2), \\ u\big|_{t=0} = 0, \quad \dot{u}\big|_{t=0} = 0, \end{cases}$$

where the function $F(t, u_1, u_2)$ can be estimated as follows:

$$\|F(t, u_1, u_2)\| \le C_1 \|Au\| + C_2\Big(\big\|(L_1 - L_2)A^{-1}\big\| + \|h_1 - h_2\|\Big).$$

As in the proof of Lemma 6.1, we now use energy equality (6.11) and Gronwall's lemma to obtain the estimate

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le C\Big(\big\|(L_1 - L_2)A^{-1}\big\|^2 + \|h_1 - h_2\|^2\Big). \tag{6.16}$$

This implies inequality (6.13). Estimate (6.14) can be obtained in a similar way. In its proof equations (6.12), (6.15), and (6.16) are used. We suggest the reader to carry out the corresponding reasonings himself/herself. Lemma 6.2 is proved.

Let us now prove that there exist an operator L and a vector h such that the corresponding mapping G possesses a hyperbolic homoclinic trajectory. To do that, we use the following well-known result (see, e.g., [1], [13], as well as Section 7) related to the Duffing equation.

Theorem 6.1.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ ***be a monodromy operator corresponding to the Duffing equation***

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(f \cdot \cos\omega t - \delta\dot{x}), \tag{6.17}$$

i.e. the mapping of the plane $\mathbb{R}^2$ ***into itself acting according to the formula***

$$g(x_0, x_1) = \left(x\left(\frac{2\pi}{\omega}\right);\ \dot{x}\left(\frac{2\pi}{\omega}\right)\right), \tag{6.18}$$

where $x(t)$ ***is a solution to equation*** (6.17) ***such that*** $x(0) = x_0$ ***and*** $\dot{x}(0) = x_1$. ***All the parameters contained in*** (6.17) ***are assumed to be positive. Let us also assume that***

$$f > f_{cr} \equiv \delta \cdot \frac{2\beta^{3/2}}{3\omega\sqrt{2\alpha}} \cdot \cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right). \tag{6.19}$$

Then there exists $\varepsilon_0 > 0$ ***such that for every*** $\varepsilon \in (0, \varepsilon_0)$ ***the mapping*** g ***possesses a fixed point*** z ***and a homoclinic trajectory*** $\{y_n : n \in \mathbb{Z}\}$ ***to it,*** $y_n \neq z$***, therewith the set*** $\{z\} \cup \{y_n : n \in \mathbb{Z}\}$ ***is hyperbolic.***

Let P_k be the orthoprojector onto the one-dimensional subspace generated by the eigenvector e_k in H. We consider problem (6.1) and (6.2) with $L_k = L(1-P_k)$ instead of L and $h = h_k \cdot e_k$, where h_k is a positive number. Then it is evident that every solution to problem (6.1) and (6.2) with the initial conditions $u_0 = c_0 e_k$ and $u_1 = c_1 e_k$ has the form

$$u(t) = x(t)\, e_k ,$$

where $x(t)$ is a solution to the Duffing equation

$$\ddot{x} + \gamma\dot{x} - \lambda_k(\Gamma - \lambda_k)x + \varkappa\lambda_k^2 x^3 = h_k \cos\omega t \tag{6.20}$$

with the initial conditions $x(0) = c_0$ and $\dot{x}(0) = c_1$. In particular, this means that the two-dimensional subspace $\mathcal{L}_k = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k)\}$ of the space $\mathcal{H}$ is strictly invariant with respect to the corresponding monodromy operator G_k while the restriction of G_k to $\mathcal{L}_k$ coincides with the monodromy operator corresponding to the Duffing equation (6.20). Therefore, if h_k is small enough and the conditions

$$0 < \lambda_k < \Gamma, \qquad \frac{h_k}{\gamma} > \frac{2\lambda_k(\Gamma-\lambda_k)}{3\omega\sqrt{2\varkappa}}\left(\frac{\Gamma}{\lambda_k} - 1\right)^{1/2} \cosh\frac{\pi\omega}{2\sqrt{\lambda_k(\Gamma-\lambda_k)}}$$

hold, then the mapping G_k possesses a hyperbolic invariant set

$$\Lambda_k = \{z e_k\} \cup \{y_n \cdot e_k : n \in \mathbb{Z}\}$$

consisting of the fixed point $(z_0 e_k;\ z_1 e_k)$ and its homoclinic trajectory

$$\{y_n e_k : n \in \mathbb{Z}\}, \quad \text{where } y_n = (y_n^0;\ y_n^1) \in \mathbb{R}^2 .$$

Thus, if $\omega > 0$ and for some k the condition $0 < \lambda_k < \Gamma$ holds, then there exists an open set $\mathcal{P}$ in the space of parameters $\{\gamma, h_k\}$ such that for every $(\gamma, h_k) \in \mathcal{P}$ the monodromy operator G_k corresponding to problem (6.1) and (6.2) with $L_k = L(1-P_k)$ instead of L and $h = h_k e_k$ possesses a hyperbolic set consisting of a fixed point and a homoclinic trajectory. This fact as well as Lemmata 6.1 and 6.2 enables us to apply the Birkhoff-Smale theorem and prove the following assertion.

