 first draft: fall 1998

CHAPTER 4

WALLPAPER PATTERNS

4.0 The Crystallographic Restriction

4.0.1 Planar repetition. Even if you don't have one in your own room, you probably see one often at a friend's place or your favorite restaurant: it fills a whole wall with the same motif **repeated** all over in an **'orderly manner'**, creating various visual impressions depending on the particular motif(s) depicted, the background color, etc. And likewise you must have noticed the **tilings** in many bathrooms you have been in: they typically consist of one square tile repeated all over the **bathroom wall**, right? Well, as you are going to find out in this chapter, we can do much better than simply repeating square tiles (plain or not) all over: tilings (and other **repeating designs** as well), be it on Roman mosaics, African baskets, Chinese windows or Escher drawings, can be wonderfully complicated!

You can certainly imagine the wall in front of which you are standing right now extended in **'all directions'** without a bound, thus turning the wallpaper or tiling you are looking at into an **infinite design**; for the sake of simplicity we call any and all such **two-dimensional** (planar) infinite designs that repeat themselves in all directions and 'in an orderly manner' **wallpaper patterns**. For example, the familiar **beehive**, consisting of hexagonal 'tiles', is still viewed here as a wallpaper pattern -- once extended to cover the entire plane, that is. More technically, a wallpaper pattern is a design that covers the entire plane and is **invariant under translation in two distinct**, **non-opposite**, **directions**; check also our definition at the end of 4.0.7 and discussion in section 4.1. Notice at this point that motif repetitions, however 'imperfect' mathematically, are not that rare in nature: think of a leopard's skin or certain butterflies' wings, for example. Moreover, there are zillions of such repetitions and 'orderly packings' to be seen in three dimensions, and in particular when one looks at a **crystal** through a microscope: although this is where this section's title (but not content) comes from, we will not dare venture into three-dimensional symmetry in this book!

4.0.2 Taming the infinite. As we have seen in 2.0.1, infinite border patterns may be 'finitely represented' by strips going around the lateral side of a **'short' cylinder**. Notice at this point that, precisely because they do have a certain finite width, border patterns are, strictly speaking, **'one-and-half'** dimensional: a truly one-dimensional pattern would be something as dull as the infinite repetition of a **Morse signal** (__ . __ . _ _ _ _ . . .), while two-dimensional patterns could only be represented on the lateral side of a **cylinder of infinite height**. But, in the same way an infinite strip can be 'wrapped around' into a 'short' cylinder (of finite height equal to the strip's width), a cylinder of infinite height can be 'wrapped around' into a **torus**: before you get somewhat intimidated by this 'abstract' geometrical term, be aware that this is a familiar item on the breakfast table, be it in the form of a doughnut or a bagel! Yes, you could draw all the wallpaper patterns you will see in this book on a bagel!

Representations of wallpaper patterns by polyhedra may at least be considered. Think of the **soccer ball**, for example, which looks like a beehive consisting of pentagonal and hexagonal 'tiles'; known to chemists as "carbon molecule C_{60}", it does not correspond to a planar (wallpaper) pattern: it is in fact **impossible** to tile the plane with such a combination of regular pentagons and hexagons! Another trick, familiar to map makers and crystallographers, is the **stereographic projection**, that is the representation of the entire plane on a sphere (as in figure 4.1); it clearly maps every point on the plane to a point on the sphere (hence every wallpaper pattern to a 'spherical design'), but it leads to great distortion and problems around the **'north pole'**:

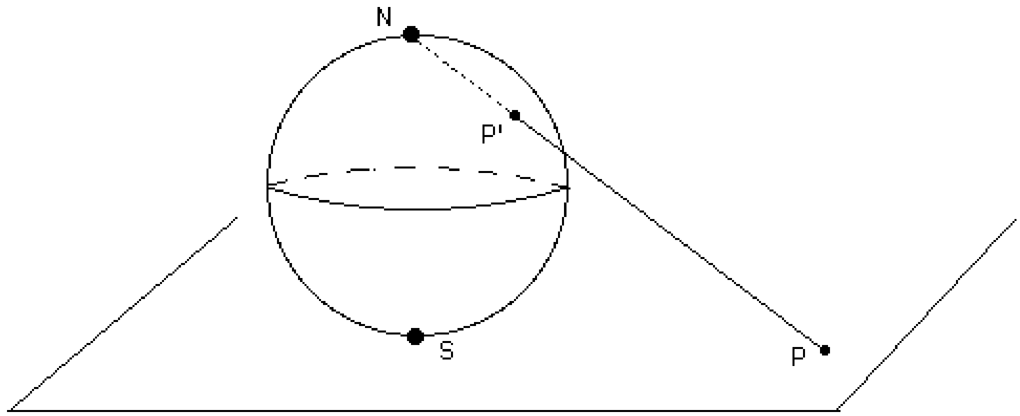


Fig. 4.1

Well, our brief excursion into the three dimensions is over. From here on you will have to keep in mind that, unless otherwise stated, the finite-looking designs in this book are in fact infinite, extending in every direction **around the page** you are looking at; it may not be easy at first, but sooner or later you will get used to the concept!

4.0.3 How about rotation? Let's have a look at the beehive and bathroom wall patterns we mentioned in 4.0.1:

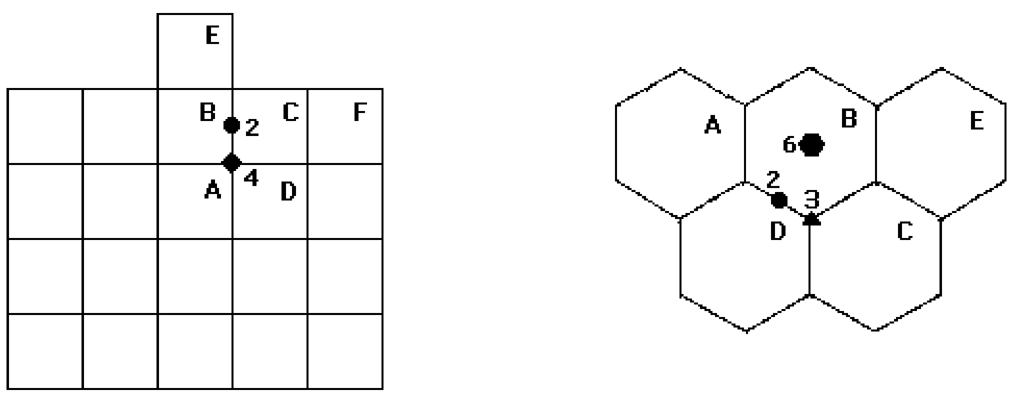


Fig. 4.2

Clearly, a sixfold (**60^0**) clockwise rotation about **6** maps the entire (infinite!) beehive to itself: B is mapped to itself, E to C, C to D, D to A, etc; every hexagonal tile is clearly mapped to another one, and, overall, the entire beehive remains **invariant**. Likewise, a threefold (**120^0**) clockwise rotation about **3** leaves the beehive invariant, mapping B to C, C to D, D to B, A to E, etc; and, a twofold (**180^0**) rotation about **2** does just the same, mapping D to B (again!), A to C, etc. We describe these facts by saying that the beehive has 60^0 rotation (about **6** and all other hexagon centers), 120^0 rotation (about **3** and all other hexagon vertices), and 180^0 rotation (about **2** and all other midpoints of hexagon edges). Notice that the existence of 60^0 rotation in a wallpaper pattern always implies the existence of 120^0 and 180^0 rotations about the same center: for example, applying **twice** the 60^0 rotation centered at **6** yields a 120^0 rotation (mapping E to D, C to A, etc), while a **triple** application leads to a 180^0 rotation (mapping E to A, etc). Further, the existence of 60^0 centers **implies** the existence of **'genuine'** 120^0 and 180^0 centers (7.5.4).

Visiting the bathroom wall now, we see that it has both **90^0** and **180^0** rotation. Indeed a clockwise fourfold (90^0) rotation about **4** leaves it invariant (mapping A to B, B to C, C to D, D to A, E to F, etc), and so does a twofold (180^0) rotation about **2** (mapping B to C, D to E, etc). In fact the middle of every square is **also** the center of a 90^0 rotation (as well as a 180^0 rotation via a **double** application of the 90^0 rotation), while the midpoint of every square edge is the center of a 180^0 rotation (but **not** a 90^0 rotation!).

So, we have just seen that wallpaper patterns can have twofold, threefold, fourfold, and sixfold rotations (by 180^0, 120^0, 90^0, and 60^0, respectively). More precisely, we have seen examples of wallpaper patterns where the **smallest rotation** is 60^0 (beehive) or 90^0 (bathroom wall). As we will see in the rest of this chapter, there also exist wallpaper patterns with smallest rotation 120^0 (somewhat exotic) and 180^0 (very common), as well as wallpaper patterns with no rotation at all. A very important question is: are there any other 'smallest' rotations besides those by 60^0, 90^0, 120^0, and 180^0? Are there any wallpaper patterns with **fivefold** rotation (**72^0**), for example? The answer to these questions is negative, and we devote the rest of this section to establish this important fact, known in the literature as the **Crystallographic Restriction** and central in proving that there exist **precisely seventeen types of wallpaper patterns**. (We describe these types in the rest of chapter 4, but we defer their classification to chapter 8.)

4.0.4 Rotation centers translated. In section 1.4 we defined glide reflection as the combination of a reflection and a translation parallel to each other, and we observed that the two operations commute with each other only when the reflection axis and the gliding vector are parallel to each other (1.4.2).

Asking the same question about rotation and translation leads **always** to a negative answer. We may confirm this in the context of the bathroom wall of figure 4.2 placed now in a coordinate axis (figure 4.3): consider for example **R**, the clockwise 90^0 rotation about (0, 0), and **T**, the translation by the vector <1, 1>; it can be verified, using techniques from either chapter 3 (see right below) or chapter 1, that **R∗T** (**T** followed by **R**) is the clockwise 90^0 rotation about (–1, 0), while **T∗R** (**R** followed by **T**) is the clockwise 90^0 rotation about (1, 0).

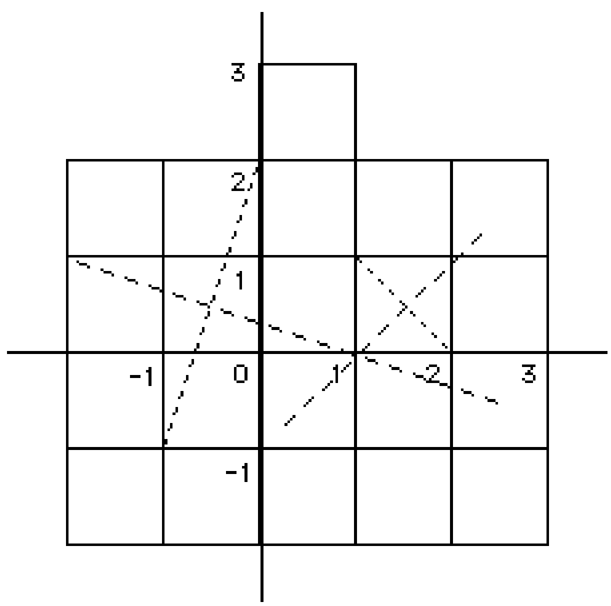


Fig. 4.3

Concerning the latter, notice that **R** maps (–1, –1) to (–1, 1), and subsequently **T** maps (–1, 1) to (0, 2); likewise, (1, 1) is mapped by **R** to (1, −1), which is in turn mapped by **T** to (2, 0). So **T∗R**, which **must** be a rotation (why?), maps (−1, −1) to (0, 2) and (1, 1) to (2, 0). With the perpendicular bisectors of the segments joining (−1, −1), (0, 2) and (1, 1), (2, 0) intersecting each other at (1, 0) (figure 4.3), it is easy from here on to verify that **T∗R** is indeed the clockwise 90^0 rotation about (1, 0).

At this point, you may ask: how come **T∗R** is **not** $\mathbf{R_T}$, the **'translated'** clockwise 90^0 **rotation** about (1, 1)? Shouldn't the translation **T** 'translate' the entire rotation **R** the same way it **translates** its **center** (0, 0) to (1, 1)? Well, we already proved in the preceding paragraph, and may further confirm here, that this is not the case: for example, $\mathbf{R_T}$ maps (0, 2) to (2, 2) instead of its image under **T∗R**, which is (3, 1). But **observe** at this point that the one point that $\mathbf{R_T}$ maps to (3, 1) is no other than the point (1, 3), which happens to be the image of (0, 2) under translation by **T**! Likewise, if we **first translate** (0, 1) **by T** to (1, 2) and **then rotate** (1, 2) **by** $\mathbf{R_T}$ we end up mapping (0, 1) to (2, 1), exactly as **T∗R** does! And so on.

Putting everything together, it seems that $\mathbf{R_T * T}$, that is **T** followed by $\mathbf{R_T}$, has the same effect as **R** followed by **T**, that is **T∗R**: in the language of Abstract Algebra, $\mathbf{R_T * T} = \mathbf{T * R}$. 'Multiplying' both sides by $\mathbf{T^{-1}}$ (**T**'s **inverse**, that is a translation by a vector **opposite** -- see 1.1.2 -- to that of **T** that **cancels T**'s effect), we obtain $\mathbf{R_T = T * R * T^{-1}}$; in even more algebraic terms, we have shown that $\mathbf{R_T}$ is the **conjugate** of **R** by **T**. Switching to Geometry and moving away from the bathroom wall, we offer a 'proof without words' (figure 4.4) of the following fact: for every translation **T** and every rotation $\mathbf{R = (K, \phi)}$, the **'product'** $\mathbf{T * R * T^{-1}}$ is indeed the rotation $\mathbf{R_T = (T(K), \phi)}$, that is, **R 'translated' by T**. (You may of course provide a rigorous geometrical proof, especially in case you are aware of the fact that any two isosceles triangles of equal bases and equal top angles must be congruent!)

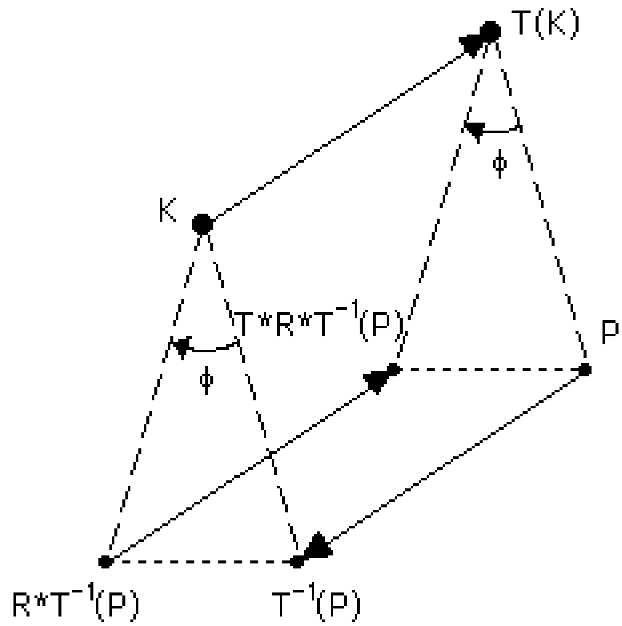


Fig. 4.4

Since compositions of isometries leaving a wallpaper pattern invariant leave it invariant, too, we conclude that we may indeed assume the following: in every wallpaper pattern, the image of the center of a rotation **R** by a translation **T** is the center for a new rotation ($\mathbf{T*R*T^{-1}}$ rather than **T∗R** or **R∗T**) by the **same angle**. This follows from the more general fact depicted in figure 4.4, was empirically confirmed in the case of the bathroom wall, and may be further verified in the cases of the beehive and the other wallpaper patterns you are going to see in this chapter.

It follows that the existence of a single rotation center in a wallpaper pattern implies the existence of infinitely many rotation centers all over the plane! Indeed, there exist two distinct, non-opposite translations in our pattern, say <p, q> and <r, s>, hence translating the rotation center by the four distinct translations <p, q>, <r, s>, <–p, –q>, and <–r, –s> -- notice that if a translation leaves a wallpaper pattern invariant then so does its opposite -- we produce four new rotation centers around the old one. Repeating this process to all new centers again and again we end up with an **infinite lattice** of rotation centers, shown in figure 4.5 below for the cases of the beehive and the bathroom wall. Observe that there exist in fact **three lattices in one** in the case of the beehive, consisting of 60^0, 120^0, and 180^0 centers, and **two lattices in one** in the case of the bathroom wall, consisting of 90^0 centers and 180^0 centers. (There is more than meets the eye here: there really are two kinds of 90^0 centers in the bathroom wall -- only one of which was shown in figure 4.2 -- the translations of which may transport us from one kind to another only in a rather ‘indirect’ manner (7.6.3); and similar remarks apply to the beehive’s 120^0 centers and to the 180^0 centers of both the beehive and the bathroom wall.)

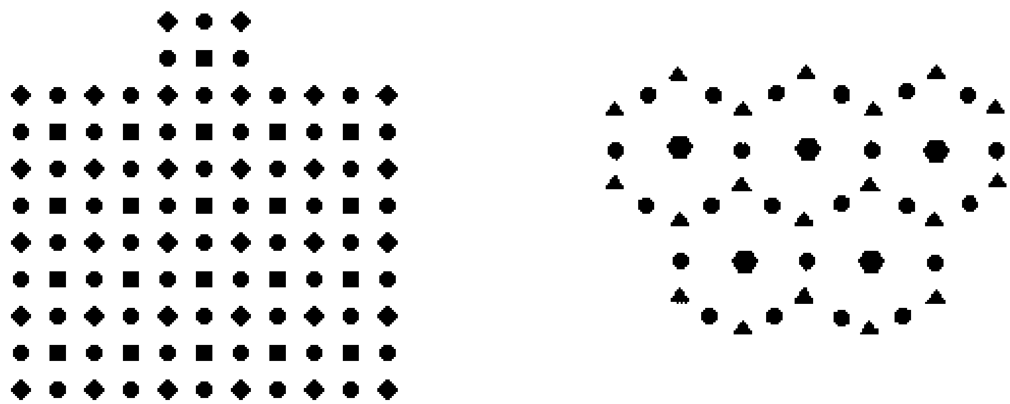


Fig. 4.5

Notice the lack of rotation centers other than the ones shown in figure 4.5: in a wallpaper pattern rotation centers cannot be arbitrarily close to each other, in the same way that translation vectors **cannot be arbitrarily small** -- this is what Arthur L. Loeb calls **Postulate of Closest Approach** in his ***Concepts and Images*** (Birkhauser, 1992). For a challenge to this principle and further discussion you may like, if adventurous enough, to have a look at 4.0.7. It seems in fact that there is an interplay between translation vectors and distances between rotation centers, to the extend that you might venture to guess that every vector starting at a rotation center and ending at a center for a rotation by the **same angle** is in fact a translation vector for the entire pattern: this is true for 60^0 centers but not for 90^0, 120^0, or 180^0 centers, as you may verify for yourself (and has been hinted on at the end of the preceding paragraph); still, there are interesting facts relating the **distances** between rotation centers to the **lengths** of translation vectors that you should perhaps explore on your own!

4.0.5 Rotation centers rotated. Let's have another look at the beehive pattern and its various rotation centers, as featured in figure 4.2 (entire pattern) or figure 4.5 (centers only). It seems clear that the rotation about a randomly chosen center (be it for 60^0, 120^0, or 180^0) of **every** other center (be it for 60^0, 120^0, or 180^0) moves it to another center (for a rotation by the **same angle**); for example, rotating a 120^0 center about a 60^0 center (by 60^0, of course) we get another 120^0 center, rotating a 60^0 center about a 180^0 center (by 180^0) we get another 60^0 center, etc. Similar observations may be made for the bathroom wall and, in fact, every wallpaper pattern that has one, therefore infinitely many, rotation centers: wallpaper patterns are indeed wonderful!

Moving away from the harmonious world of wallpaper patterns, we must ask: is it true in general that rotations always rotate rotation centers to rotation centers? To be more specific, consider two rotations, $\mathbf{R_1 = (K_1, \phi_1)}$ and $\mathbf{R_2 = (K_2, \phi_2)}$: is it true that $\mathbf{R_1(K_2)}$, that is $\mathbf{K_2}$ rotated about $\mathbf{K_1}$ by $\boldsymbol{\phi_1}$, is a center for a rotation by $\boldsymbol{\phi_2}$? The answer is "yes", and the rotation in question is no other than $\mathbf{R_1 * R_2 * R_1^{-1}}$, the **conjugate** of $\mathbf{R_2}$ by $\mathbf{R_1}$: the same algebraic operation employed in 4.0.4 to express the translation of a rotation works here for the **rotation of a rotation**! While a computational proof using the rotation formulas of section 1.3 certainly works, the easiest way to demonstrate this wonderful fact is a **geometrical** 'proof without words' (figure 4.6 below) in the spirit of figure 4.4; we take both $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ to be clockwise, but you may certainly verify that this is an unnecessary restriction.

We should note in passing that 4.0.4 and 4.0.5 (and figures 4.4 & 4.6 in particular) are special cases of a broader phenomenon that we will encounter again and again in chapter 6 (starting at 6.4.4) and section 8.1 (and the rest of chapter 8): the **'image' of an isometry** by another isometry is again an isometry; we should probably remember this fact under a name like **Mapping Principle**, but we will later call it **Conjugacy Principle** on account of the algebraic realities discussed above.