Theorem 6.2.

Let $\omega > 0$ ***and let the condition*** $0 < \lambda_k < \Gamma$ ***hold for some*** k***. Then there exist*** $\mu > 0$ ***and an open set*** $\mathcal{P}$ ***in the metric space*** $\mathbb{R}_+ \times H$ ***such that if***

$$\|Le_k\|\,\lambda_k^{-1} < \mu, \qquad (\gamma, h) \in \mathcal{P},$$

then some degree G^l ***of the monodromy operator*** G ***of problem*** (6.1) ***and*** (6.2) ***possesses a compact strictly invariant set*** Y $(G^l Y = Y)$ ***in the space*** $\mathcal{H}$ ***in which the mapping*** G^l ***is topologically conjugate to the Bernoulli shift of***

sequences of m ***symbols, i.e. there exists a homeomorphism*** $\varphi\colon \Sigma_m \to Y$ ***such that***

$$G^l(\varphi(a)) = \varphi(Sa), \qquad a \in \Sigma_m .$$

— Exercise *6.1* Prove that if the hypotheses of Theorem 6.2 hold, then equation (6.1) possesses an infinite number of periodic solutions with periods multiple to ω .

— Exercise *6.2* Apply Theorem 6.2 to the Berger approximation of the problem of nonlinear plate oscillations:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \gamma \dfrac{\partial u}{\partial t} + \Delta^2 u - \left(\varkappa \displaystyle\int_\Omega |\nabla u(x,\, t)|^2 \, dx - \Gamma \right) \Delta u \; + \\[2ex] \qquad + \; \rho \dfrac{\partial u}{\partial x_1} \; = \; h(x) \cos \omega t , \qquad x = (x_1,\, x_2) \in \Omega \subset \mathbb{R}^2 , \quad t > 0 \; , \\[2ex] u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0 , \quad u\big|_{t=0} = u_0(x) , \quad \dfrac{\partial u}{\partial t}\bigg|_{t=0} = u_1(x) \; . \end{cases}$$

§ 7 On the Existence of Transversal Homoclinic Trajectories

Undoubtedly, Theorem 6.1 on the existence of a transversal (hyperbolic) homoclinic trajectory of the monodromy operator for the periodic perturbation of the Duffing equation is the main fact which makes it possible to apply the Birkhoff-Smale theorem and to prove the possibility of chaotic dynamics in the problem of plate oscillations. In this connection, the question as to what kind of generic condition guarantees the existence of a transversal homoclinic orbit of monodromy operators generated by ordinary differential equations gains importance. Extensive literature is devoted to this question (see, e.g., [1], [2] and the references therein). There are several approaches to this problem. All of them enable us to construct systems with transversal homoclinic trajectories as small perturbations of “simple” systems with homoclinic (not transversal!) orbits. In some cases the corresponding conditions on perturbations can be formulated in terms of the Melnikov function.

This section is devoted to the exposition and discussion of the results obtained by K. Palmer [14]. These results help us to describe some classes of systems of ordinary differential equations which generate dynamical systems with transversal homoclinic orbits. Such differential equations are obtained as periodic perturbations of autonomous equations with homoclinic trajectories.

In the space $\mathbb{R}^n$ let us consider a system of equations

$$\dot{x}(t) = g(x(t)), \qquad x(t) \in \mathbb{R}^n, \tag{7.1}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable mapping. Assume that the Cauchy problem for equation (7.1) is uniquely solvable for any initial condition $x(0) = x_0$. Let us also assume that there exist a fixed point z_0 $(g(z_0)=0)$ and a trajectory $z(t) \neq z_0$ homoclinic to z_0, i.e. a solution to equation (7.1) such that $z(t) \to z_0$ as $t \to \pm\infty$. Exercises 7.1 and 7.2 given below give us the examples of the cases when these conditions hold. We remind that every second order equation $\ddot{x} + V(x) = 0$ can be rewritten as a system of the form (7.1) if we take $x_1 = x$ and $x_2 = \dot{x}$.