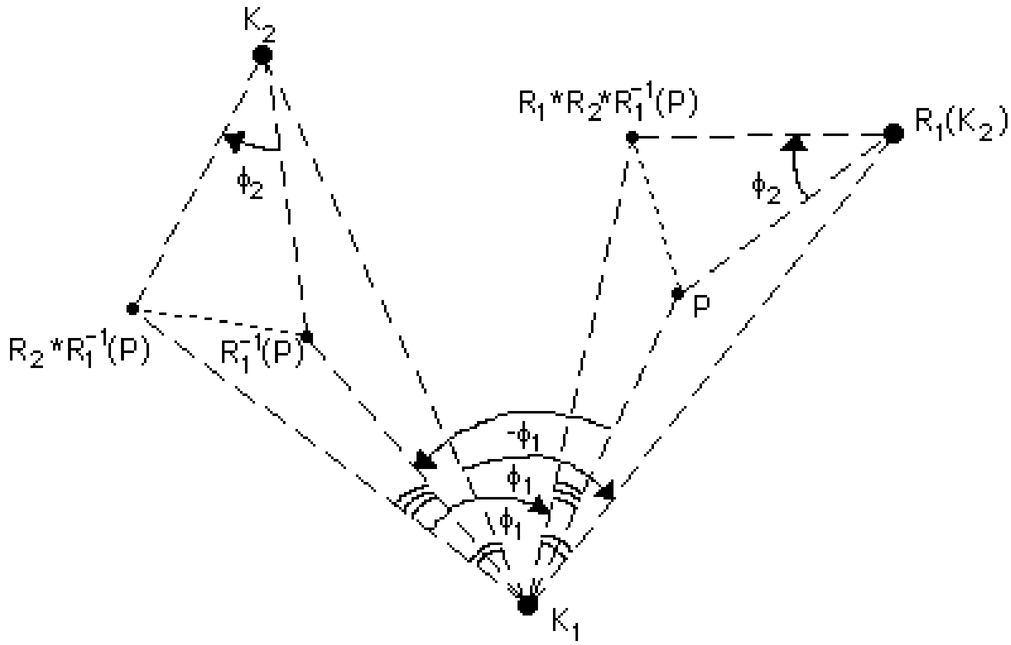


Fig. 4.6

As in 4.0.4, we must stress that the rotations $\mathbf{R_1 * R_2}$, $\mathbf{R_2 * R_1}$, and $\mathbf{R_1 * R_2 * R_1^{-1}}$ are all distinct: we will thoroughly examine such compositions of rotations and other isometries in chapter 7.

4.0.6 Only four angles are possible! At long last, we are ready to **establish** the Crystallographic Restriction. Assume that a certain wallpaper pattern remains invariant under rotation by an angle ϕ, and pick two centers at the **shortest possible distance** (4.0.4) from each other, K_0 and K_1. Let us also assume that $\mathbf{0^0 < \phi \leq 180^0}$: in case $\phi > 180^0$ we can work with the angle $360^0 - \phi$, which also leaves the wallpaper pattern invariant. We may now (4.0.5) **rotate $\mathbf{K_1}$** by a counterclockwise ϕ **about $\mathbf{K_0}$** in order to get a new center K_2, and then rotate K_2 (about K_0 and by counterclockwise ϕ always) to obtain yet another center K_3, and so on. For how long can we continue this way, producing new centers on the **'rotation center circle'** (figure 4.7) of center K_0 and radius $|K_0K_1|$? In theory (and absence of the assumption that $|K_0K_1|$ is the minimal possible distance between any two distinct rotation centers) for ever; in practice not for too long, as **no** center is allowed to fall within an **'arc distance'** of **less**

than 60^0 from K_1, unless it returns to K_1: otherwise we would have two rotation centers at a **distance smaller than $|K_0K_1|$** from each other! (Think of an isosceles triangle K_0K_1K where K is the multi-rotated K_1 and $|K_0K_1| = |K_0K|$; if the angle $\angle K_1K_0K$ is smaller than 60^0 then the other two angles are bigger than 60^0, therefore $|KK_1|$ would be smaller than $|K_0K_1|$.)

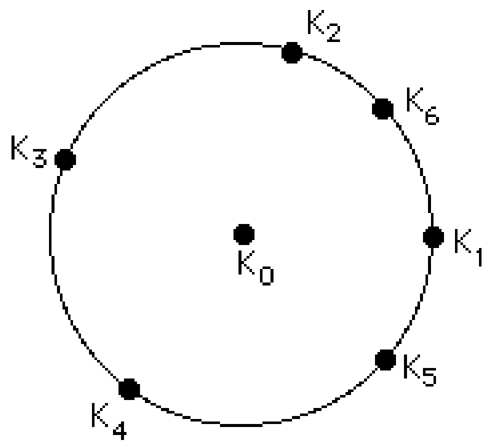


Fig. 4.7

Let now N be the unique integer such that **$N\times\phi \leq 360^0 < (N+1)\times\phi$**: that is, N records how many rotations are required for K_1 to either **return** to K_1 ($N\times\phi = 360^0$) or **bypass** K_1 ($N\times\phi < 360^0 < (N+1)\times\phi$).

In the latter case (**$N\times\phi < 360^0$**) we must also assume, in order to avoid the **'forbidden arc'**, the inequalities **$360^0 - N\times\phi \geq 60^0$** and **$(N+1)\times\phi - 360^0 \geq 60^0$**; these inequalities lead to $300^0/N \geq \phi$ and $\phi \geq 420^0/(N+1)$, respectively. It follows that $300/N \geq 420/(N+1)$, so $300\times(N+1) \geq 420\times N$ and $300 \geq 120\times N$; we end up with **$N \leq 2.5$**, hence either N = 1 or N = 2. The case N = 1 is ruled out by $\phi \leq 180^0$, while in the case N = 2 the inequalities $300^0/N \geq \phi$ and $\phi \geq 420^0/(N+1)$ yield **$140^0 \leq \phi \leq 150^0$**. But if K_2 lies on the arc $[140^0, 150^0]$ then K_3 lies on the arc $[280^0, 300^0]$, K_4 on the arc $[60^0, 90^0]$, K_5 on $[200^0, 240^0]$, and K_6 on $[340^0, 390^0]$ = **$[-20^0, 30^0]$**, which is part of the 'forbidden arc': K_1's trip ends up in a disaster, **unless perhaps $\phi = 144^0$** (the solution of the **'return equation'** $5\times\phi = 2\times360$), in which case K_1 quietly returns to itself with $K_6 \equiv K_1$ (figure 4.8). But in that case a (counterclockwise) rotation by 144^0 applied twice certainly yields a (counterclockwise) rotation by 288^0, hence a (clockwise) rotation by $360^0 - 288^0 =$ **72^0**, a rotation that will be ruled out further below.

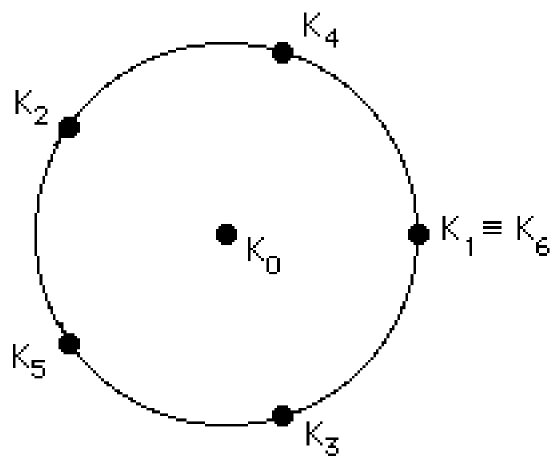


Fig. 4.8

In the former case (**$N\times\phi = 360^0$**) we substitute $\phi = 360^0/N$ into the inequality $(N+1)\times\phi - 360^0 \geq 60^0$ to get $(N+1)\times360/N - 360 \geq 60$ and, eventually, **$N \leq 6$**; more intuitively, we must have **$\phi \geq 60^0$** or else K_2 would fall into that 'forbidden arc' discussed above. After discarding the case N = 1 ($\phi = 360^0$ -- no rotation), we are left with the cases N = 2 ($\phi = 180^0$), N = 3 ($\phi = 120^0$), N = 4 ($\phi = 90^0$), N = 5 ($\phi = 72^0$), and N = 6 ($\phi = 60^0$); 'global rotations' by all these angles are possible and familiar to you by now, **except for $\phi = 72^0$** (the angle that tormented many artists only a few centuries ago!). To render a rotation by 72^0 impossible for a wallpaper pattern, we simply rotate **K_0 about K_1** by **clockwise 72^0** to a rotation center **K'_0** (figure 4.9): it is obvious now that **$|K_2K'_0|$ is smaller than $|K_0K_1|$**, thus violating the assumption on the minimality of $|K_0K_1|$! (To be precise, trigonometry yields $|K_2K'_0| = (\sin18^0/\sin54^0) \times |K_0K_1| \approx .38 \times |K_0K_1|$.)

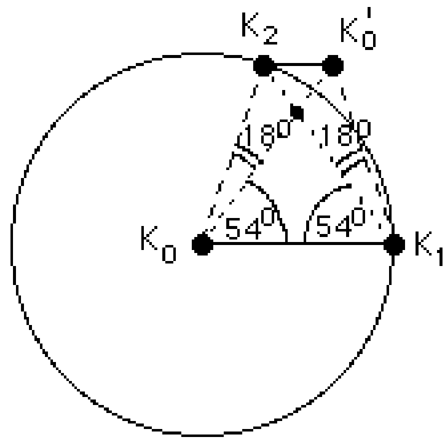


Fig. 4.9

We conclude that a wallpaper pattern either has no rotation at all or that the smallest rotation that leaves it invariant can only be by one of the four angles that we couldn't rule out: **60^0**, **90^0**, **120^0**, **180^0**. Based on this fact, we naturally split wallpaper patterns into **five families**: those that have no rotation at all (or, equivalently, smallest rotation 360^0), and those of smallest rotation 60^0, 90^0, 120^0, and 180^0, respectively; this greatly facilitates their classification into **seventeen distinct types** (chapter 8), as well as their descriptions in this chapter (sections 4.1 through 4.17).

4.0.7* An 'exotic' pattern and a definition. Most available proofs of the crystallographic restriction seem to follow, in one way or another, **W. Barlow**'s proof, published in Philosophical Magazine in 1901; such is the case, for example, with both H. S. M. Coxeter's ***Introduction to Geometry*** (Wiley, 1961) and David W. Farmer's ***Groups and Symmetry: A Guide to Discovering Mathematics*** (American Mathematical Society, 1996). These proofs assume **both** Loeb's Postulate of Closest Approach (4.0.4), which guarantees a minimum distance between rotation centers, also assumed in our proof, and the fact that a wallpaper pattern's smallest rotation angle is of the form **$360^0/n$** (where n is an integer), which we did **not** assume. Our example below presents a clear challenge to both these assumptions.

Let **S** be the set of all **rational points** in the plane, that is, the set of all points both coordinates of which are rational numbers; notice that S is **dense** in the plane, in the sense that every circular disk, no matter how small, contains **infinitely many** elements of S. What if we consider S to be a wallpaper pattern? It certainly has **translations in infinitely many directions**: for every pair of rational numbers, a and b, T(x, y) = (a+x, b+y) defines a translation <a, b> that leaves S invariant! Observe also that S has 180^0 rotation about every point (c, d) in S, defined by R(x, y) = (2c–x, 2d–y); this already shows that rotation centers of S can indeed be arbitrarily close to each other. Moreover, S has **rotation about each point of S by infinitely many angles**: every angle both the sine and the cosine of which are rational would map a rational point to a rational point, as the rotation formulas of 1.3.7 would demonstrate; and for every pair of integers m, n, such an angle is actually defined via $\sin\phi = \frac{2mn}{m^2+n^2}$ and $\cos\phi = \frac{m^2-n^2}{m^2+n^2}$, thanks to $|2mn| \leq m^2+n^2$, $|m^2-n^2| \leq m^2+n^2$, and the **Pythagorean identity** $(2mn)^2 + (m^2-n^2)^2 = (m^2+n^2)^2$!

So, S is indeed a pattern invariant under rotation by angles other than $360^0/n$ that has rotation centers at arbitrarily small distances from each other. In case you protest the fact that S consists of single points, we can easily modify it to look more ‘pattern-like’. For example, we can augment every rational point (a, b) to a square ‘frame’ defined by the points (a–r, b–r), (a–r, b+r), (a+r, b–r), and (a+r, b+r), where r is an arbitrary **rational** number. As each such ‘frame’ contains many points with one or two irrational coordinates, you may protest that the union of all the ‘frames’ (over all (a, b) and all r) is no other than the entire plane: that turns out not to be the case, because each ‘frame’ is **‘thin’** (in the sense that it contains no full disks) and a theorem in **Topology** -- many thanks to **Robert Israel**, who helped this former topologist recall his first love by way of a sci.math discussion! -- called **Baire Category Theorem** states that the plane cannot be a **countably infinite** union of such ‘thin’ sets. This much you could perhaps see even without this heavy-duty theorem -- the union of all ‘frames’ contains no points **both** coordinates of which are irrational! -- but you would need the theorem in case our ‘extended pattern’ contains not only the ‘frames’ described above but their images by **all** rotations of S described in the preceding paragraph as well: yes, this extended pattern **S#** that inherits all the translations and rotations of S and seems to be everywhere is **still a countable union of ‘thin’ sets**, hence not the entire plane! (There is of course a bit more to this Baire Category Theorem, as you may find out by checking any undergraduate Topology book; one suggestion is George F. Simmons’ ***Introduction to Topology and Modern Analysis*** (McGraw-Hill, 1963).)

‘In practical terms’ now, exotic wallpaper patterns such as S and S# cannot quite exist (as art works) in the real world: for every bit of paint (or even ink) contains a miniscule full disk -- recall **Buckminster Fuller**’s statements about “every line having some width and structure” and “every circle being a polygon with enormously many sides” (***Loeb***, p. 126) -- and, **‘reversing’** the Baire Category Theorem, we easily conclude that every pattern containing such disks **and** having arbitrarily small translations must equal/blacken the entire plane! That is, art works -- which cannot be infinite to begin with -- cannot have arbitrarily small translations, hence, less obviously, must also satisfy that **Postulate of Closest Approach** (no arbitrarily small distances between rotation centers): indeed, as we will see in 7.5.2, one can always ‘combine’ two rotations (by the same angle but of opposite orientations) to produce a translation (of vector length not exceeding twice the distance between the two centers).

A broader way of ruling out arbitrarily small translations is the following definition (certainly satisfied by art works): a wallpaper pattern S is a **countable union of congruent sets S_n** that is invariant under translation in two distinct, non-opposite directions, and has also the property that **every disk intersects at most finitely many S_ns**. (In the case of the beehive and the bathroom wall the S_ns are (boundaries of) regular hexagons and squares, respectively; and in the case of the sets S and S# -- not accepted as wallpaper patterns under this definition due to failure of the **finite intersection property** -- the S_ns are rational points and rational points surrounded by those rationally rotated concentric rational square frames, respectively.)

4.1 360^0, translations only (p1)

4.1.1 Stacking **p111**s. What happens when we fill the plane with copies of a **p111** border pattern placed right above/below each other in **'orderly'** fashion, whatever that means? We obtain wallpaper patterns like the ones shown below:

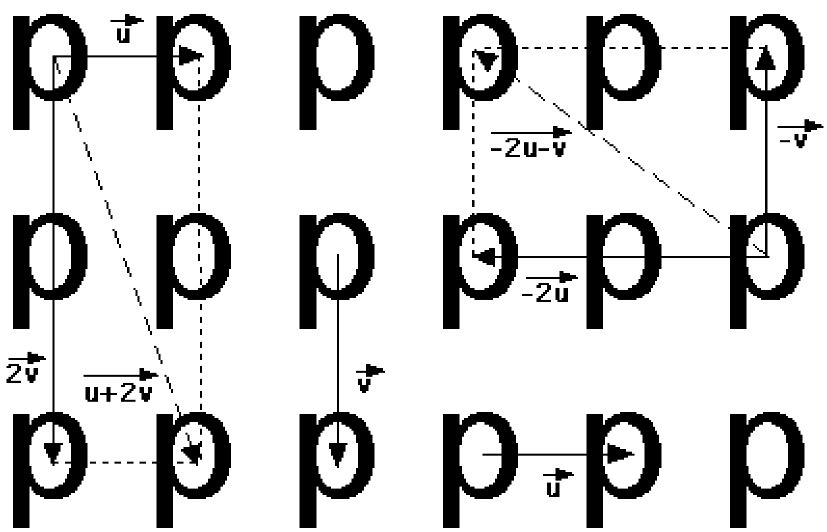


Fig. 4.10

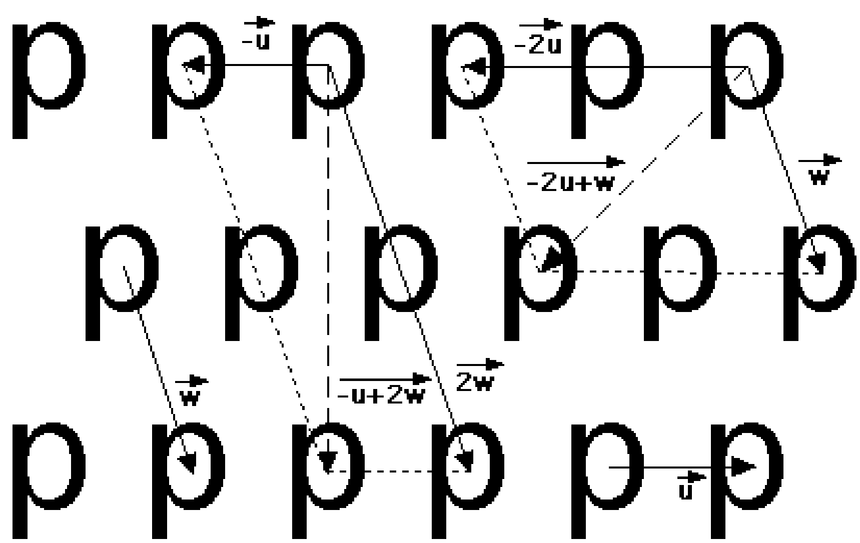


Fig. 4.11

While looking different from each other, these two wallpaper patterns are ‘mathematically identical’: they both have translation in two, thus **infinitely many** directions, and no other isometries. It is only their **minimal translation vectors** that separate them: horizontal $\vec{u}$ and vertical $\vec{v}$ (figure 4.10) versus horizontal $\vec{u}$ and diagonal $\vec{w}$ (figure 4.11); notice that the two patterns share many translation vectors, like $\vec{u}$, $\vec{u}+2\vec{v} = 2\vec{w}$, $-\vec{u}+2\vec{w} = 2\vec{v}$, etc. (Vectors are added following the **parallelogram rule** familiar from Physics, see figures 4.10 & 4.11; and it is this addition’s nature that leads to the infinitude of translations alluded to right above.) Such wallpaper patterns are denoted by **p1** and are the simplest of all.

Is it possible to stack copies of the **“p ”** border pattern in some kind of **‘disorderly’** fashion so that the end result is **not** a wallpaper pattern? The answer is “yes”, and here is an example:

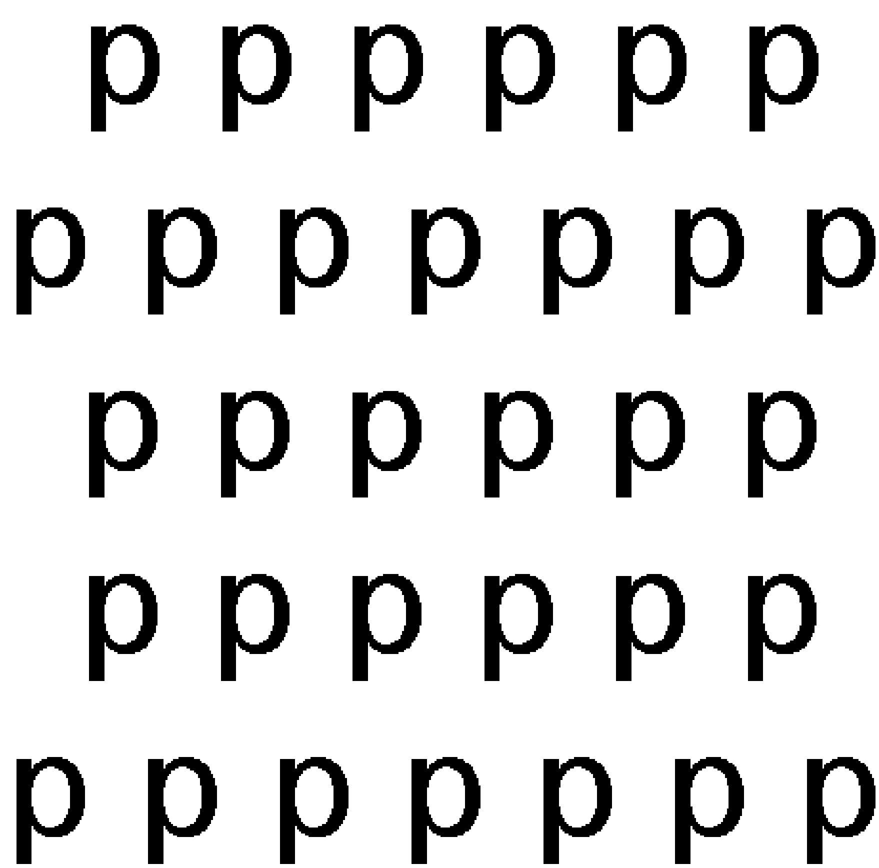


Fig. 4.12

What went wrong with the design in figure 4.12? To answer this question you have to know, if you have not guessed it already, what the rest of the design is! Recall that all wallpaper patterns are infinite, and you must always be able to imagine their extension beyond the page you are reading! This ‘extension’ is normally not that difficult to see (as long as you remember that you have to, of course), but in the case of the ‘pathological’ pattern of figure 4.12 you may need some help: we start with a copy of the **“p ”** border pattern, then place **one** ‘shifted’ copy right below it, continue with **two** ‘straight’ copies underneath, then **one** shifted copy below them, then **three** straight copies again, then **one** shifted copy, and so on; the same process applies to all rows above the top one in figure 4.12. We leave it to you to verify that this ...32123...-like design is **not** a wallpaper pattern: all you have to do is to verify that it has **translations in only one direction**, the horizontal one.

4.1.2 Pis all the way! Below you find another design that fails to be a wallpaper pattern by having translation in only one direction, in this case the vertical one; unlike the one in figure 4.12, built by disorderly stacking of a border pattern, this one is built by orderly stacking of an one-dimensional design that is **not** a border pattern:

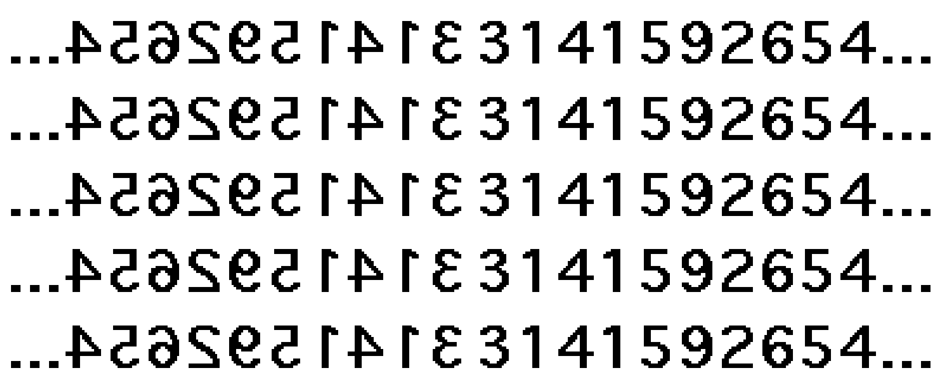


Fig. 4.13

In case you haven’t noticed, the protagonist here is no other than **π ≈ 3.141592654...**, well known to have an infinite, non-repeating decimal expansion: don’t be fooled, **one** reflection alone (right in the middle) cannot produce a translation!

4.1.3 <u>From the land of the Incas.</u> Here is a very geometrical Inca design that, in spite of its geometrical beauty and complexity, has no isometries other than translations, therefore it is classified as a **p1** wallpaper pattern (***Stevens***, p. 180):

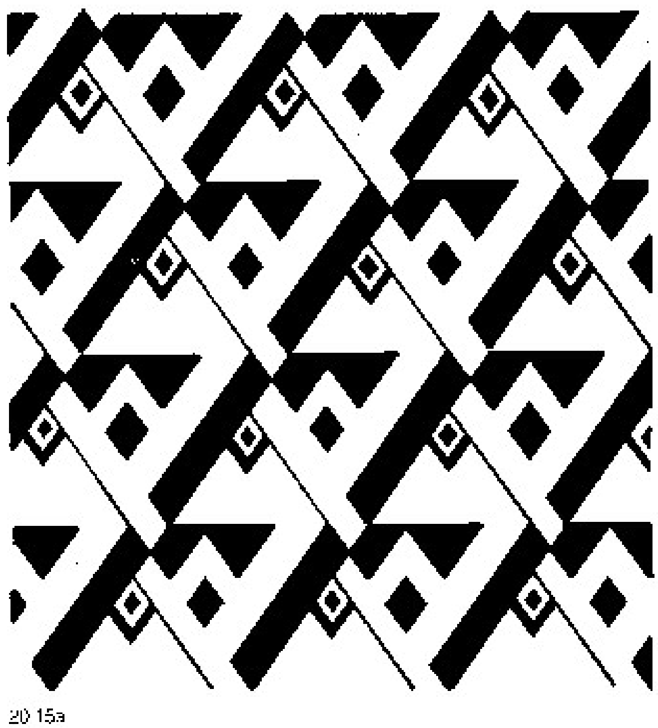


Fig. 4.14

4.2 360^0 with reflection (pm)

4.2.1 <u>Straight stacking of **pm11**s.</u> You have certainly noticed that the design in figure 4.13 has mirror symmetry. Due to the lack of horizontal translation, however, there exists one and only one reflection axis that works. To obtain a wallpaper pattern with infinitely many reflection axes (all **parallel** to each other), we can resort to the process of 4.1.1, stacking copies of a **pm11** border pattern this time:

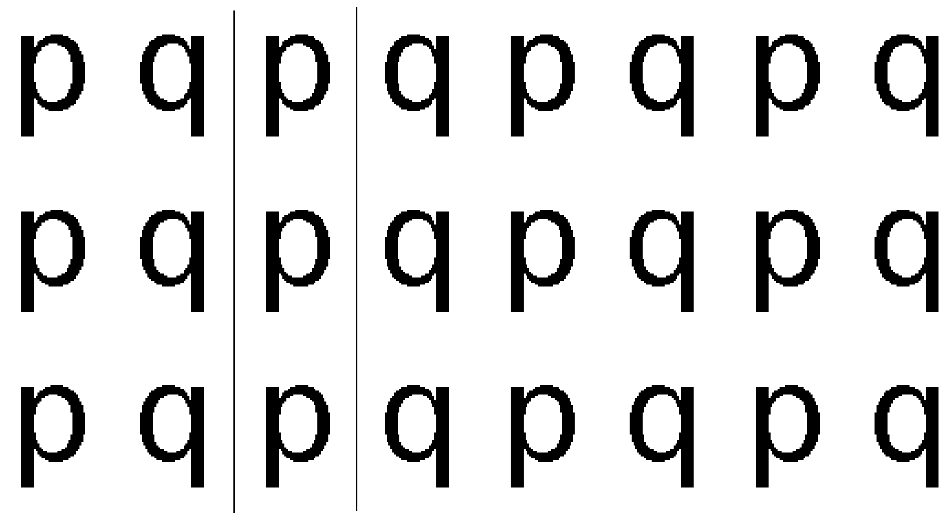


Fig. 4.15

You recognize of course the **"p q "** border pattern of 2.2.2 and figure 2.6. The wallpaper pattern in figure 4.15 has automatically inherited all its symmetries (like vertical reflection and horizontal translation) in a rather obvious manner; in addition to those, **'straight'** stacking -- **every p straight above a p and every q straight above a q** -- has created vertical translation.

Such wallpaper patterns generated by straight stacking of a **pm11** border pattern and having reflection in one direction (and no rotation of course) are denoted by **pm**.

4.2.2 Two kinds of mirrors. Just like **pm11** border patterns, all **pm** wallpaper patterns have two kinds of reflection axes; this is for example the case with the wallpaper pattern of figure 4.15. We illustrate this phenomenon with a more geometrical example, stressing once again the fact that reflection axes are allowed to go **through** the motifs (in this case being identical to the trapezoids' own reflection axes):

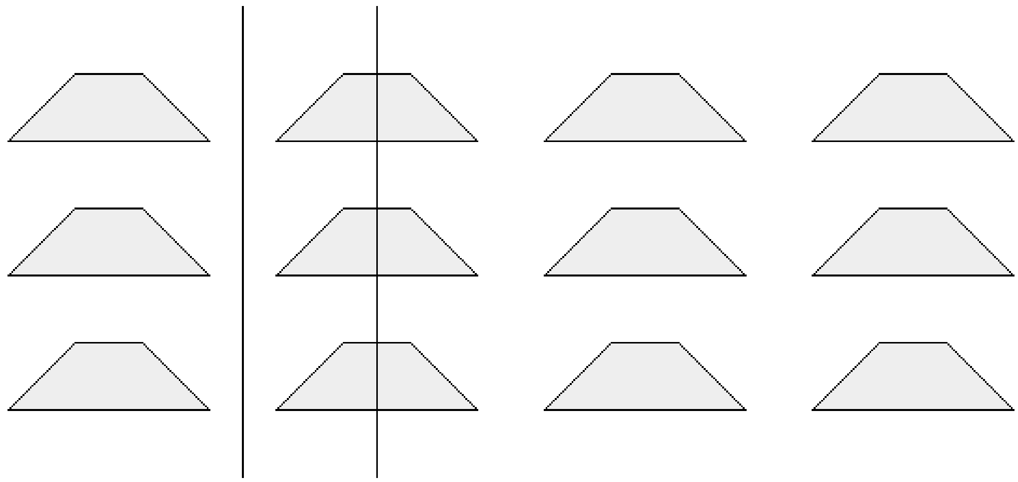


Fig. 4.16

4.2.3 Ancient Egyptian oxen. The following example of a **pm** pattern (***Stevens***, p. 193) is dominated by the stillness that tends to characterize the **pm** patterns (as well as oxen in general):

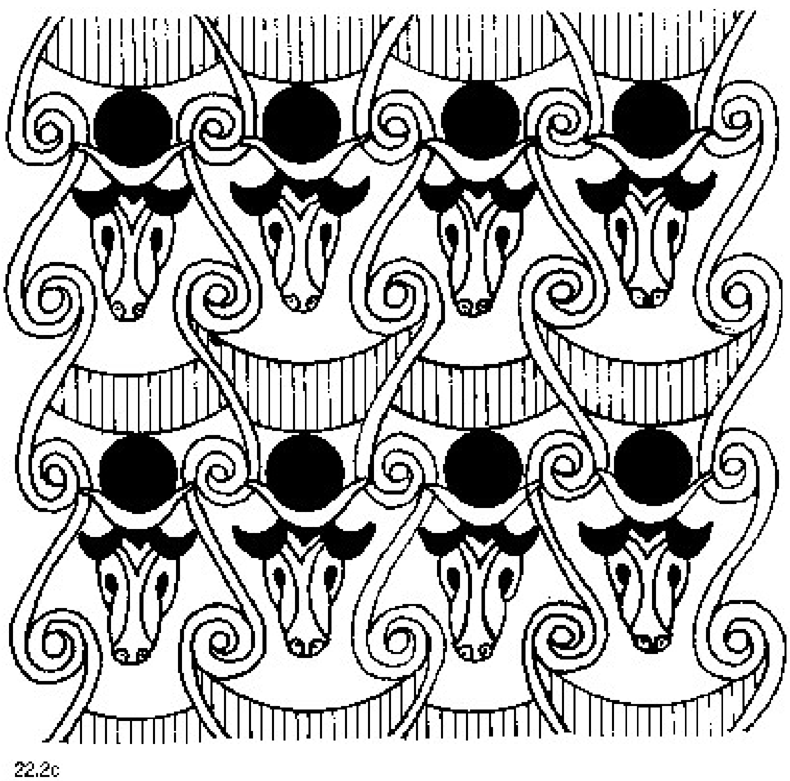


Fig. 4.17

Thanks to the **cascading spires** between the oxen, there are still **two kinds** of reflection axes, even though they all ‘dissect’ the oxen!

4.3 360^0 with glide reflection (pg)

4.3.1 Shifted stackings of **pm11**s. What happens when we stack the **“p q ”** pattern in a ‘disorderly’ manner, as shown right below?

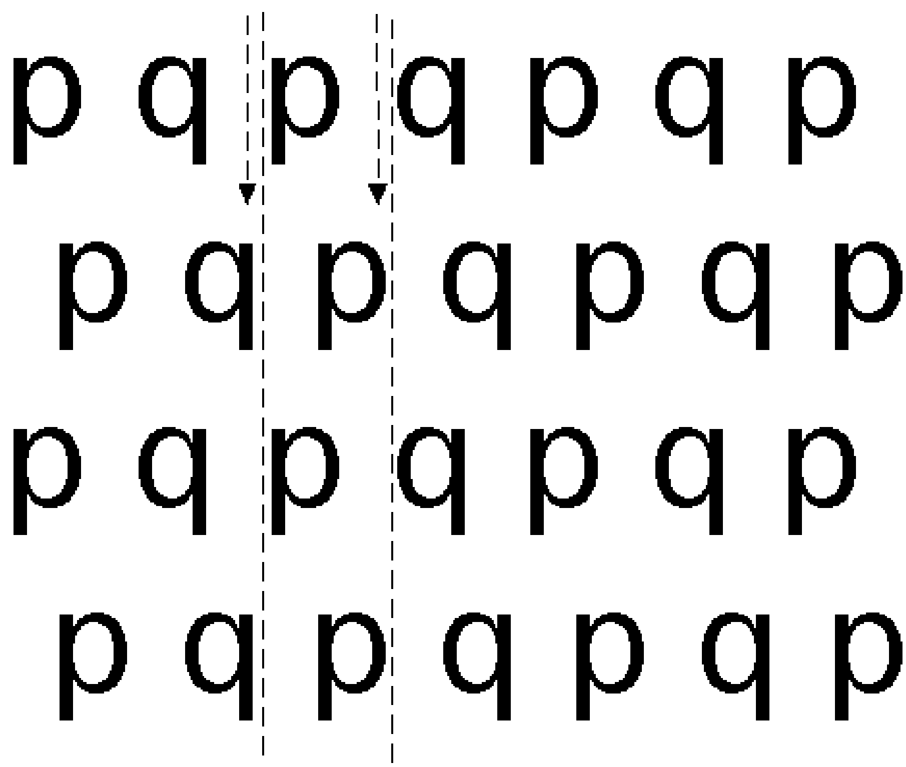


Fig. 4.18

Clearly, the **shifting of every other row** has eliminated any possibility for reflection, but it has generated **two kinds** of vertical **glide reflection**, as shown in figure 4.18. Such rotationless wallpaper patterns with glide reflection are denoted by **pg**; they may be obtained either as a shifted stacking of a **pm11** border pattern (figure 4.18) or by shifting every other row in a **pm** wallpaper pattern, as the following modification of figure 4.16 (and determination of glide reflection axes based on chapter 3 methods) demonstrates:

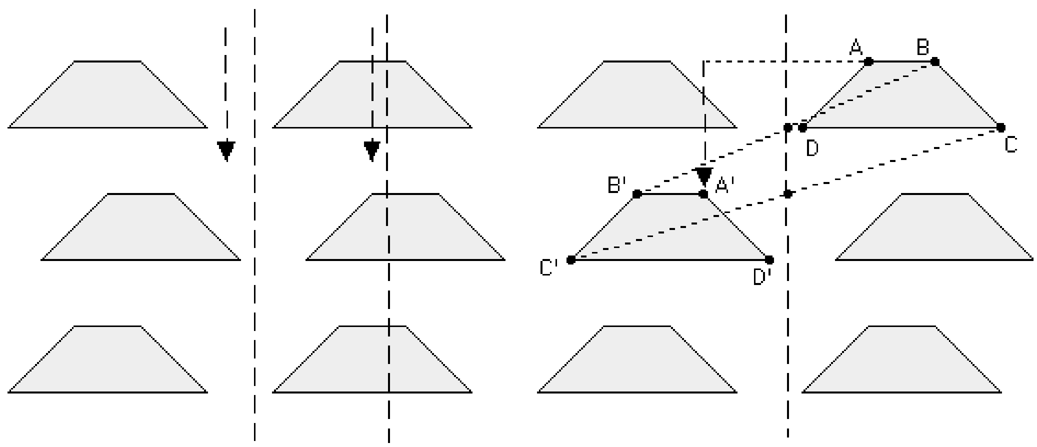


Fig. 4.19

4.3.2 <u>Straight stackings of **p1a1**s.</u> Can we get a **pg** wallpaper pattern by stacking copies of a border pattern with glide reflection?

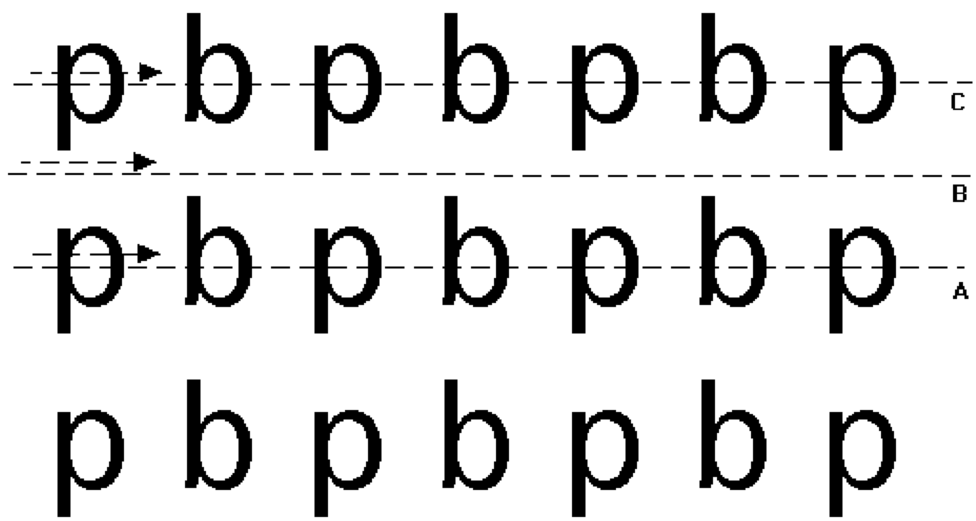


Fig. 4.20

As figure 4.20 illustrates, this is certainly possible: our pattern inherits the **horizontal** glide reflection from the **p1a1** border pattern of figure 2.15 (crossing right **through** the stacks, like lines **A** and **C**), and it has its own, ‘stack-gluing horizontal glide reflection (with axes running right **between** the stacks, like line **B**).

4.3.3 Between **pg** and **p1**. What kind of wallpaper pattern is the one obtained via a **shifted** stacking of copies of **"p b "**?

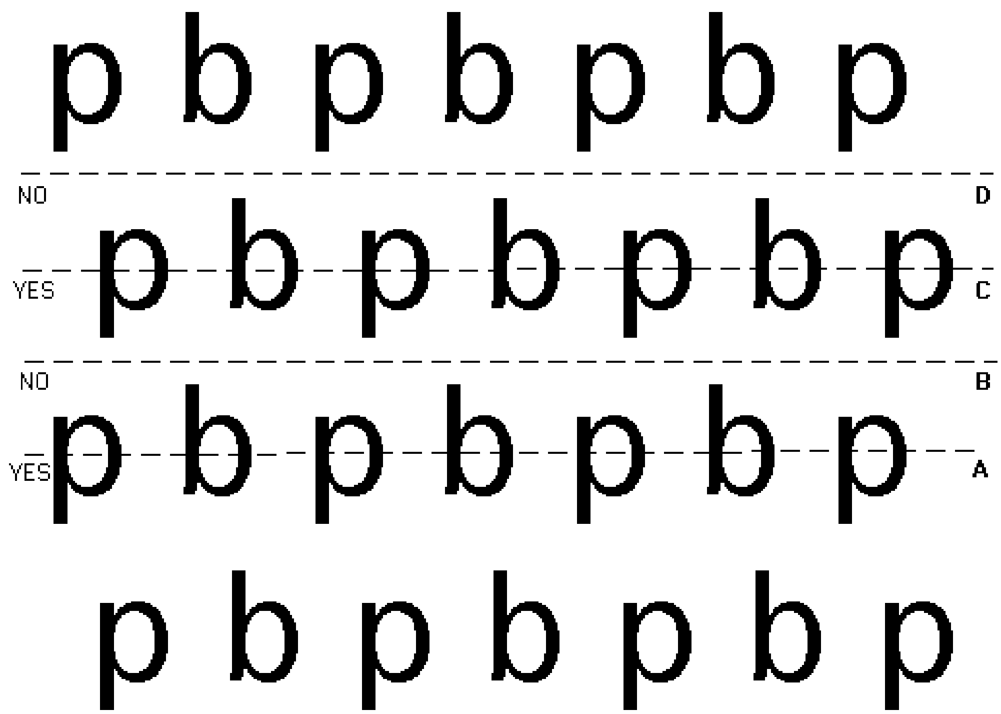


Fig. 4.21

The wallpaper pattern shown in figure 4.21 is a 'complicated' one: it has glide reflection along the lines **A** and **C**, exactly as the pattern in figure 4.20, but **not** along lines **B** or **D**! Indeed line **B** (or **D**) fails to be a glide reflection axis for the same reason that the border pattern in figure 2.16 does not have glide reflection: it would require **two distinct vectors** -- short vector sending C-letters to A-letters, long vector sending A-letters to C-letters -- in order to work as a glide reflection axis! So, and unless one checks **only** axes like **B** or **D**, our pattern is classified as a **pg** rather than a **p1**. Interestingly, this **pg** pattern may be viewed as a straight stacking of a **p111** border pattern (consisting of the strip between two **B**-like axes, for example)!