— Exercise *7.1* Consider the Duffing equation

$$\ddot{x} - \beta x + \alpha x^3 = 0, \qquad \alpha, \beta > 0.$$

Prove that the curve $z(t) = (\eta(t), \dot{\eta}(t)) \in \mathbb{R}^2$ is an orbit of the corresponding system (7.1) homoclinic to 0. Here $\eta(t) = \sqrt{2\beta/\alpha}\ \mathrm{sech}\sqrt{\beta}\,t$.

— Exercise *7.2* Assume that for a function $U(x) \in C^3(\mathbb{R})$ there exist a number E and a pair of points $a < b$ such that

$$U(a) = U(b) = E; \qquad U(x) < E, \quad x \in (a, b);$$
$$U'(a) = 0; \qquad U''(a) < 0; \qquad U'(b) > 0.$$

Then system (7.1) corresponding to $\ddot{x} + U'(x) = 0$ possesses an orbit homoclinic to $(a, 0)$ that passes through the point $(b, 0)$.

Unfortunately, as the cycle of Exercises 7.3–7.5 shows, the homoclinic orbit of autonomous equation (7.1) cannot be used directly to construct a discrete dynamical system with a transversal homoclinic trajectory.

— Exercise *7.3* For every $\tau > 0$ define the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by the formula $f(x_0) = x(\tau; x_0)$, where $x(t; x_0)$ is a solution to equation (7.1) with the initial condition x_0. Show that f is a diffeomorphism in $\mathbb{R}^n$ with a fixed point z_0 and a family of homoclinic orbits $\{y_n^s\colon\ n \in \mathbb{Z}\}$, where $y_n = z(s + n\tau)$, $s \in [0, \tau)$.

— Exercise *7.4* Prove that the derivative f' of the mapping f constructed in Exercise 7.3 can be evaluated using the formula $f'(x_0)w = y(\tau, w)$, where $y(t, w)$ is a solution to problem

$$\dot{y}(t) = g'(x(t))\,y(t), \qquad y(0) = w_0.$$

Here $x(t) = x(t; x_0)$ is a solution to equation (7.1) with the initial condition x_0.

Exercise 7.5 Let f be the mapping constructed in Exercise 7.3 and let $\{z(t):\ t \in \mathbb{Z}\}$ be a homoclinic orbit of equation (7.1). Show that $\{w_n = \dot{z}(n\tau):\ n \in \mathbb{Z}\}$ is a bounded solution to the difference equation $w_{n+1} = f'(y_n)w_n$, where $y_n = z(n\tau)$ (*Hint*: the function $w(t) = \dot{z}(t)$ satisfies the equation $\dot{w} = g'(z(t))w$).

Thus, due to Theorems 2.1 and 3.1 the result of Exercise 7.5 implies that the set

$$\Lambda = \{z_0\} \cup \{z(n\tau):\ n \in \mathbb{Z}\}$$

cannot be hyperbolic with respect to the mapping f defined by the formula $f(x_0) = x(\tau;\ x_0)$, where $x(t;\ x_0)$ is a solution to equation (7.1) with the initial condition x_0. Nevertheless we can indicate some quite simple conditions on the class of perturbations $\{h(t,\ x,\ \mu)\}$ periodic with respect to t under which the monodromy operator of the problem

$$\dot{x}(t) = g(x(t)) + \mu h(t,\ x(t),\ \mu), \qquad x(t) \in \mathbb{R}^n, \tag{7.2}$$

possesses a transversal (hyperbolic) homoclinic trajectory for μ small enough.

Further we will use the notion of exponential dichotomy for ordinary differential equations (see [15], [16] as well as [5] and the references therein)

Let $A(t)$ be a continuous and bounded $n \times n$ matrix function on the real axis. We consider the problem

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{R}, \quad x\big|_{t=s} = x_0, \tag{7.3}$$

in the space $X = \mathbb{R}^n$. It is easy to see that it is solvable for every initial condition. Therefore, we can define the evolutionary operator $\Phi(t,\ s)$, $t,\ s \in \mathbb{R}$, by the formula

$$\Phi(t,\ s)x_0 = x(t) \equiv x(t,\ s;\ x_0), \qquad t,\ s \in \mathbb{R},$$

where $x(t)$ is a solution to problem (7.3).