How does one 'see' glide reflection in a wallpaper pattern where **not** all motifs are **homostrophic**, distinguishing between **pg** and **p1**? One trick is suggested by our observation in 2.4.2 that remains valid for wallpaper patterns, too: the glide reflection vector is always equal to **half** of a translation vector -- but not vice versa, as the **pg** has glide reflection in **only one** direction and translation in **infinitely many** directions... So, first you use your intuition to pick the **'right' direction**, next you translate a motif by half the minimal translation vector in that direction, and finally you look for a reflection axis that maps it to another motif: for example, trapezoid ABCD in figure 4.19 is first vertically translated right across the reflection axis from trapezoid A′B′C′D′.

4.3.4 Peruvian birds. We conclude this section with an example of a Peruvian **pg** pattern from ***Stevens*** (p. 188):

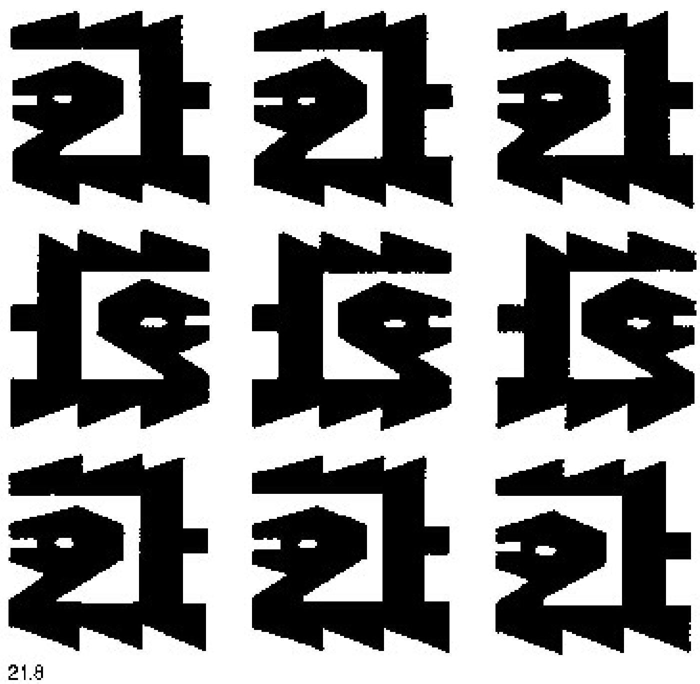


Fig. 4.22

Clearly, there are two flocks of birds 'flying' in opposite directions, and that feeling of 'opposite' movements **perpendicular** to the direction of the glide reflection is quite common in **pg** patterns; you can see that in the wallpaper pattern of figure 4.20, for example (especially if you turn the page sideways), but not quite in those of figures 4.18 or 4.19 -- can you tell why?

4.4 360^0 with reflection and glide reflection (cm)

4.4.1 A 'perfectly shifted' stacking of **pm11**s. What if we shift every other row in the pattern of figure 4.18 a bit further, pushing **every p straight above a q and vice versa**? Here is the result:

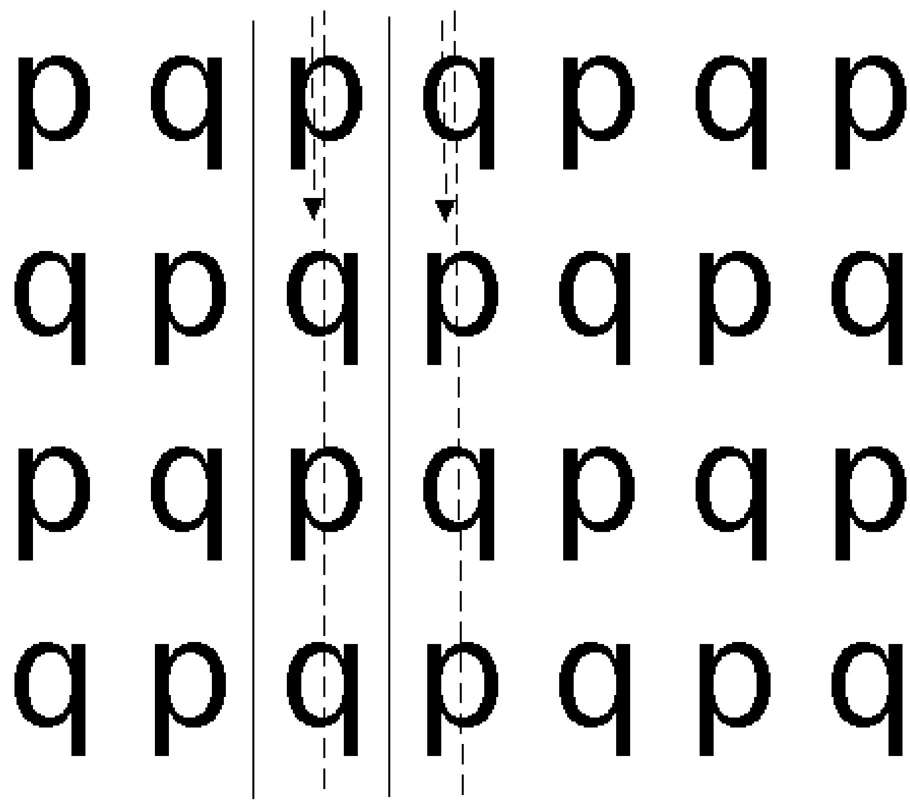


Fig. 4.23

The wallpaper pattern in figure 4.23 looks like **'both'** a **pm** and a **pg**, as reflection axes alternate with glide reflection axes: it is in fact a 'new' type, known as **cm**.

4.4.2 More perfectly shifted stackings. What has made the patterns of figures 4.15, 4.18, and 4.23 different? Well, a straight stacking of the **pm11 "p q "** border pattern simply preserved the border pattern's reflection and created a **pm** wallpaper pattern in figure 4.15; a **'random'** shifting of every other row 'replaced' the reflection of the **pm** pattern by glide reflection and created a **pg** pattern in figure 4.18; and, finally, a **'perfect'** shifting of every other row 'preserved' the glide reflection of the **pg** pattern **and** 'revived' the lost reflection, creating a **cm** pattern. But, what do we mean by "perfect shifting"? Well, the following example may help you answer this question:

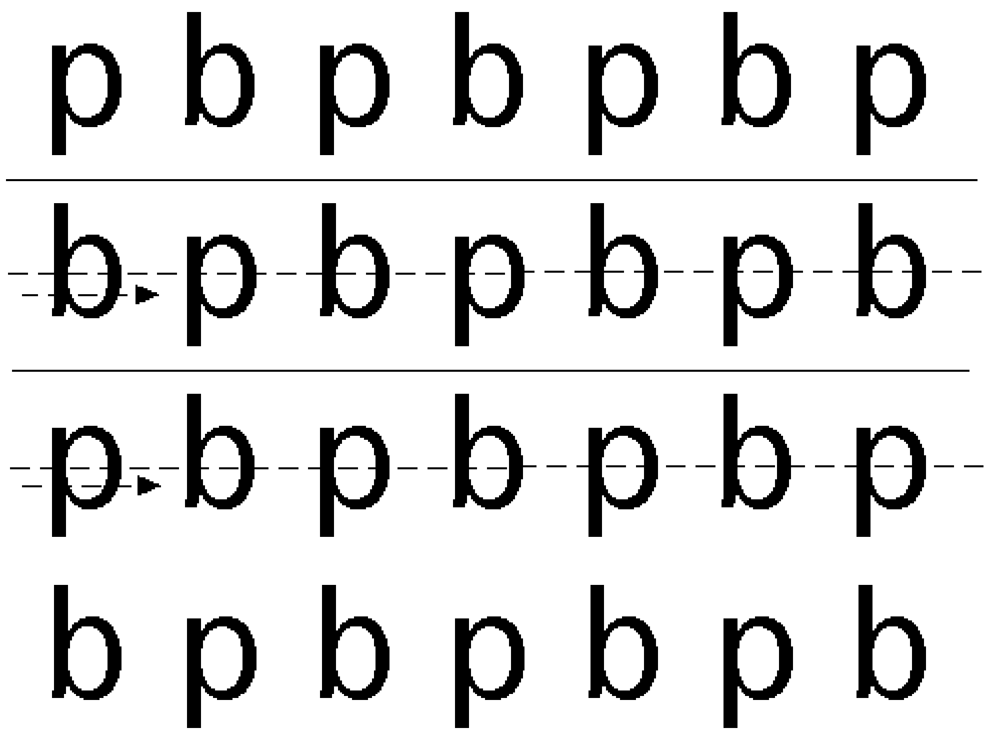


Fig. 4.24

We just obtained another **cm** wallpaper pattern, this one with horizontal reflection and glide reflection, stacking copies of the **"p b " p1a1** border pattern: just as in figure 4.23, placing every **p** straight below a **b** and vice versa allows for some reflection that we couldn't possibly have in the patterns of figures 4.20 & 4.21 (consisting of straight stackings and randomly shifted stackings of that **"p b "** border pattern, respectively). A closer look reveals that it was crucial to shift every other row by a vector equal to **half the minimal translation vector** of the original border pattern! That's what we mean by "perfect shifting", as opposed to "random shifting" (by a vector of length either strictly smaller or strictly bigger than half the minimal translation vector's length). By the way, the pattern in figure 4.11 is the result of a perfect shifting!

We leave it to you to check that perfectly shifted stackings of **p1m1** border patterns are **cm** wallpaper patterns, while their randomly shifted stackings are **pm** wallpaper patterns: you may of course use the **D**-pattern of figure 2.8 to verify this.

4.4.3 <u>In-between glide reflection.</u> Consider the following trapezoid-based wallpaper pattern:

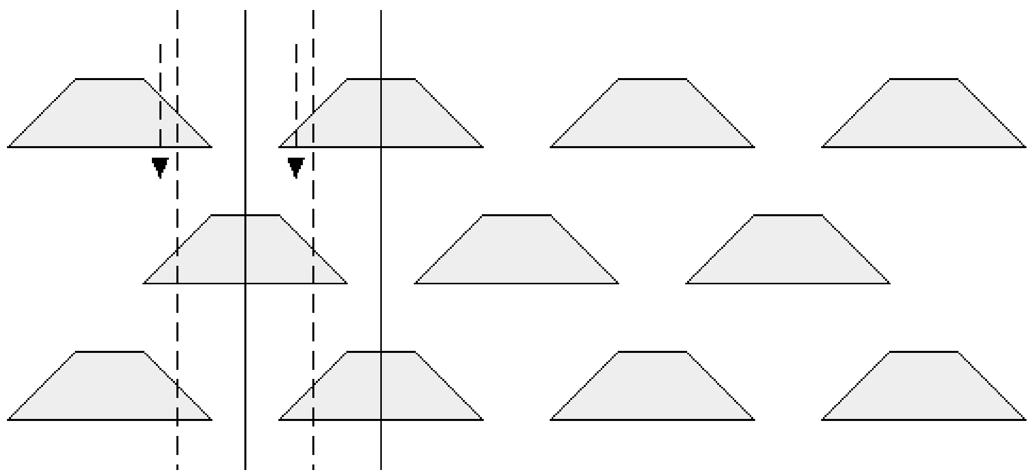
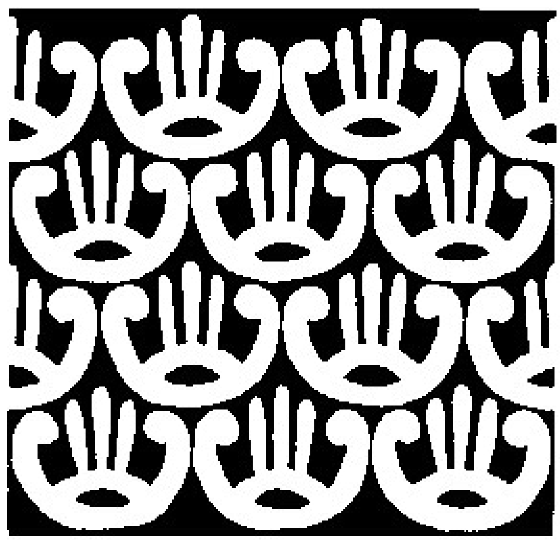


Fig. 4.25

Many students will typically see either all the reflections or half of them and quickly classify it as a **pm** pattern. Having just gone through 4.4.2, you are of course likely to recognize it as either a perfectly shifted version of the **pm** pattern of figure 4.16 or a perfectly shifted stacking of a **pm11** border pattern: either way, it is clearly a **cm** pattern!

Are there any ways of seeing the glide reflection **'directly'**? One could employ the machinery of chapter 3, as we did in 4.3.1, or resort to the idea discussed in 4.3.3. An easier approach takes advantage of the very structure of the **cm** type and the fact that its glide reflection axes always run **half way** between two nearest reflection axes: once you have determined the reflection axes in what seems to be a **pm** pattern, draw a line half way between them and check whether or not there is a vector that makes it work: if yes your pattern is a **cm**, if not your pattern ‘remains’ a **pm**. In short, every time you see reflections in a wallpaper pattern **check** whether or not there exists **in-between glide reflection**.

4.4.4 Phoenician funerary ‘crowns’. The following design from a Phoenician tomb in Syria (***Stevens***, p. 202) shows that the **cm** type has been with us for a very long time; but this is the case with most, if not all, types of wallpaper patterns...



23.5b

Fig. 4.26

The Phoenicians were a naval superpower more than twenty five centuries ago, but the **cm** remains popular with our times’ superpower: next time you stand **close** to the Star-Spangled Banner, have a **careful** look at its stars!

4.4.5 Diagonal axes. Reflection and glide reflection axes do not always have to be ‘vertical’ or ‘horizontal’; they may certainly run in every possible direction, and the concept of direction is a **relative** one, as it changes every time you rotate the page a bit! Here we present an interesting example of a **cm** pattern with easy-to-see **‘diagonal’** reflection and more subtle in-between glide reflection:

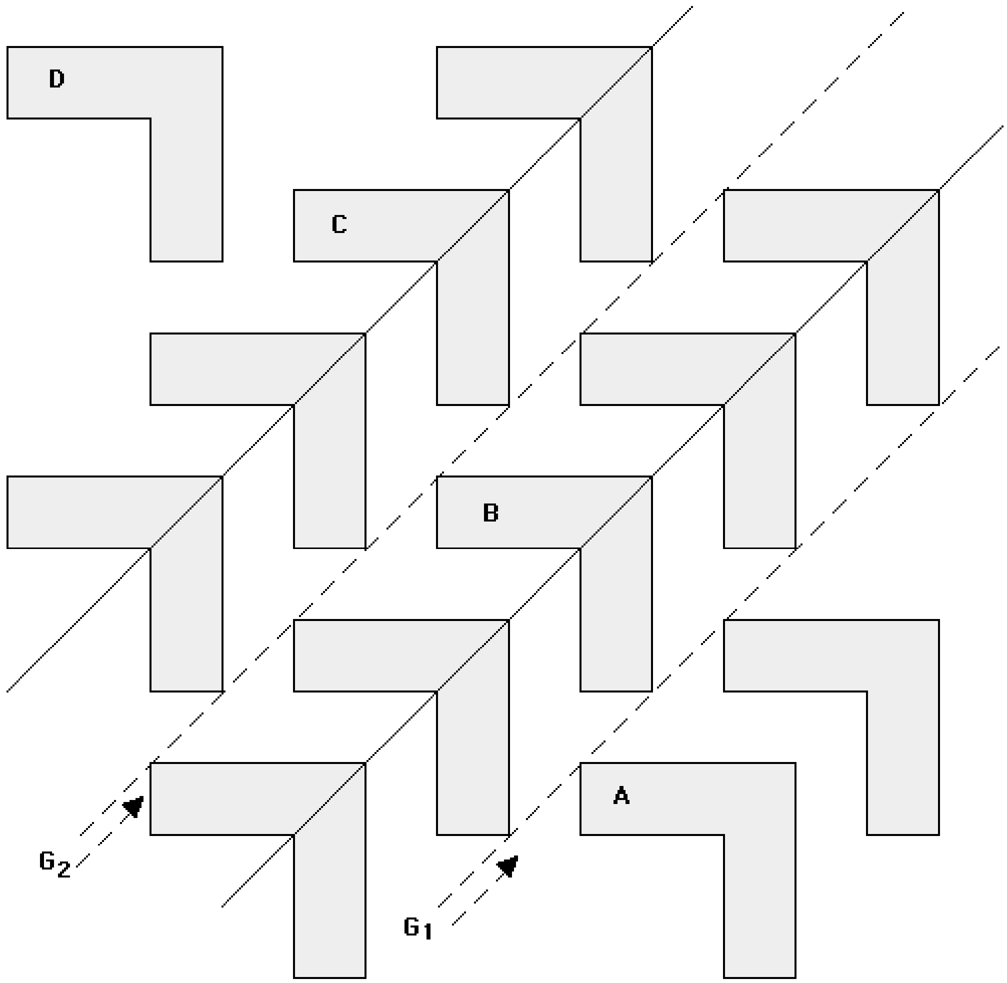


Fig. 4.27

Under glide reflection $\mathbf{G_1}$, for example, **A** is mapped to **B**, while glide reflection $\mathbf{G_2}$ maps **A** to **D** and **B** to **C**, etc.

4.4.6 <u>Only one kind of axes!</u> While our examples in sections 4.2 and 4.3 show that there are always two kinds of reflection and glide reflection axes in **pm** and **pg** wallpaper patterns, respectively (both in the same direction, of course), all the examples in this section clearly indicate that every **cm** wallpaper pattern has only **one** kind of reflection axes and only **one** kind of glide reflection axes as well. We elaborate on this observation in 6.4.4 and 8.1.5, as well as in 4.11.2. For the time being we would like to point out that, in the case of the **cm**, it seems that whatever we **gained** in terms of symmetry we **lost** in terms of diversity! In other words, whenever all vertical reflection axes look the same to you, look out for that in-between glide reflection: your pattern is probably not a **pm** but a **cm**! Likewise, if all the glide reflection axes in a seemingly **pg** pattern look the same to you, then either you have **missed the 'other half'** of the glide reflection or you have **achieved the impossible**: you saw the glide reflection without seeing the reflection ... and your pattern is probably a **cm** rather than a **pg**!

4.5 180^0, translations only (p2)

4.5.1 <u>Stacking **p112**s.</u> Replacing the **"p "** border pattern of 4.1.1 by the **"p d "** border pattern, we obtain the following wallpaper patterns, direct analogues of the **p1** patterns in figures 4.10 & 4.11:

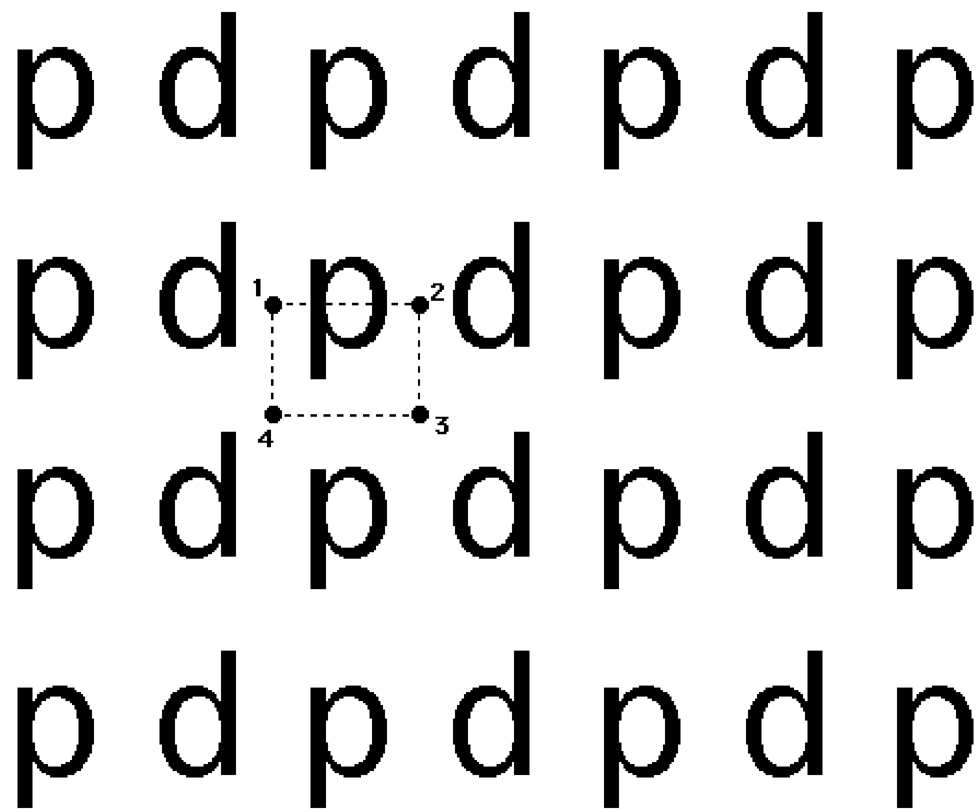


Fig. 4.28

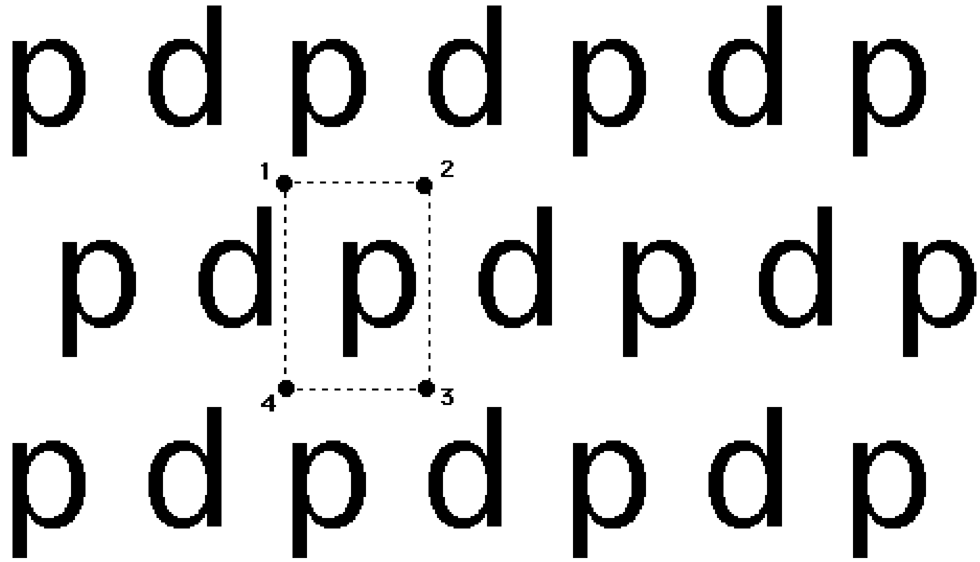


Fig. 4.29

Such patterns, having nothing but **half turn** -- in addition to translation, always -- are known as **p2**. As you can see, there exist **four kinds** of rotation centers, nicely arranged at the **vertices of rectangles** (and numbered **1**, **2**, **3**, **4**). These rectangles are usually mere **parallelograms** -- as in figure 8.18, think for example of a **p2 tiling** of the plane by copies of a single parallelogram -- but they may on occasion be rhombuses or even squares:

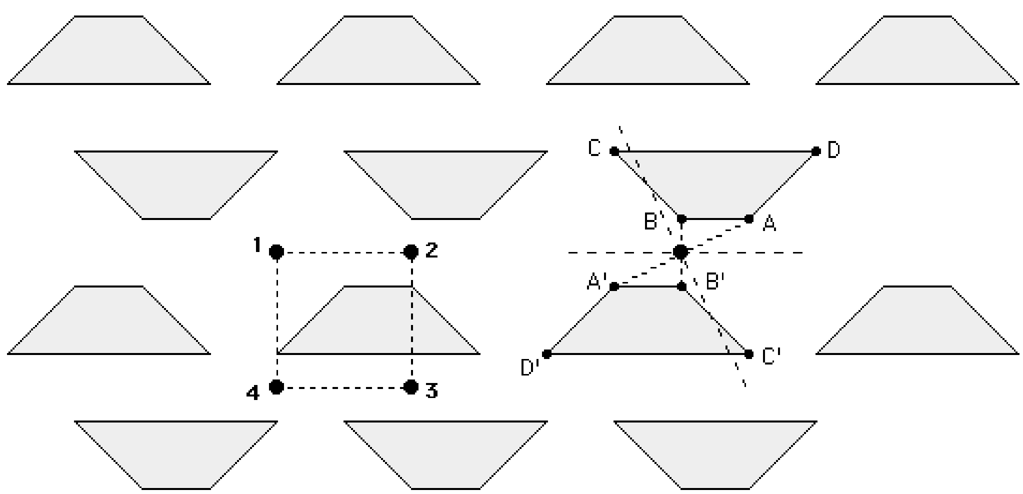


Fig. 4.30

4.5.2 <u>When is the twofold rotation there?</u> How could you tell that the wallpaper pattern in figure 4.28 has 180^0 rotation without some familiarity with the **p112** border pattern that created it? Well, the easiest way is to turn the page upside down and decide whether or not the pattern still looks the same ... keeping always in mind the fact that all patterns are **infinite**. It is always better, on the other hand, to be able to determine some 180^0 rotation centers: this you can do either based on your intuition and experience or following methods from chapter 3, as shown in figure 4.30; and then you can always confirm your findings using tracing paper!

As we will see in the next four sections, it is easier to find the twofold rotation centers when the given pattern happens to have some (glide) reflection: then the location of the rotation centers is, more or less, **predictable**. Within the **p2**, once you have found **one** center, you can use the pattern's **translations** to locate **all** the others: indeed a look at figures 4.28-4.31 will convince you that the lengths of the sides of those 'center parallelograms' are equal to **half the length** of the pattern's **minimal translation vectors** (to which the sides themselves are **parallel**); more on this in 7.6.4!

4.5.3 <u>Italian curves.</u> How about finding all four kinds of 180^0 rotation centers in this modern Italian ceramic (***Stevens***, p. 213)?

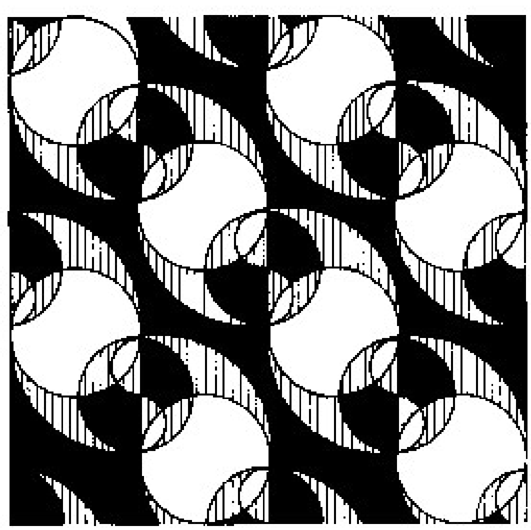


Fig. 4.31

4.6 180^0, reflection in two directions (pmm)

4.6.1 Stacking **pmm2**s. Rather predictably in view of what we saw in earlier sections, straight stackings of **pmm2** border patterns have both 180^0 rotation and reflection in **two** directions:

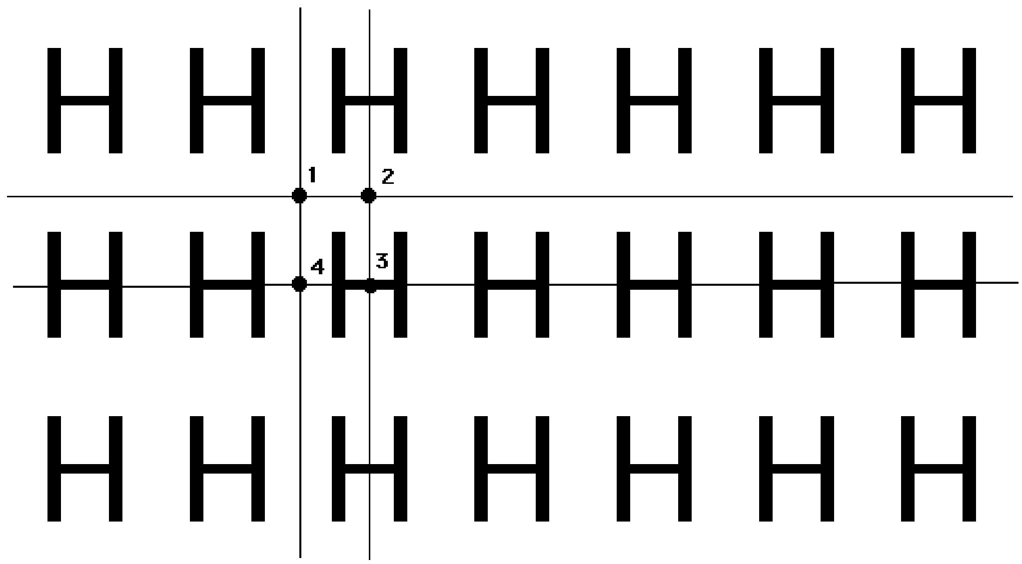


Fig. 4.32

There is nothing too tricky about this new type of wallpaper pattern, known as **pmm**: it has reflection axes (of two kinds) in **two perpendicular directions** and four kinds of 180^0 rotation centers, all of them located at the **intersections of reflection axes**. This last observation gives you a chance to practice your geometry a bit and try to explain why, as first noticed in 2.7.1, the intersection of two perpendicular reflection axes yields a 180^0 rotation center: this is a special case of a more general fact discussed in 7.2.2!

4.6.2 Native American 'gates'. Here is a Nez Perce′ **pmm** pattern from ***Stevens*** (p. 244), not quite dominated by the **pmm**'s stillness:

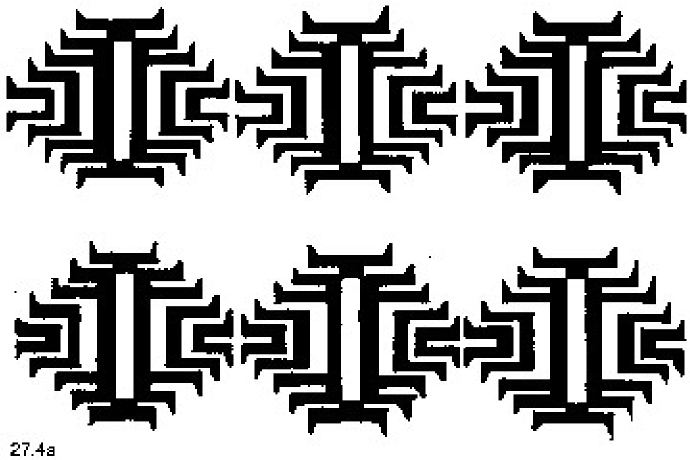


Fig. 4.33

4.6.3 More examples. While the ‘building blocks’ in the wallpaper patterns of figures 4.32 & 4.33 had a lot of symmetry themselves ($\mathbf{D_2}$ sets), it is certainly possible to build **pmm** patterns employing less symmetrical motifs (still creating $\mathbf{D_2}$ fundamental regions though), as figures 4.34 & 4.35 demonstrate:

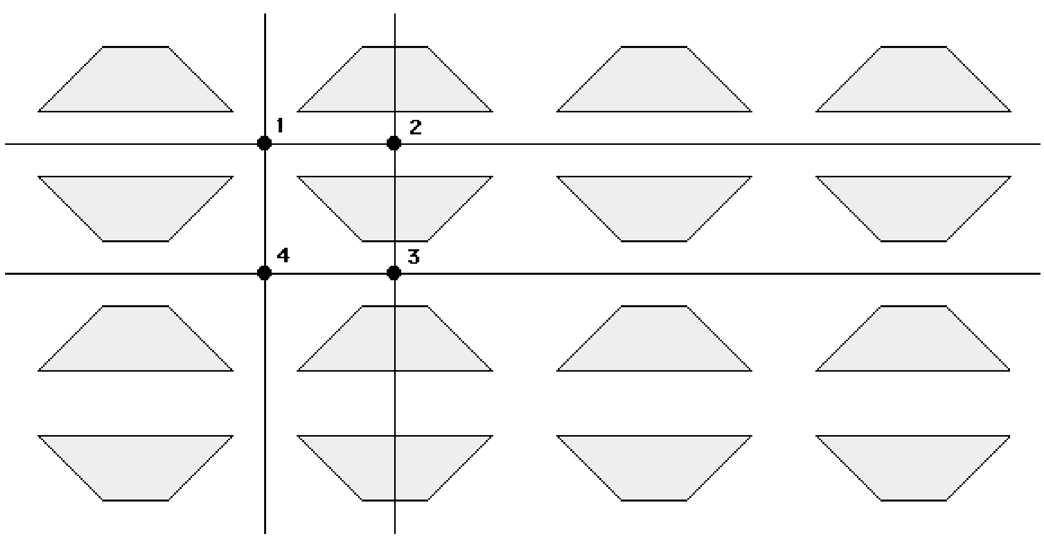


Fig. 4.34

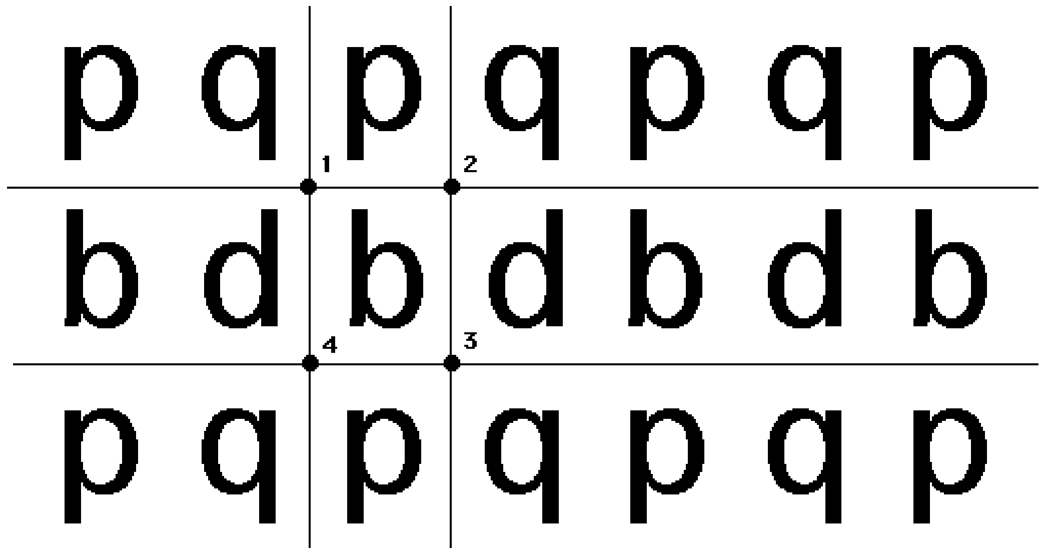


Fig. 4.35

4.7 180^0, reflection in one direction with perpendicular glide reflection (pmg)

4.7.1 <u>Shifted stackings of **pmm2**s.</u> Let's look at a randomly shifted stacking of the **"H "** border pattern employed in figure 4.32:

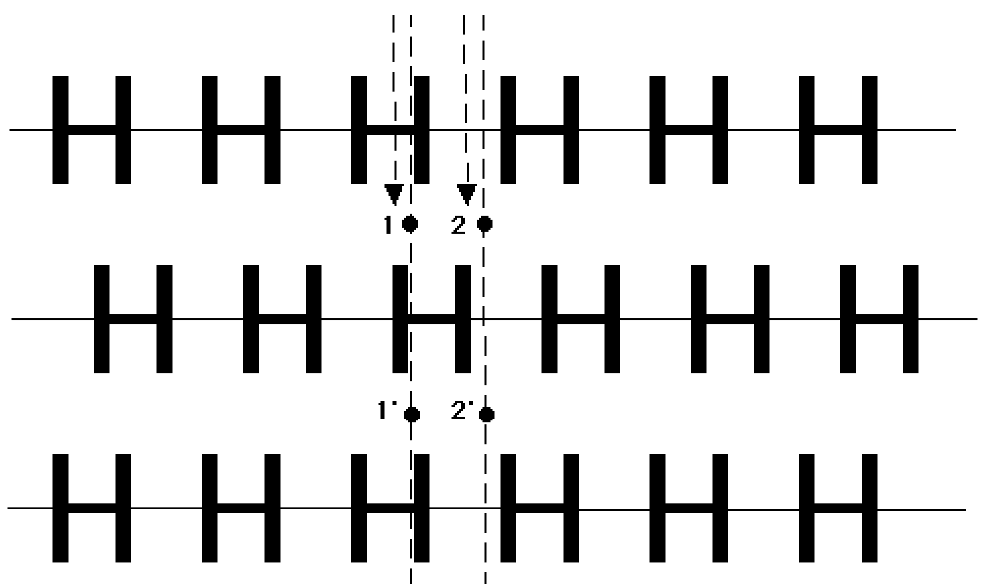


Fig. 4.36

4.7.2 <u>Straight stackings of **pma2**s.</u> Let's also look at a straight stacking of a **pma2** border pattern similar to the one in figure 2.24:

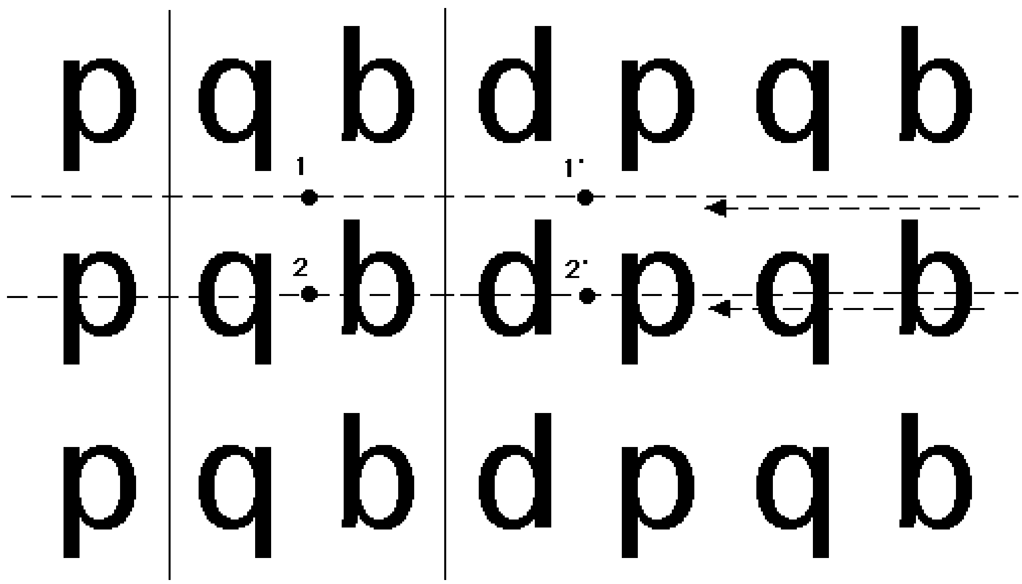


Fig. 4.37

4.7.3 <u>What is going on?</u> As we have indicated above, both wallpaper patterns created have 180^0 rotation, reflection in one direction (horizontal in figure 4.36, vertical in figure 4.37), and glide reflection in one direction as well (vertical in figure 4.36, horizontal in figure 4.37). In both cases, the directions of reflection and glide reflection are **perpendicular** to each other, with **all the rotation centers on glide reflection axes**, **half way between two reflection axes**: this type of 180^0 wallpaper pattern is known as **pmg**. Here are two more examples employing, once again, trapezoids:

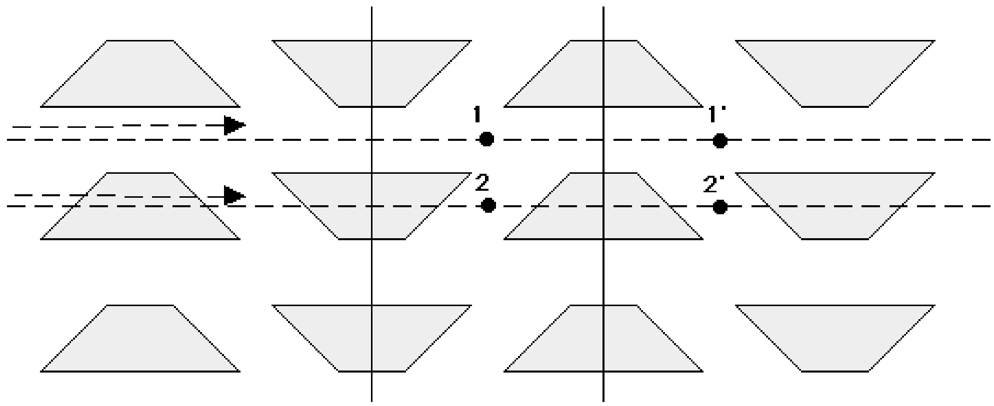


Fig. 4.38

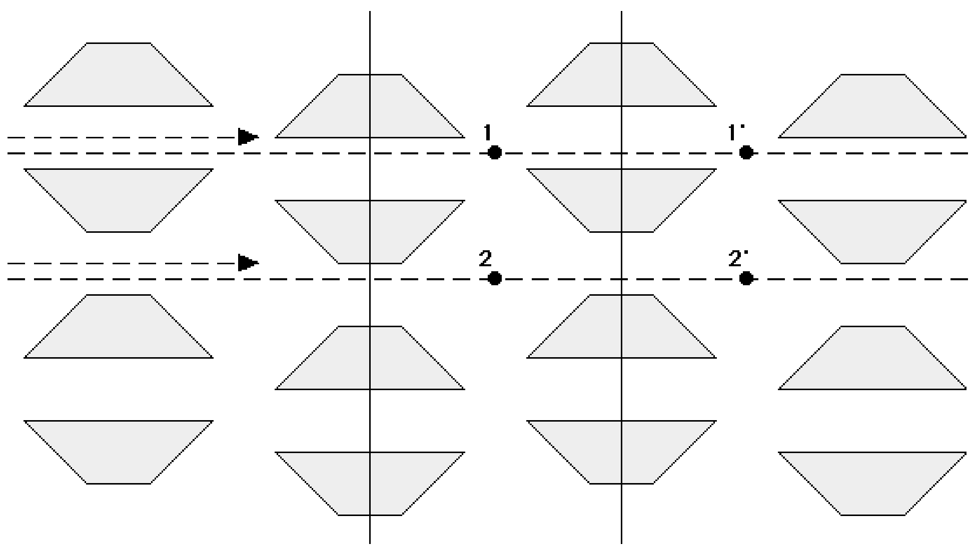
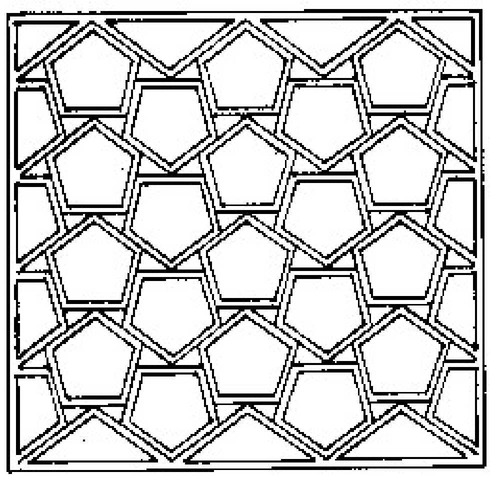


Fig. 4.39

While all four wallpaper patterns in figures 4.36-4.39 belong to the same type (**pmg**), they do not necessarily 'look' the same; for example, the ones in figures 4.36 & 4.39 (**randomly** shifted stackings of a **pmm2** border pattern) create a feeling of a wave-like motion, while the ones in figures 4.37 & 4.38 (straight stackings of a **pma2** border pattern) create an impression of two flows in opposite directions. More significantly, there are **glide reflection axes of two kinds** in all four examples. It is tempting to say the same about reflection axes (especially in figures 4.37 & 4.38), but not quite so if we are 'cautious' enough to **turn** the patterns upside down: we elaborate further on this in 4.11.2. (Likewise concerning the **numbering** of half turn centers in figures 4.36-4.39!)