Exercise 7.6 Prove that

$$\Phi(t,\ s) = \Phi(t,\ \tau)\Phi(\tau,\ s), \qquad \Phi(t,\ t) = I$$

for all $t,\ s,\ \tau \in \mathbb{R}$ and the following matrix equations hold:

$$\frac{d}{dt}\Phi(t,\ s) = A(t)\,\Phi(t,\ s), \qquad \frac{d}{ds}\Phi(t,\ s) = -\Phi(t,\ s)\,A(s). \tag{7.4}$$

Exercise 7.7 Prove the inequality

$$\|\Phi(t,\ s)\| \le \exp\left\{\int_s^t \|A(\tau)\|\,d\tau\right\}, \qquad t \ge s.$$

Let $\mathcal{J}$ be some interval of the real axis. We say that equation (7.3) admits an ***exponential dichotomy*** over the interval $\mathcal{J}$ if there exist constants $K, \alpha > 0$ and a family of projectors $\{P(t):\ t \in \mathcal{J}\}$ continuously depending on t and such that

$$P(t)\,\Phi(t,\,s) = \Phi(t,\,s)\,P(s)\,, \qquad t \ge s\,; \tag{7.5}$$

$$\|\Phi(t,\,s)P(s)\| \le Ke^{-\alpha(t-s)}, \qquad t \ge s\,, \tag{7.6}$$

$$\|\Phi(t,\,s)\,(I-P(s))\| \le Ke^{-\alpha(s-t)}, \qquad t \le s\,, \tag{7.7}$$

for $t, s \in \mathcal{J}$.

— Exercise *7.8* Let $A(t) \equiv A$ be a constant matrix. Prove that equation (7.3) admits an exponential dichotomy over $\mathbb{R}$ if and only if the eigenvalues on A do not lie on the imaginary axis.

The assertion contained in Exercise 7.8 as well as the following theorem on the roughness enables us to construct examples of equations possessing an exponential dichotomy.

Theorem 7.1.

Assume that problem (7.3) ***possesses an exponential dichotomy over an interval*** $\mathcal{J}$***. Then there exists*** $\varepsilon > 0$ ***such that equation***

$$\dot{x}(t) = (A(t)+B(t))\,x(t) \tag{7.8}$$

possesses an exponential dichotomy over $\mathcal{J}$***, provided*** $\|B(t)\| \le \varepsilon$ ***for*** $t \in \mathcal{J}$***. Moreover, the dimensions of the corresponding projectors for*** (7.3) ***and*** (7.8) ***are the same.***

The proof of this theorem can be found in [15] or [16], for example.

The exercises given below contain some simple facts on systems possessing an exponential dichotomy. We will use them in our further considerations.

— Exercise *7.9* Prove that equations (7.5)–(7.7) imply the estimates

$$\|\Phi(t,\,s)\,\xi\| \ge K^{-1}e^{\alpha(t-s)}\|(1-P(s))\,\xi\|\,, \qquad t \ge s\,,$$

$$\|\Phi(t,\,s)\,\xi\| \ge K^{-1}e^{\alpha(s-t)}\|P(s)\,\xi\|\,, \qquad t \le s\,,$$

for any $\xi \in X \equiv \mathbb{R}^n$.

— Exercise *7.10* Assume that equation (7.3) admits an exponential dichotomy over $\mathbb{R}_+ = [0,\ +\infty)$. Prove that $P(0)X = V_+$, where

$$V_+ \equiv \left\{\xi \in X \equiv \mathbb{R}^n:\ \sup_{t>0}\|\Phi(t,\,0)\,\xi\| < \infty\right\} \tag{7.9}$$

(*Hint*: $\|\Phi(t,\,0)\,\xi\| \ge K^{-1}e^{\alpha t}\|(1-P(0))\,\xi\|\,,\quad t \ge 0$).

Exercise *7.11* If equation (7.3) possesses an exponential dichotomy over $\mathbb{R}_- = (-\infty, 0]$, then $(I - P(0))X = V_-$, where

$$V_- = \left\{ \xi \in X \equiv \mathbb{R}^n : \sup_{t<0} \|\Phi(t, 0)\xi\| < \infty \right\} \tag{7.10}$$

(*Hint*: $\|\Phi(t, 0)\xi\| \ge K^{-1} e^{\alpha t} \|P(0)\xi\|, \quad t \le 0$).

Exercise *7.12* Assume that equation (7.3) possesses an exponential dichotomy over $\mathbb{R}_+$ (over $\mathbb{R}_-$, respectively). Show that any solution to problem (7.3) bounded on $\mathbb{R}_+$ (on $\mathbb{R}_-$, respectively) decreases at exponential velocity as $t \to +\infty$ (as $t \to -\infty$, respectively).

Exercise *7.13* Assume that equation (7.3) possesses an exponential dichotomy over the half-interval $[a, +\infty)$, where a is a real number. Prove that equation (7.3) possesses an exponential dichotomy over any semiaxis of the form $[b, +\infty)$.
(*Hint*: $P(t) = \Phi(t, a) P(a) \Phi(a, t)$).