How does one recognize a **pmg** pattern? Basically, look for a 180^0 pattern with **reflection in only one direction** -- as we are going to see the **pmg** is the **only** 180^0 wallpaper pattern with reflection in only one direction -- and then use all the other observations made in this section for confirmation.

4.7.4 Chinese pentagons. The following **pmg** example of a Chinese window lattice (***Stevens***, p. 221) comes close to a famous **impossibility** (tiling the plane with **regular** pentagons):



25.7n

Fig. 4.40

4.8 180^0, glide reflection in two directions (pgg)

4.8.1 Shifted stackings of **pma2**s. In the same way that going from straight to shifted stackings of **pmm2**s substituted reflection by glide reflection in **one** of the two directions (and 'reduced' the symmetry type from **pmm** to **pmg**), going from straight to shifted stackings of **pma2**s replaces the reflection by glide reflection and 'reduces' the symmetry type from **pmg** to **pgg**:

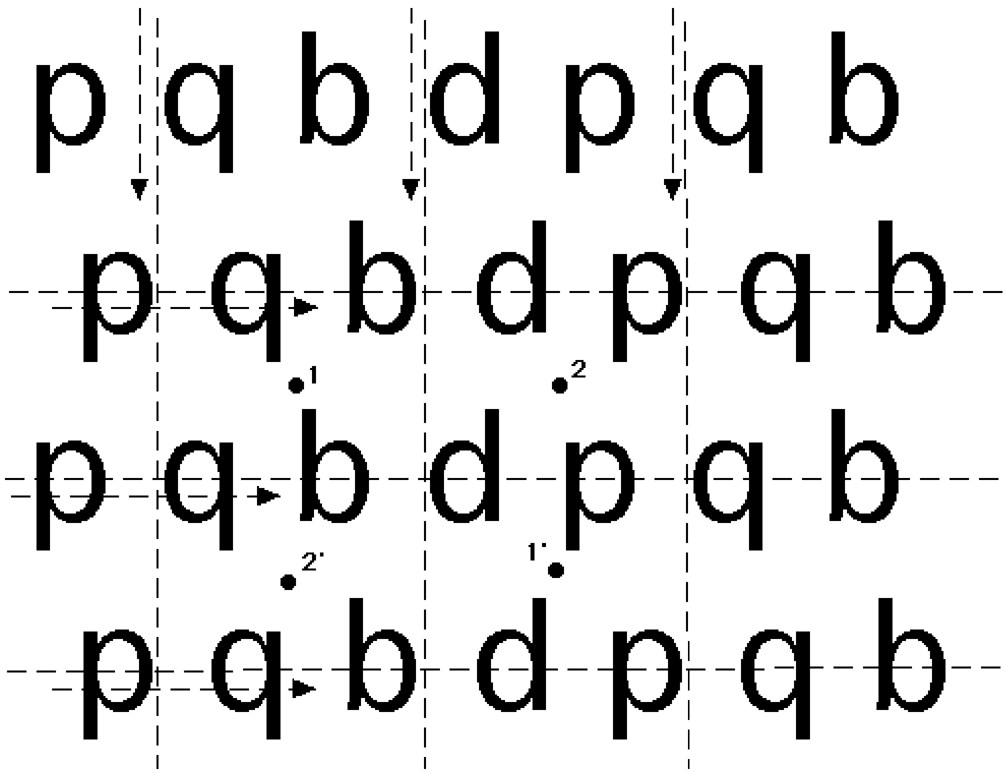


Fig. 4.41

A brief review of figures 4.18 & 4.21 would make it easier for you to realize that the wallpaper pattern in figure 4.41 has the indicated glide reflections, in **two perpendicular directions**. It should also be easy for you by now to locate the 180^0 rotation centers (of **two** kinds, actually) and confirm that each of them lies **right between four glide reflection axes**. There is no reflection. Wallpaper patterns of this type are known as **pgg**.

4.8.2 <u>Between **p2** and **pgg**.</u> Distinguishing between **p2** and **pgg** -- especially in the presence of **'rectangularly ruled'** half turn centers characteristic of glide reflection (8.2.2) -- is not that easy. Reversing our advice in 4.8.1, we suggest that every time you determine **all** the 180^0 rotation centers in a wallpaper pattern you should subsequently check the lines passing **right between rows or columns of rotation centers**: those **could** be glide reflection axes! In general, the presence of **heterostrophic motifs** in a pattern (such as **p** and **q** in figures 4.18 & 4.41) is a major indication in favor of glide reflection (4.3.3). Things can get a bit trickier in case the pattern's 'building blocks' are $\mathbf{D_1}$ (rather than $\mathbf{C_1}$) sets:

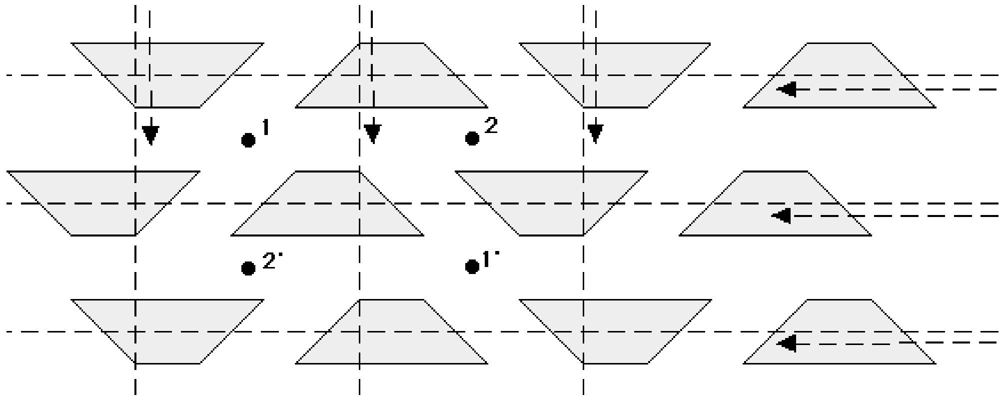


Fig. 4.42

4.8.3 <u>Two kinds of axes?</u> Observe that the **pgg** patterns in figures 4.41 & 4.42 **appear** to have two kinds of vertical glide reflection axes: we must stress at this point that remarks similar to the ones made in 4.7.3 do apply! Anyway, returning now to $\mathbf{C_1}$ motifs, or cutting the trapezoids of the pattern in figure 4.42 in half if you wish, here is a **pgg** pattern that **appears** to have two kinds of glide reflection axes in **both** directions:

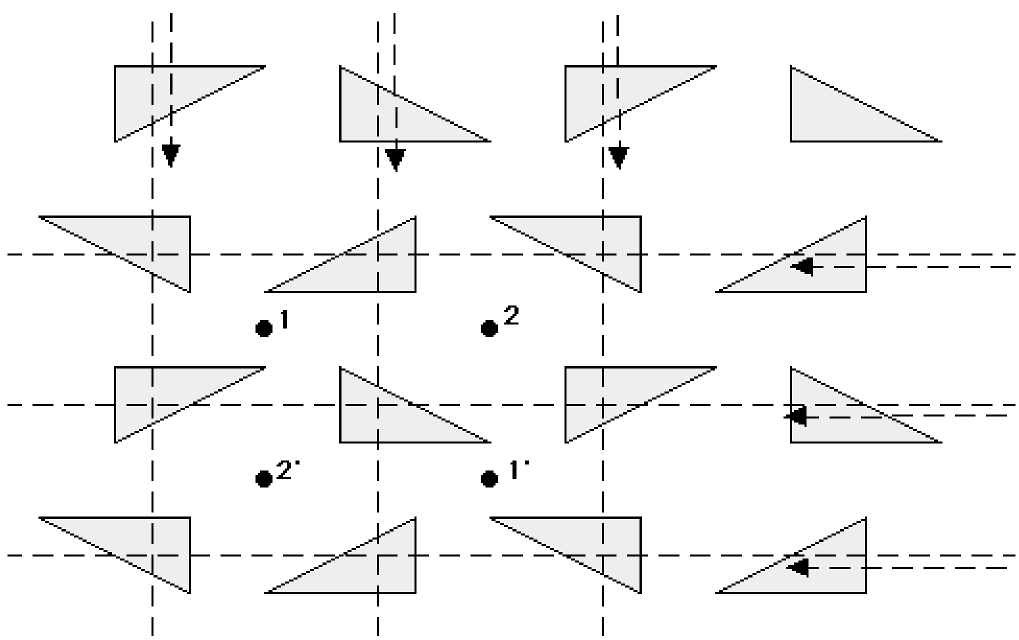


Fig. 4.43

4.8.4 Congolese parallelograms. The following **pgg** example from ***Stevens*** (p. 236), full of **heterostrophic parallelograms**, should allow you to practice your skills in determining glide reflection axes:

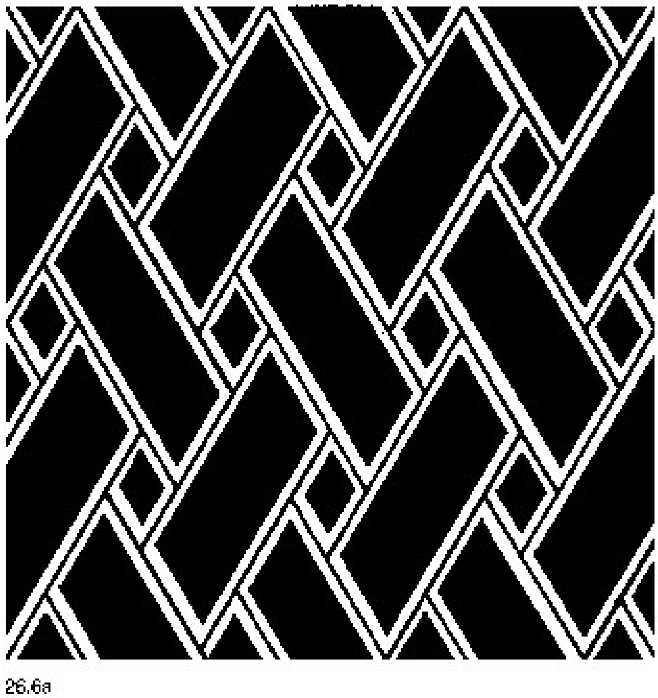


Fig. 4.44

4.9 180^0, reflection in two directions with in-between glide reflections (cmm)

4.9.1 Perfectly shifted stackings of **pma2**s and **pmm2**s. In the same way perfectly shifted stackings of **pm11**, **p1a1**, and **p1m1** border patterns created a 'new' type of wallpaper pattern (**cm**) in section 4.4, perfectly shifted stackings of **pma2** and **pmm2** border patterns create a two-directional analogue of **cm** as shown below:

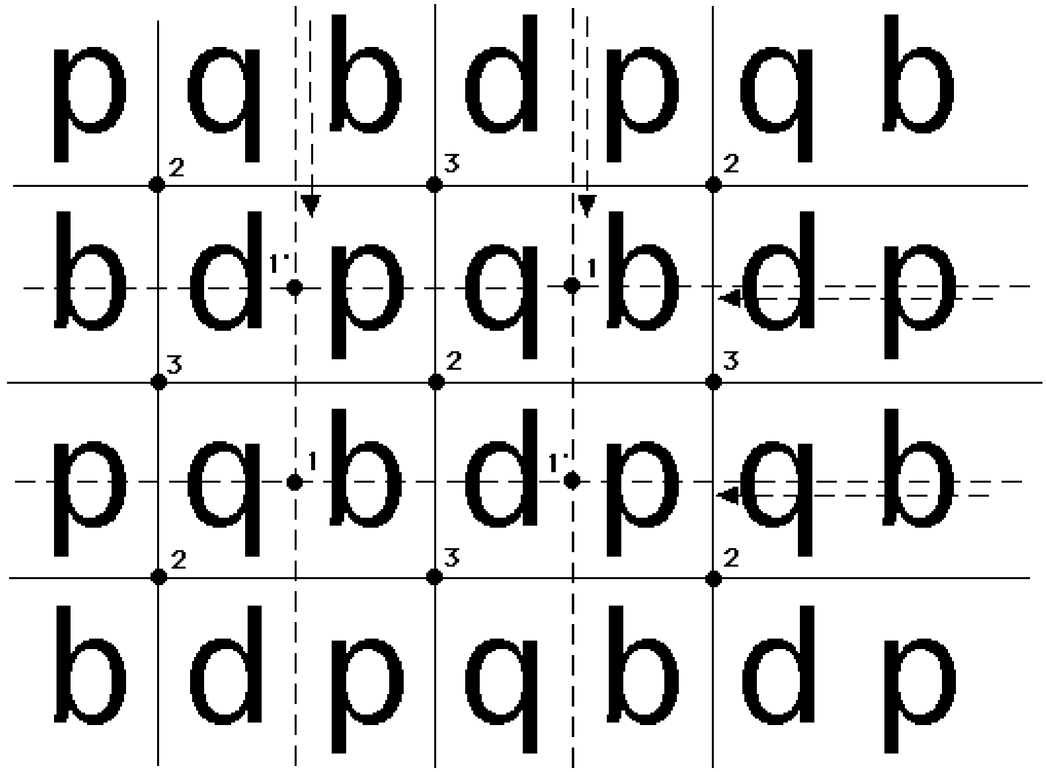


Fig. 4.45

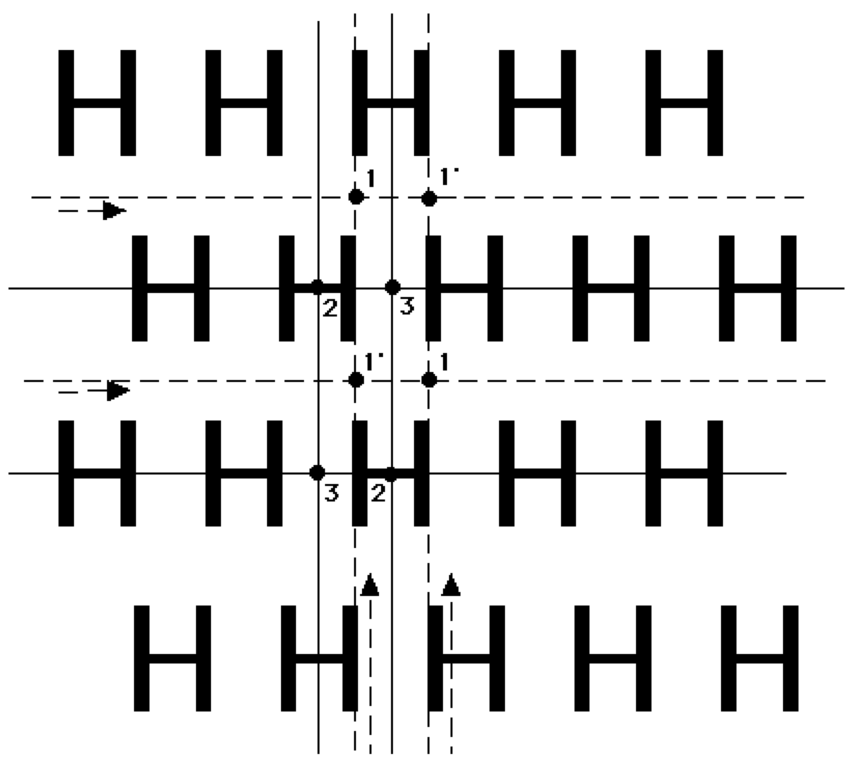


Fig. 4.46

There is nothing too surprising about this new type of wallpaper pattern to those familiar with the **cm** and **pmm** patterns: there is reflection and in-between glide reflection in **two perpendicular directions**; within each direction, both reflection and glide reflection axes are of **one kind only**; 180^0 rotation centers are found at the intersections of reflection axes and -- the only new element -- at the **intersections of glide reflection axes** as well. This new type, very rich in terms of symmetry, is known as **cmm**, and its last property is perhaps the easiest way to distinguish it from the **pmm** type: in the **pmm** pattern **all** rotation centers lie on reflection axes, in the **cmm** pattern **half** of them do **not**. Since locating the rotation centers can at times be trickier than finding the glide reflection axes, another obvious way of distinguishing between **pmm** and **cmm** is the latter's in-between glide reflection. Either way, once **all** reflection axes have been determined, you know where to look for both glide reflection axes and rotation centers!

4.9.2 Shifting back and forth to other types. Quite clearly, the **cmm** pattern of figure 4.45 is a close relative, or a 'shifted version', if you wish, of the **pmg** pattern in figure 4.37 and the **pgg** pattern in figure 4.41. Likewise, the **cmm** pattern of figure 4.46 is related to the **pmm** pattern of figure 4.32 and the **pmg** pattern of figure 4.36. Here are two more, trapezoid-based, **cmm** patterns the 'shifting relations' of which to previously presented examples you may like to investigate:

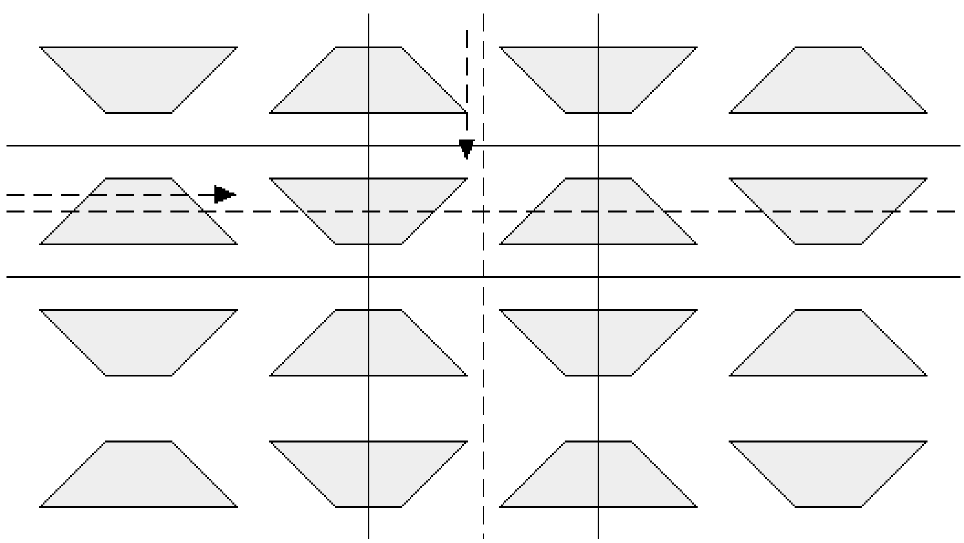


Fig. 4.47

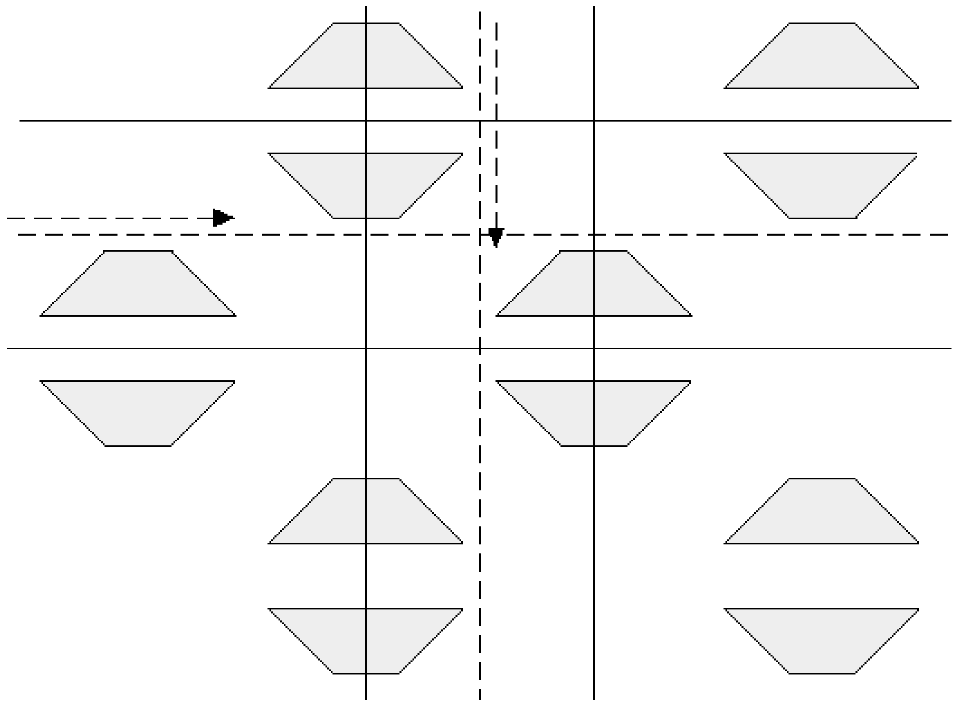


Fig. 4.48

4.9.3 Turkish arrows. Here comes our long-awaited real-world example of a **cmm** pattern, a 16th century Turkish design from ***Stevens*** (p. 250); make sure you can find all the rotation centers!

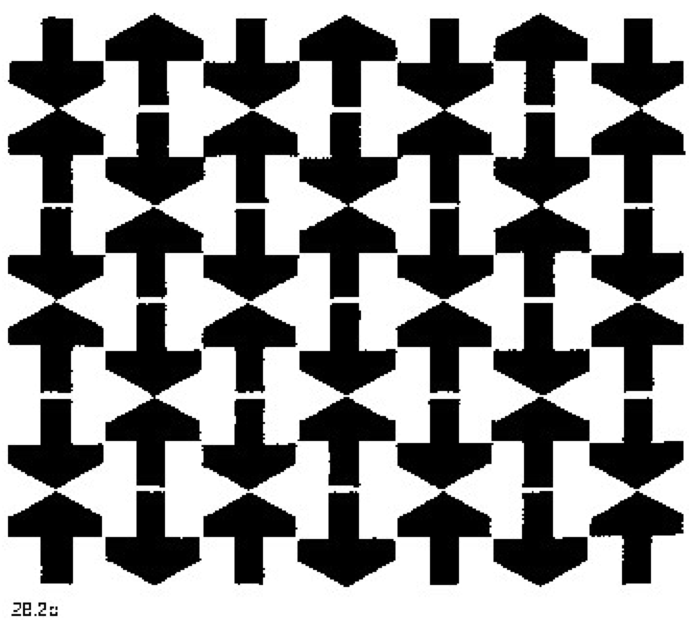


Fig. 4.49

4.9.4 The world's most famous **cmm** pattern ... is no other than the all-too-familiar **brick wall**:

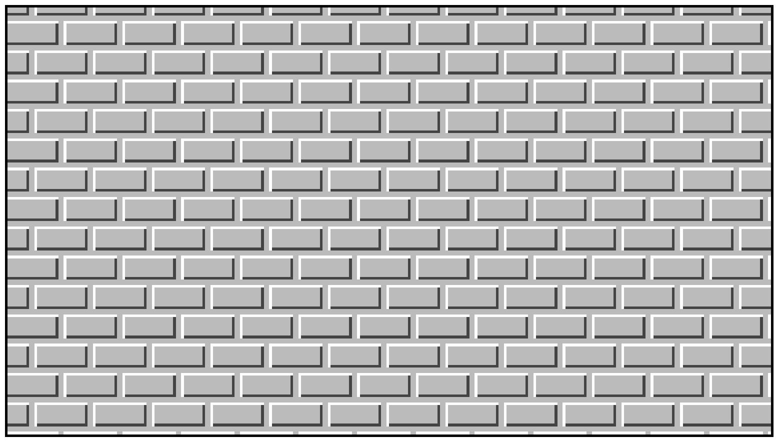


Fig. 4.50

We have already discussed the **bathroom wall** in section 4.0**:** as you will see right below, the two walls **are** mathematically distinct!

4.10 90^0, four reflections, two glide reflections (p4m)

4.10.1 The bathroom wall revisited. How would one classify the bathroom wall in case he or she misses its 90^0 rotation, already discussed in 4.0.3? It all depends on which reflections one goes by! Indeed, looking at its vertical and horizontal reflections only, the bathroom wall would certainly look like a **pmm**: two kinds of axes, no in-between glide reflection... If, on the other hand, one focuses only on the diagonal reflections, then the bathroom wall looks like a **cmm**: for there does indeed exist some 'unexpected' (yet **in-between**) glide reflection, as demonstrated in figure 4.51:

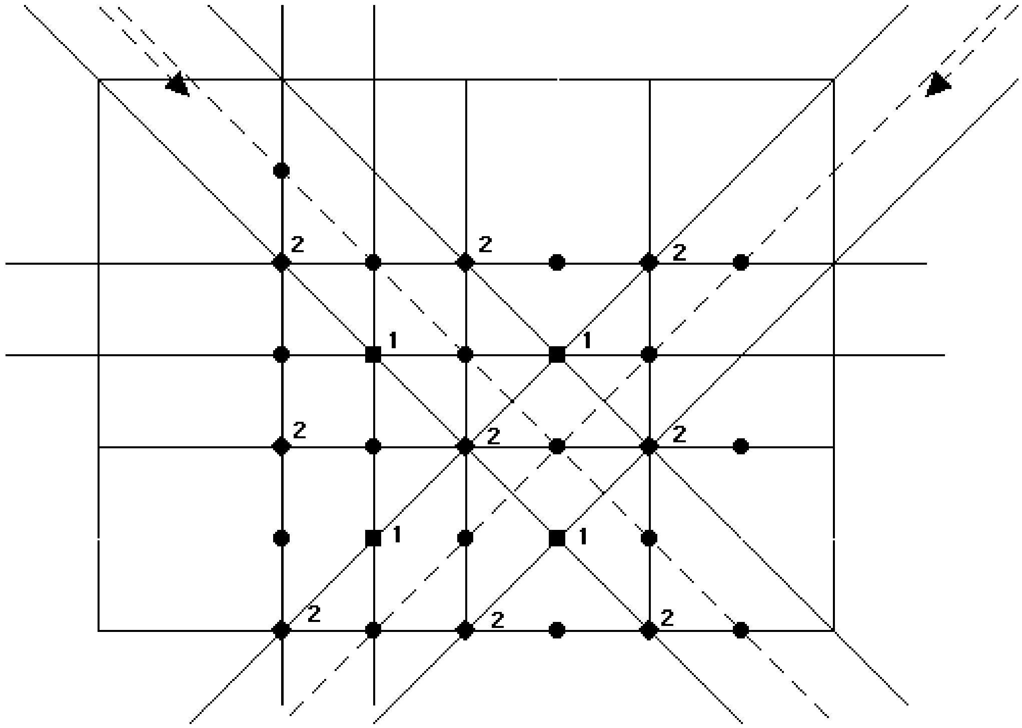


Fig. 4.51

Well, our 180^0 dreams are over! The bathroom wall is clearly full of 90^0 rotation centers, as pointed out in 4.0.3 and shown in figures 4.5 & 4.51. Moreover, 180^0 patterns may have reflection in at most two directions, and, as we will see in section 7.2, the intersection point of two reflection axes intersecting each other at a 45^0 angle is always a center for a 90^0 rotation. On the other hand, the bathroom wall has many 180^0 rotation centers, too: again, we first noticed that in 4.0.3, where it was also pointed out that there are **two kinds** of fourfold (90^0) centers, as opposed to only **one kind** of 180^0 centers; notice also the 90^0-45^0-45^0 triangles formed by two fourfold centers (one of each kind) and one twofold center (figure 4.51), something that will be further analysed in 6.10.1 and 7.5.1. Finally, observe that 90^0 and 180^0 centers are always at the intersection of **four** and **two** reflection axes, respectively. Wallpaper patterns having all these remarkable properties are known as **p4m**, and they are the only ones having **reflection in precisely four directions**.

4.10.2 The role of the squares. Do we always get 90^0 rotation in wallpaper patterns formed by square motifs? The answer is a flat "no", as demonstrated by a familiar floor tiling:

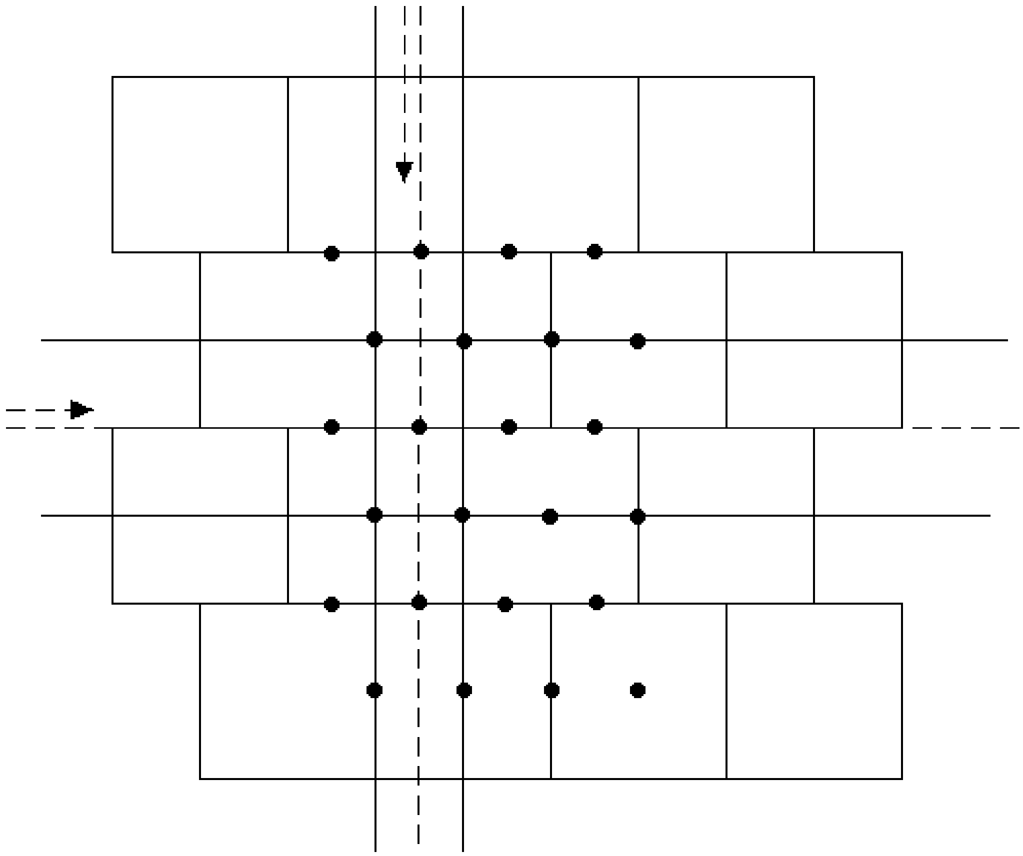


Fig. 4.52

The pattern in figure 4.52 is somewhere between the bathroom wall and the brick wall of 4.9.4, a perfectly shifted version of the former yet much closer to the latter mathematically: they both belong to the **cmm** type. Other shiftings of the bathroom wall will easily produce **pmg** patterns, and you should also be able to produce the other 180^0 (or even 360^0) wallpaper patterns using square motifs by being a bit more imaginative!

Reversing the question asked two paragraphs above, can we say that **p4m** patterns are always formed by square motifs? The answer is again "no", and the following modification of the **cmm** pattern of figure 4.48 provides an easy counterexample:

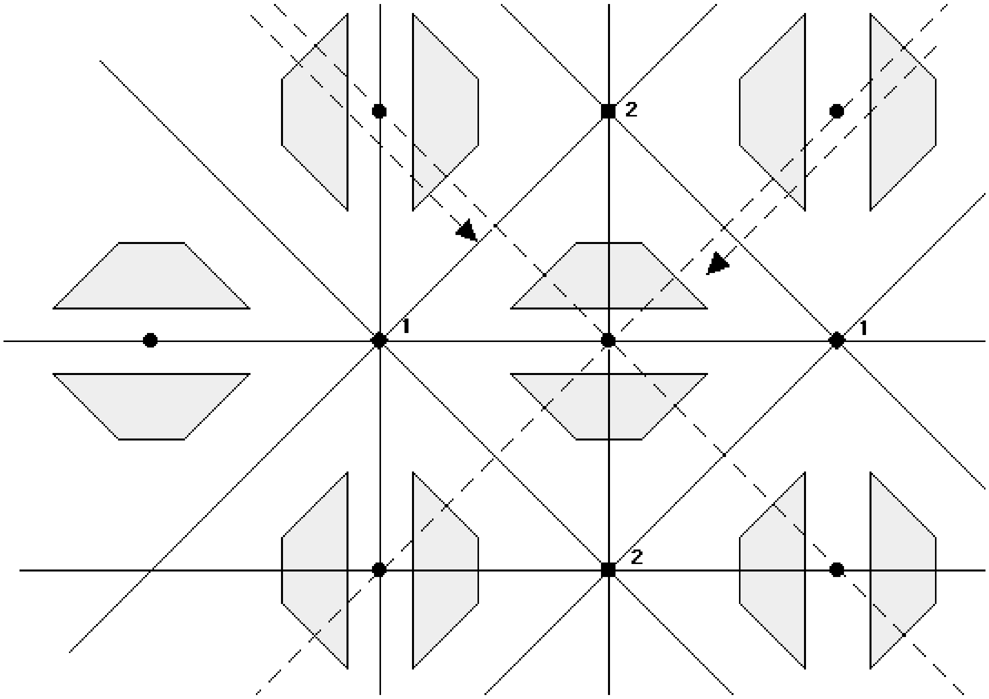


Fig. 4.53

4.10.3 Byzantine squares. The following example from ***Stevens*** (p. 308) stresses the **p4m**'s glide reflections:

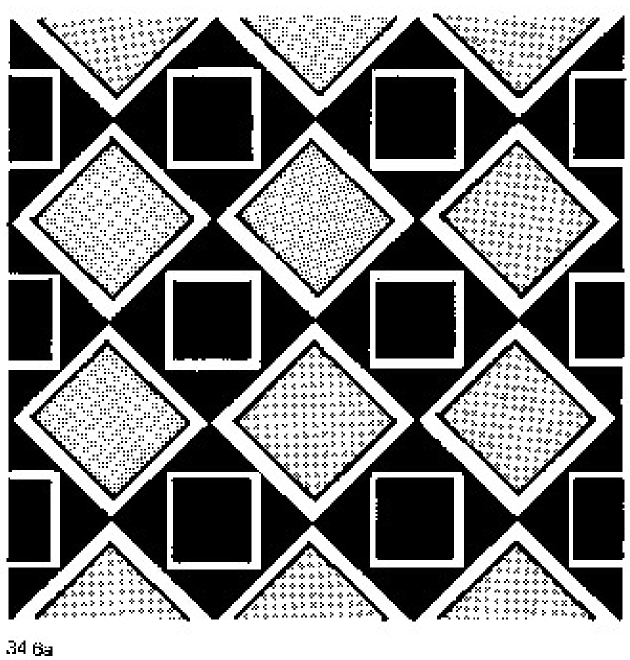


Fig. 4.54

4.11 90^0, two reflections, four glide reflections (p4g)

4.11.1 Similar processes, different outcomes. In 4.10.2 we rotated the motifs in **every other column** of the 'squarish' **cmm** pattern of figure 4.48 by 90^0 and ended up with the **p4m** pattern in figure 4.53. Here is what happens when we rotate the motifs in **every other 'diagonal'** of the **pmm** pattern of figure 4.34:

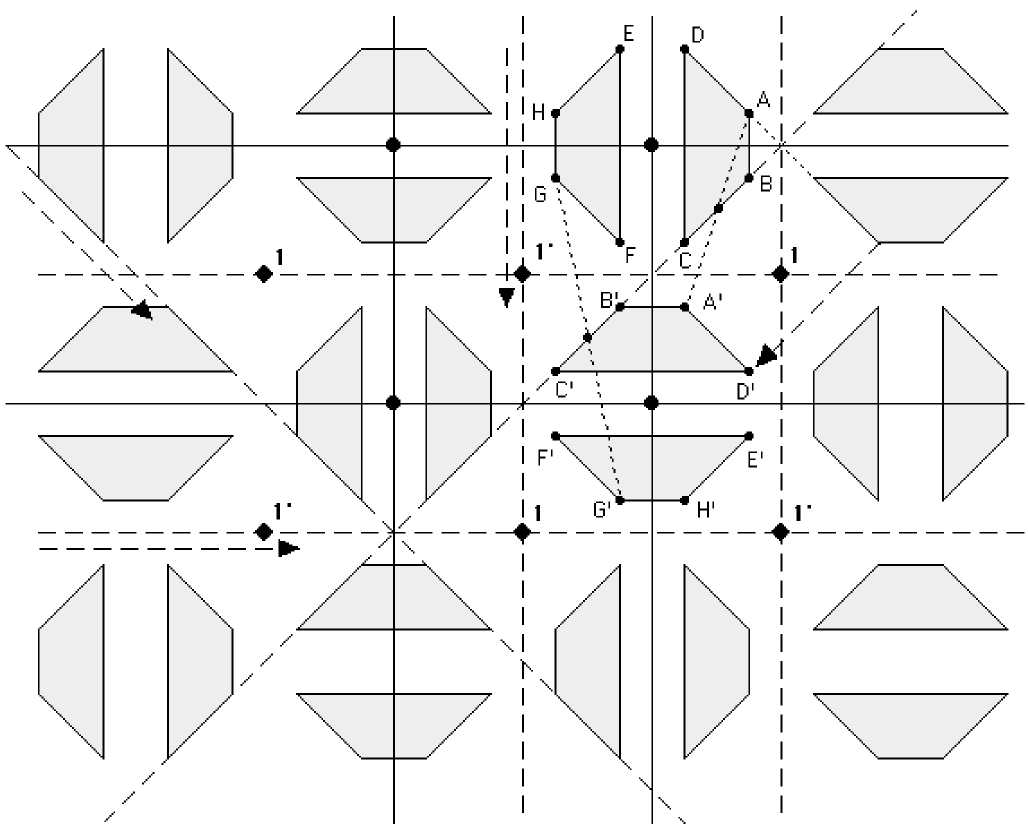


Fig. 4.55

The derived pattern looks very much like a **cmm**, having vertical and horizontal reflection and in-between glide reflection. There are, **as always** (4.6.1), 180^0 rotation centers at the intersections of perpendicular reflection axes. What happens at the **much less predictable** intersections of perpendicular **glide** reflection axes?

Well, those intersections are centers for **90^0**, rather than just 180^0, rotations: no chance for a **cmm**, which has by definition a **smallest** rotation of 180^0! And the surprises are not over yet: as in figure 4.53, every pair of trapezoids is a **'square' D_2** set, hence (3.6.3) there exist two rotations **and** two glide reflections between every two adjacent, perpendicular pairs of trapezoids (such as ABCD/EFGH and A′B′C′D′/E′F′G′H′); we already know the two 90^0 rotations that do the job, but where are the glide reflections? Using methods from chapter 3 once again (figure 4.55), we see that our new pattern has **glide reflection in two diagonal directions**; these are the 'subtle' glide reflections we were looking for, passing half way through two adjacent rotation centers (for 90^0 and 180^0, alternatingly) **in every single row and column of centers**!

Wallpaper patterns with the properties discussed above are known as **p4g**; they are the only 90^0 patterns with **reflection in precisely two directions**. They are easy to distinguish from the **p4m** patterns: one has to simply look at the number of directions of reflection (or even glide reflection, if adventurous enough).

4.11.2 How about rotation centers? Although there seem to be two kinds of 90^0 rotation centers in figure 4.55, marked by **1** and **1′**, we still declare that, unlike **p4m** patterns, every **p4g** pattern has just **one kind** of fourfold centers: indeed every 90^0 rotation center of **type 1′** is the **image** of a **type 1** 90^0 rotation center under one of the pattern's isometries (glide reflection or reflection), and vice versa; and, for reasons that will become clear in 6.4.4, but have also been discussed in 4.0.5, we tend to view any two isometries that are images of each other as 'equivalent' (read **"conjugate"**).

Likewise, we view all the 180^0 centers in either a **p4g** or a **p4m** pattern as being of the same kind: any two of them are images of each other by either a 180^0 rotation (possibly about a 90^0 center) or a 90^0 rotation! This also confirms that the **cm** has only 'one kind' of reflection axes (4.4.6): every two adjacent reflection axes are images of each other under the **cm**'s glide reflection or translation! More subtly, all reflections in the **pmg** (4.7.3) and all the glide reflections (of same direction) in the **pgg** (4.8.3) are 'of the same kind': indeed every two adjacent **pmg** reflection axes and every two adjacent, parallel **pgg** glide reflection axes are, as we indicated in 4.7.3, images of each other under a 180^0 rotation! Finally, we leave it to you to confirm that there exist **two**, rather than four, kinds of 180^0 centers in the **pgg** and **pmg** types, and **three** kinds of 180^0 centers in the **cmm**.

4.11.3 <u>More on 'diagonal' glide reflections.</u> The **p4g** wallpaper pattern in figure 4.56 should be compared to the **pgg** pattern of 4.8.4, which may be viewed as a 'compressed' version of it. On the other hand, every **p4g** pattern may be viewed, with some forgiving imagination, as a **'special case'** of a **pgg** pattern: just 'overlook' the 90^0 rotation and all reflections and in-between glide reflections ... and focus on the 180^0 rotations and the diagonal glide reflections!