Exercise *7.14* Prove the analogue of the assertion of Exercise 7.13 for the semiaxis $(-\infty, a]$.

Exercise *7.15* Prove that for problem (7.3) to possess an exponential dichotomy over $\mathbb{R}$ it is necessary and sufficient that equation (7.3) possesses an exponential dichotomy both over $\mathbb{R}_+$ and $\mathbb{R}_-$ and has no nontrivial solutions bounded on the whole axis $\mathbb{R}$.

Exercise *7.16* Prove that the spaces V_+ and V_- (see (7.9) and (7.10)) possess the properties

$$V_+ \cap V_- = \{0\}, \qquad V_+ + V_- = X \equiv \mathbb{R}^n,$$

provided problem (7.3) possesses an exponential dichotomy over $\mathbb{R}$.

Exercise *7.17* Consider the following equation adjoint to (7.3):

$$\dot{y}(t) = -A^*(t) y(t), \tag{7.11}$$

where $A^*(t)$ is the transposed matrix. Prove that the evolutionary operator $\Psi(t, s)$ of problem (7.11) has the form $\Psi(t, s) = [\Phi(s, t)]^*$.

Exercise *7.18* Assume that problem (7.3) possesses an exponential dichotomy over an interval $\mathcal{I}$. Then equation (7.11) possesses exponential dichotomy over $\mathcal{I}$ with the same constants K, $\alpha > 0$ and projectors $Q(t) = I - P(t)^*$.

Exercise *7.19* Assume that problem (7.3) possesses an exponential dichotomy both over $\mathbb{R}_+$ and $\mathbb{R}_-$. Let $\dim V_+ + \dim V_- = n$, where $V_\pm$ are defined by equalities (7.9) and (7.10). Show that the dimensions of the spaces of solutions to problems (7.3) and (7.11) bounded on the whole axis are finite and coincide.

— Exercise *7.20* Assume that problem (7.3) possesses an exponential dichotomy over $\mathbb{R}$. Then for any $s \in \mathbb{R}$ and $\tau > 0$ the difference equation $x_n = \Phi(s+\tau n, s)x_{n-1}$ possesses an exponential dichotomy over $\mathbb{Z}$ (for the definition see Section 2).

Let us now return to problem (7.1). Assume that z_0 is a hyperbolic fixed point for (7.1), i.e. the matrix $g'(z_0)$ does not have any eigenvalues on the imaginary axis. Let $z(t)$ be a trajectory homoclinic to z_0. Using Theorem 7.1 on the roughness and the results of Exercises 7.13 and 7.14 we can prove that the equation

$$\dot{y} = g'(z(t))\,y \tag{7.12}$$

possesses an exponential dichotomy over both semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$. Moreover, the dimensions of the corresponding projectors are the same and coincide with the dimension of the spectral subspace of the matrix $g'(z_0)$ corresponding to the spectrum in the left semiplane. Therewith it is easy to prove that $\dim V_+ + \dim V_- = n$, where $V_\pm$ have form (7.9) and (7.10). The result of Exercise 7.15 implies that equation (7.12) cannot possess an exponential dichotomy over $\mathbb{R}$ ($y(t) = \dot{z}(t)$ is a solution to (7.12) bounded on $\mathbb{R}$) while Exercise 7.19 gives that the number of linearly independent bounded (on $\mathbb{R}$) solutions to (7.12) and to the adjoint equation

$$\dot{y} = -[g'(z(t))]^* y \tag{7.13}$$

is the same. These facts enable us to formulate Palmer's theorem (see [14]) as follows.

Theorem 7.2.

Assume that** $g(x)$ **is a twice continuously differentiable function from** $\mathbb{R}^n$ **into** $\mathbb{R}^n$ **and equation

$$\dot{x} = g(x)$$

possesses a fixed hyperbolic point** z_0 **and a trajectory** $\{z(t)\colon\ t \in \mathbb{R}\}$ **homoclinic to** z_0. **We also assume that** $y(t) = \dot{z}(t)$ **is a unique (up to a scalar factor) solution to equation

$$\dot{y} = g'(z(t))\,y \tag{7.14}$$

bounded on** $\mathbb{R}$. **Let** $h(t, x, \mu)$ **be a continuously differentiable vector function** T**-periodic with respect to** t **and defined for** $t \in \mathbb{R}$, $|x - z(t)| < \Delta_0$, $|\mu| < \sigma_0$, $\mu \in \mathbb{R}$. **If

$$\int_{-\infty}^{\infty} (\psi(t),\ h(t,\ z(t),\ 0))_{\mathbb{R}^n}\,\mathrm{d}t = 0, \qquad \int_{-\infty}^{\infty} (\psi(t),\ h_t(t,\ z(t),\ 0))_{\mathbb{R}^n}\,\mathrm{d}t \neq 0, \tag{7.15}$$

where** $\psi(t)$ **is a bounded (unique up to a constant factor) solution to the equation adjoint to** (7.14), **then there exist** Δ **and** σ **such that for** $0 < |\mu| < \sigma$ **the perturbed equation

$$\dot{x} = g(x) + \mu h(t,\ x,\ \mu) \tag{7.16}$$

possesses the following properties:

(a) ***there exists a unique*** T***-periodic solution*** $\xi_0(t,\ \mu)$ ***such that***

$$|z_0 - \xi_0(t,\ \mu)| < \Delta, \qquad t \in \mathbb{R}\ ,$$

and

$$\sup_{t\ \in\ \mathbb{R}} |z_0 - \xi_0(t,\ \mu)| \to 0, \qquad \mu \to 0\ ;$$

(b) ***there exists a solution*** $\xi(t,\ \mu)$ ***bounded on*** $\mathbb{R}$ ***and such that***

$$|\xi(t,\ \mu) - z(t)| < \Delta, \qquad t \in \mathbb{R}\ ,$$

$$\sup_{t\ \in\ \mathbb{R}} |\xi(t,\ \mu) - z(t)| = 0(\mu)\ ,$$

and

$$\lim_{t\ \to\ \pm\infty} |\xi(t,\ \mu) - \xi_0(t,\ \mu)| = 0\ ;$$

(c) ***the linearized equation***

$$\dot{y} = \left\{ g'(\eta(t,\ \mu)) + \mu h'_x(t,\ \eta(t,\ \mu),\ \mu) \right\} y, \tag{7.17}$$

where $\eta(t,\ \mu)$ ***is equal to either*** $\xi(t,\ \mu)$ ***or*** $\xi_0(t,\ \mu)$, ***possesses an exponential dichotomy over*** $\mathbb{R}$.

This theorem immediately implies (see Exercise 7.20 and Theorem 3.1) that under conditions (7.15) the monodromy operator for problem (7.16) has a hyperbolic fixed point in a vicinity of the orbit $\{z(t):\ t \in \mathbb{R}\}$ and a transversal trajectory homoclinic to it.

We will not prove Theorem 7.2 here. Its proof can be found in paper [14]. We only outline the scheme of reasoning which enables us to construct a homoclinic trajectory $\xi(t,\ \omega)$. Here we pay the main attention to the role of conditions (7.15). If we change the variable $x = z(t) + \zeta$ in equation (7.16), then we obtain the equation

$$\dot{\zeta} = g(z(t) + \zeta) - g(z(t)) + \mu h(t,\ z(t) + \zeta,\ \mu)\ .$$

We use this equation to construct a mapping $\mathcal{F}$ from $C_b^1(\mathbb{R},\ \mathbb{R}^n) \times \mathbb{R}$ into $C_b^0(\mathbb{R},\ \mathbb{R}^n)$ acting according to the formula

$$(\zeta,\ \mu) \to \mathcal{F}(\zeta,\ \mu) = \dot{\zeta} - \{g(z(t) + \zeta) - g(z(t)) + \mu h(t,\ z(t) + \zeta,\ \mu)\}. \tag{7.18}$$

We remind that $C_b^k(\mathbb{R},\ X)$ is the space of k times continuously differentiable bounded functions from $\mathbb{R}$ into X with bounded derivatives with respect to t up to the k-th order, inclusive.

Thus, the existence of bounded solutions to problem (7.16) is equivalent to the solvability of the equation $\mathcal{F}(\zeta,\ \mu) = 0$. It is clear that $\mathcal{F}(0,\ 0) = 0$. Therefore, in order to construct solutions to equation $\mathcal{F}(\zeta,\ \mu) = 0$ we should apply an appropriate version of the theorem on implicit functions. Its standard statement requires that the operator $L = D_\zeta \mathcal{F}(0,\ 0)$ be invertible. However, it is easy to check that the operator L has the form

$$(Ly)(t) = \dot{y}(t) - g'(z(t))\,y(t) \ .$$

Therefore, it possesses a nonzero kernel $(L\dot{z} = 0)$. Hence, we should use the modified (nonstandard) theorem on implicit functions (see Theorem 4.1 in [14]). Roughly speaking, we should make one more change $\zeta = \mu w$ and consider the equation

$$\mathscr{F}(\mu w,\ \mu) = 0\,. \tag{7.19}$$

If this equation is solvable and the solution w depends on μ smoothly, then w satisfies the equation

$$D_\zeta \mathscr{F}(\mu w,\ \mu)[w + \mu w_\mu] + D_\mu \mathscr{F}(\mu w,\ \mu) = 0\,, \tag{7.20}$$

where w_μ is the derivative of w with respect to the parameter μ. This equation can be obtained by differentiation of the identity $\mathscr{F}(\mu w(\mu),\ \mu) = 0$ with respect to μ. Due to the smoothness properties of the mapping $\mathscr{F}$, it follows from (7.20) the solvability of the problem

$$D_\zeta \mathscr{F}(0,\ 0) w_0 + D_\mu \mathscr{F}(0,\ 0) = 0\ , \tag{7.21}$$

which is equivalent to the differential equation

$$\dot{w}_0 = g'(z(t)) w_0 + h(t,\ z(t),\ 0) \tag{7.22}$$

in the class of bounded solutions. It is easy to prove that the first condition in (7.15) is necessary for the solvability of (7.22) (it is also sufficient, as it is shown in [14]).

Further, the necessary condition of the dichotomicity of (7.17) for $\eta(t,\ \mu) = \xi(t,\ \mu)$ on $\mathbb{R}$ is the condition of the absence of nonzero solutions to equation (7.17) bounded on $\mathbb{R}$. However, this equation can be rewritten in the form

$$D_\zeta \mathscr{F}(\mu w,\ \mu)\, y(\mu) = 0\,, \tag{7.23}$$

where $w = w(\mu)$ is determined with (7.19). If we assume that equation (7.23) has nonzero solutions, then we differentiate equation (7.23) with respect to μ at zero, as above, to obtain that

$$\Big[D_{\zeta\zeta} \mathscr{F}(0,\ 0) w_0 + D_{\zeta\mu} \mathscr{F}(0,\ 0)\Big] y_0 + D_\zeta \mathscr{F}(0,\ 0) y_1 = 0\,, \tag{7.24}$$

where w_0 is a solution to equation (7.21), $y_0 = y(0)$, $y_1 = D_\mu y(0)$, and $y(\mu)$ is a solution to equation (7.23). Equation (7.17) transforms into (7.14) when $\mu = 0$. Therefore, the condition of uniqueness of bounded solutions to (7.14) gives us that $y_0 = c_0 \dot{z}(t)$, therewith we can assume that $c_0 = 1$. Hence, equation (7.24) transforms into an equation for y_1 of the form

$$\dot{y}_1 - g'(z(t)) y_1 = a(t)\,, \tag{7.25}$$

where

$$\begin{aligned} a(t) &= -\Big(\Big[D_{\zeta\zeta} \mathscr{F}(0,\ 0) w_0 + D_{\zeta\mu} \mathscr{F}(0,\ 0)\Big]\dot{z}\Big)(t) = \\ &= \Big[D_{xx}\, g(z(t))\, w_0(t) + D_x h(t,\ z(t),\ 0)\Big]\dot{z}(t)\ . \end{aligned}$$

Here $w_0(t)$ is a solution to (7.22). A simple calculation shows that equation (7.25) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(y_1(t)+\dot{w}_0(t)) - g'(z(t))(y_1(t)+\dot{w}_0(t)) = h_t(t,\ z(t),\ 0)\,. \tag{7.26}$$

The second condition in (7.15) means that equation (7.26) cannot have solutions bounded on the whole axis. It follows that equation (7.23) has no nonzero solutions, i.e. equation (7.17) is dichotomous for $\eta(t,\ \mu) = \xi(t,\ \mu)$.

Thus, the first condition in (7.15) guarantees the existence of a homoclinic trajectory $\xi(t,\ \mu)$ while the second one guarantees the exponential dichotomicity of the linearization of the equation along this trajectory.

As to the existence and properties of the periodic solution $\xi_0(t,\ \mu)$, this situation is much easier since the point z_0 is hyperbolic. The standard theorem on implicit functions works here.

It should be noted that condition (7.15) can be modified a little. If we consider a "shifted" homoclinic trajectory $z_s(t) = z(t-s)$ for $s \in \mathbb{R}$ instead of $z(t)$ in Theorem 7.2, then the first condition in (7.15) can be rewritten in the form

$$\Delta(s) \equiv \int_{-\infty}^{\infty} \Big(\psi(t-s),\ h(t,\ z(t-s),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t = 0\,.$$

If we change the variable $t \to t+s$, then we obtain that

$$\Delta(s) = \int_{-\infty}^{\infty} \Big(\psi(t),\ h(t+s,\ z(t),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t\,. \tag{7.27}$$

It is evident that

$$\Delta'(s) = \int_{-\infty}^{\infty} \Big(\psi(t-s),\ h_t(t,\ z(t-s),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t\,.$$

Therefore, the second condition in (7.15) leads us to the requirement $\Delta'(s) \neq 0$. Thus, if the function $\Delta(s)$ has a simple root s_0 ($\Delta(s_0) = 0$, $\Delta'(s_0) \neq 0$), then the assertions of Theorem 7.2 hold if we substitute the value $z(t-s_0)$ for $z(t)$ in (b). Performing the corresponding shift in the function $\xi(t,\ \mu)$, we obtain the assertions of the theorem in the original form. Thus, condition (7.15) is equivalent to the requirement

$$\Delta(s_0) = 0\,, \quad \Delta'(s_0) \neq 0 \quad \text{for some } s_0 \in \mathbb{R}\,, \tag{7.28}$$

where $\Delta(s)$ has form (7.17).

In conclusion we apply Theorem 7.2 to prove Theorem 6.1. The unperturbed Duffing equation can be rewritten in the form

$$\begin{cases} \dot{x}_1 = x_2\,, \\ \dot{x}_2 = \beta x_1 - \alpha x_1^3\,. \end{cases} \tag{7.29}$$

The equation linearized along the homoclinic orbit $z(t) = (\eta(t),\ \dot\eta(t))$ (see Exercise 7.1) has the form

$$\begin{cases} \dot y_1 = y_2 , \\ \dot y_2 = \beta y_1 - 3\alpha\eta(t)^2 y_1 . \end{cases} \tag{7.30}$$

Let us show that system (7.30) has no solutions which are bounded on the axis and not proportional to $\dot z(t)$. Indeed, if $w(t) = (v(t),\ \dot v(t))$ is another bounded solution, then due to the fact that $|\dot\eta(t)| + |\ddot\eta(t)| \to 0$ as $|t| \to \infty$, the Wronskian $W(t) = v(t)\,\ddot\eta(t) - \dot v(t)\,\dot\eta(t)$ possesses the properties

$$\frac{d}{dt} W(t) = 0 \quad \text{and} \quad \lim_{|t|\to\infty} W(t) = 0 .$$

This implies that $W(t) \equiv 0$ and therefore $v(t)$ is proportional to $\dot\eta(t)$.

Evidently, the equation adjoint to (7.30) has the form

$$\begin{cases} \dot y_1 = -\beta y_2 + 3\alpha\eta(t)^2 y_2 , \\ \dot y_2 = -y_1 . \end{cases} \tag{7.31}$$

Since we have that

$$\ddot y_2 - \beta y_2 + 3\alpha\eta(t)^2 y_2 = 0 ,$$

a solution to (7.31) bounded on $\mathbb{R}$ has the form $\psi(t) = (-\ddot\eta(t);\ \dot\eta(t))$. Let us now consider the corresponding function $\Delta(s)$. Since in this case $h(t, x, \mu) = (0;\ f\cos\omega t - \delta x_2)$, we have

$$\Delta(s) = \int_{-\infty}^{\infty} \dot\eta(t)[f\cos\omega(t+s) - \delta\dot\eta(t)]\,dt ,$$

where

$$\eta(t) = \sqrt{\frac{2\beta}{\alpha}}\,\mathrm{sech}\sqrt{\beta}\,t = \sqrt{\frac{8\beta}{\alpha}}\left(e^{\sqrt{\beta}\,t} + e^{-\sqrt{\beta}\,t}\right)^{-1} .$$

Calculations (try to do them yourself) give us that

$$\Delta(s) = 2f\omega\sqrt{\frac{2}{\alpha}} \cdot \frac{\sin\omega s}{\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right)} - \frac{4}{3\alpha}\delta\beta^{3/2} .$$

Therefore, equation $\Delta(s) = 0$ has simple roots under condition (6.19). Thus, the assertion of Theorem 6.1 follows from Theorem 7.2.

It should be noted that in this case the function $\Delta(s)$ coincides with the famous Melnikov function arising in the geometric approach to the study of the transversality (see, e.g., [1], [2] and the references therein). Therewith conditions (7.28) transform into the standard requirements on the Melnikov function which guarantee the appearance of homoclinic chaos.

Addition to the English translation:

The monographs by Piljugin [1*] and by Palmer [2*] have appeared after publication of the Russian version of the book. Both monographs contain an extensive bibliography and are closely related to the subject of Chapter 6.

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