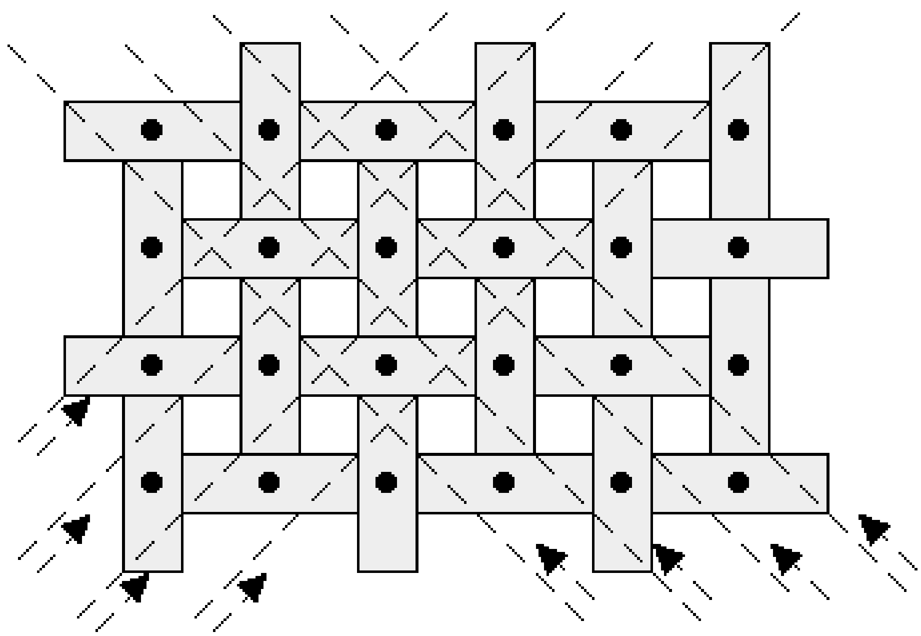


Fig. 4.56

Every **p4g** pattern may also be viewed as the union of two disjoint, **'perpendicular' cmm patterns** mapped to each other by any and all of the **p4g**'s diagonal glide reflections; this is best seen in the following 'relaxed' version of the previous **p4g** pattern (where the two **cmm**s consist of the vertical and the horizontal motifs, respectively):

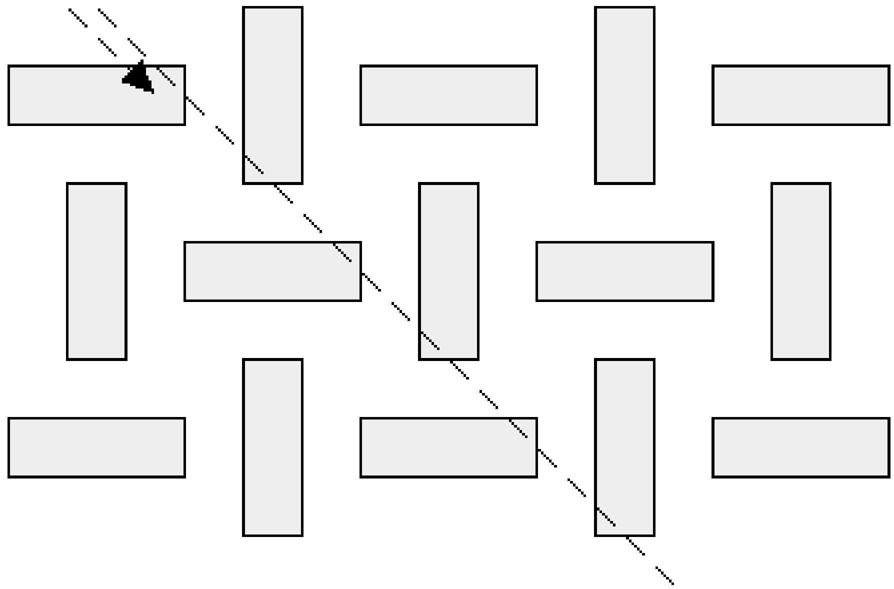


Fig. 4.57

4.11.4 Roman semicircles. In the following **p4g** example from ***Stevens*** (p. 294), every two nearest 90^0 centers are nicely placed at the centers of heterostrophic $\mathbf{C_4}$ sets:

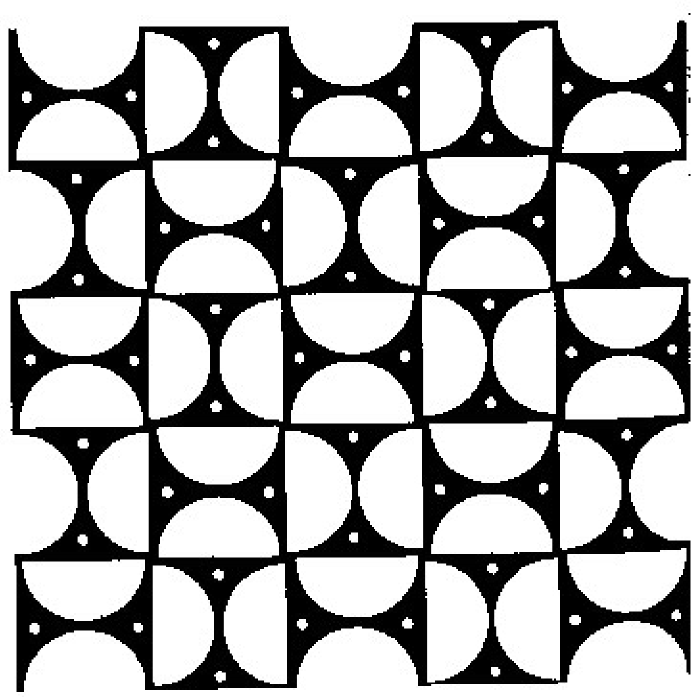


Fig. 4.58

4.12 90^0, translations only (p4)

4.12.1 Still the same centers! Let's have a look at the following 'distorted' version of the **p4g** pattern of figure 4.57, obtained via an 'up and down, left and right' process:

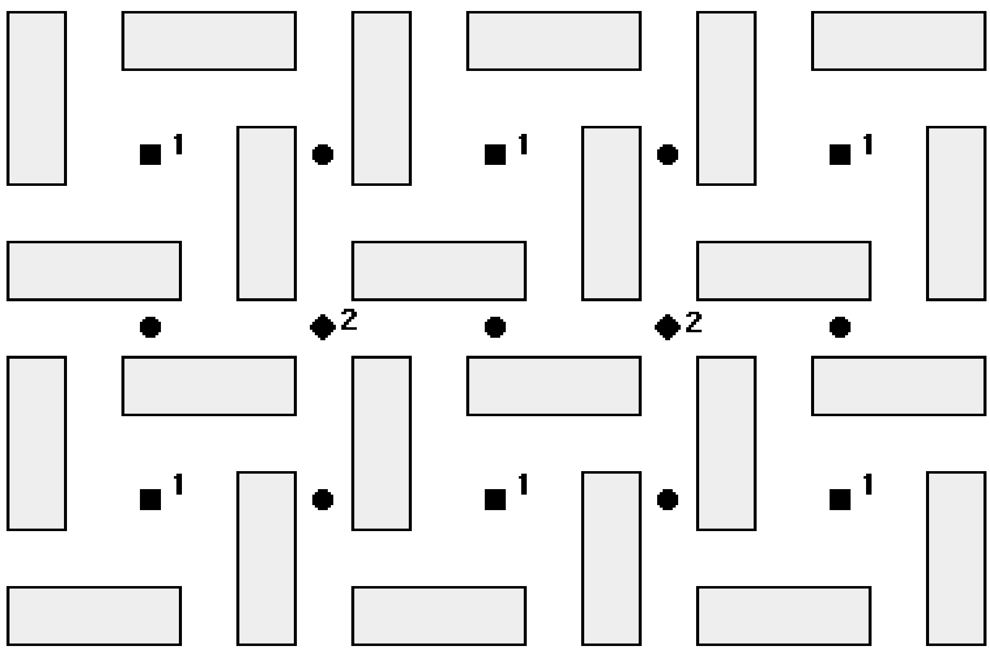


Fig. 4.59

In terms of rotations, the wallpaper pattern in figure 4.59 is identical to a **p4m** pattern: **two kinds** of 90^0 centers, **one kind** of 180^0 centers, and exactly the same lattice of rotation centers that we first saw in figure 4.5. What makes this new pattern different is that it has no reflections or glide reflections: the absence of the former is obvious, some candidates for the latter would require two or more gliding vectors each in order to work. Such patterns, having only 90^0 (and 180^0, of course) rotation (plus translation, always), are known as **p4**.

4.12.2 <u>On the way back to **p4m**.</u> Pushing the ‘process’ that led from the **p4g** pattern of figure 4.57 to the **p4** pattern of figure 4.59 one more step we obtain the following **p4** pattern:

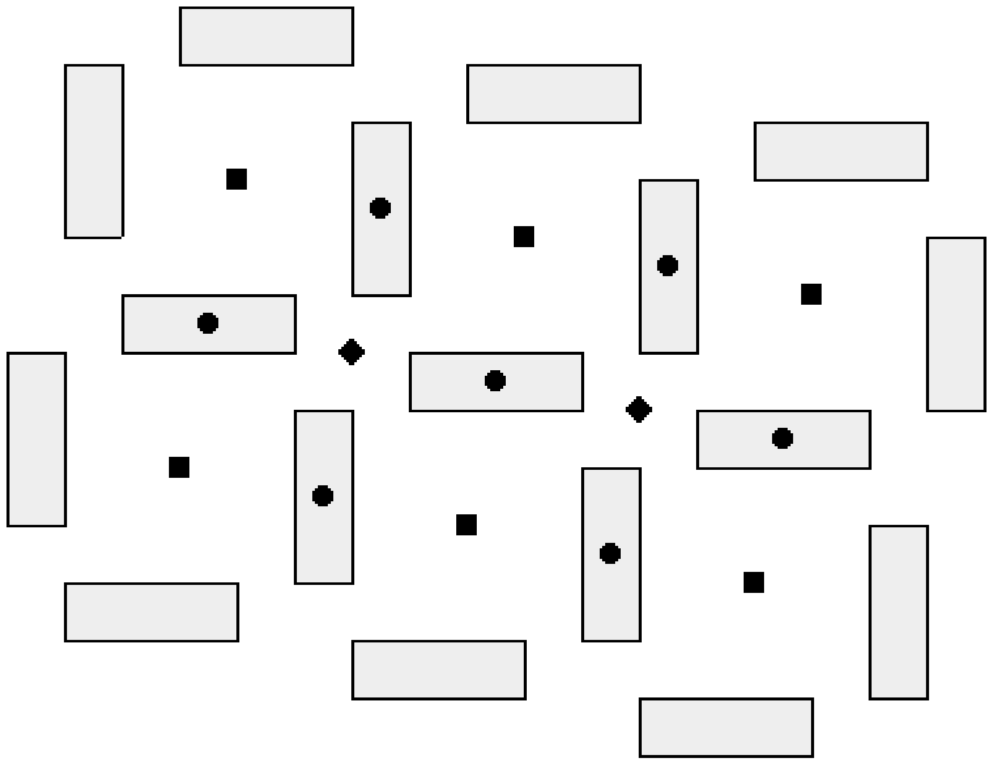


Fig. 4.60

You can probably guess at this point the next step in the process, a step that will result into a **p4m** pattern: all these 90^0 types are close relatives indeed!

4.12.3 <u>Two kinds of Egyptian ‘flowers’.</u> In this remarkable **p4** design from ancient Egypt (***Stevens***, p. 284), the two kinds of 90^0 centers are cleverly placed inside two slightly different types of flower-like D_4 figures; were the two kinds of ‘flowers’ one and the same, this design would still be a **p4**, except that the other kind of 90^0 centers would have to move to the ‘swastikas’:

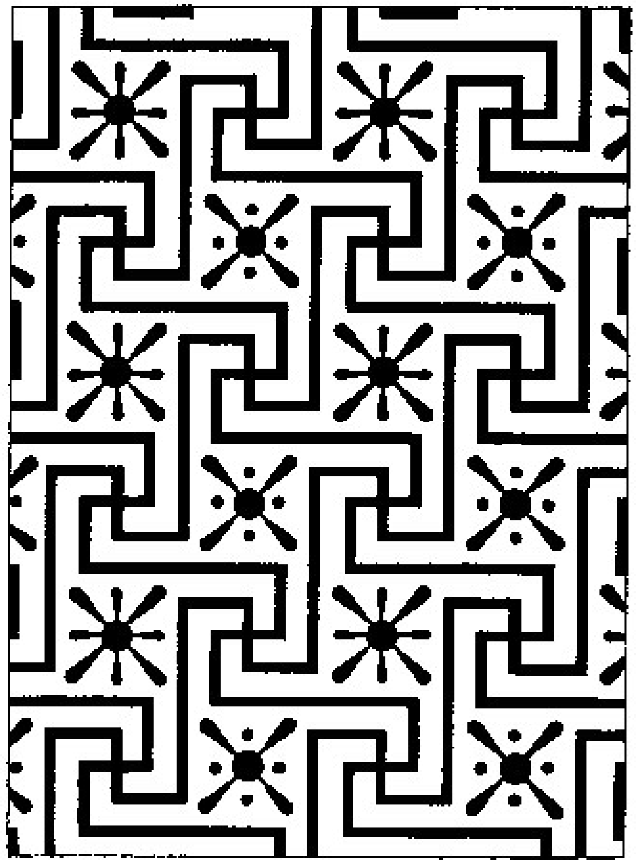


Fig. 4.61

4.13 60^0, six reflections, six glide reflections (p6m)

4.13.1 Bisecting the beehive. We have already discussed the lattice of rotation centers of the beehive (figure 4.5), and are aware of its three rotations (60^0, 120^0, 180^0). Figure 4.62 stresses **some** of its rather obvious reflections (of two kinds and in six directions), as well as its **in-between** glide reflections (again, of two kinds and in six directions):

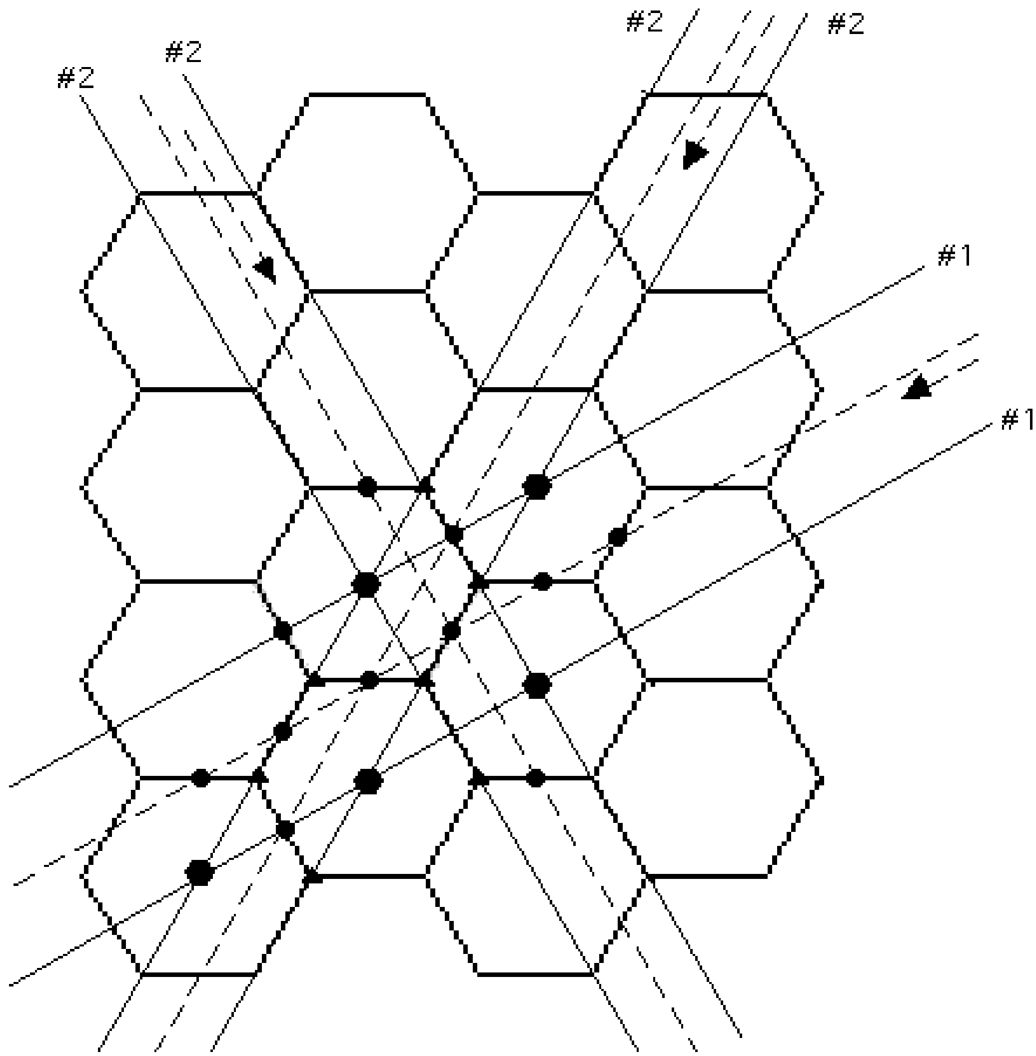


Fig. 4.62

So, while some reflection axes (#**1**) pass through sixfold and twofold centers only, others (#**2**) pass through all three kinds of centers. As for glide reflection axes, they all pass through **twofold centers only**, but they are of two kinds as well, having gliding vectors of different length (figure 4.62). Wallpaper patterns with these properties are denoted by **p6m**, a type that can justifiably be branded “the king of wallpaper patterns”: indeed not only is **p6m** very rich in terms of symmetry, but, as we will see in the coming sections, many other types are ‘contained’ in it or ‘generated’ by it. (The downside of this is that some times one may miss the 60^0 rotation and underclassify a **p6m** as a **cmm** or even **cm**).

4.13.2 From hexagons to rhombuses. It is easy to get a **'dual'** of the pattern in figure 4.62 that features **rhombuses** instead of hexagons and yet preserves all its isometries:

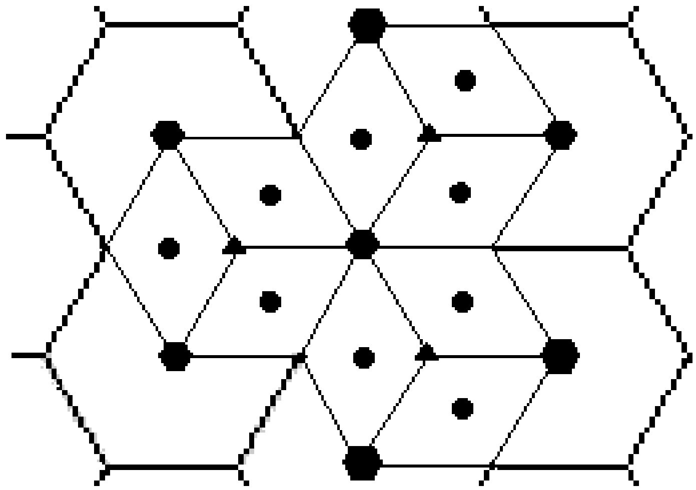


Fig. 4.63

4.13.3 Arabic rectangles. Here are two complex, 'rectangular' **p6m** patterns from ***Stevens*** (p. 330); can you see how to derive them from the beehive by attaching **rectangles** to the hexagons?

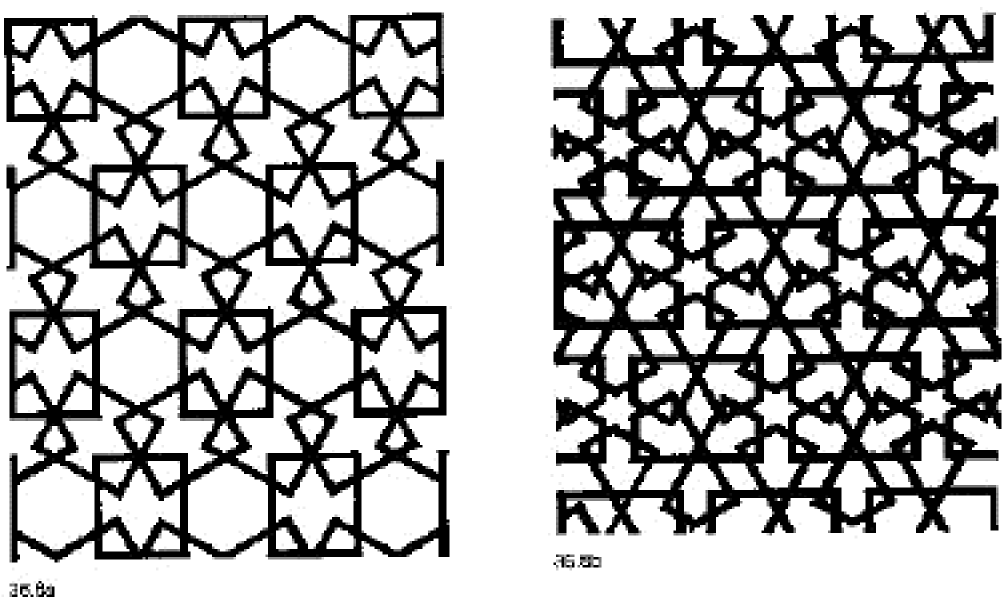


Fig. 4.64

4.14 60^0, translations only (p6)

4.14.1 ‘Adorning’ the rhombuses. What happens when one starts ‘enriching’ the ‘plain’ rhombuses in the **p6m** pattern of figure 4.63? The following Arabic design (***Stevens***, p. 318) provides an answer:

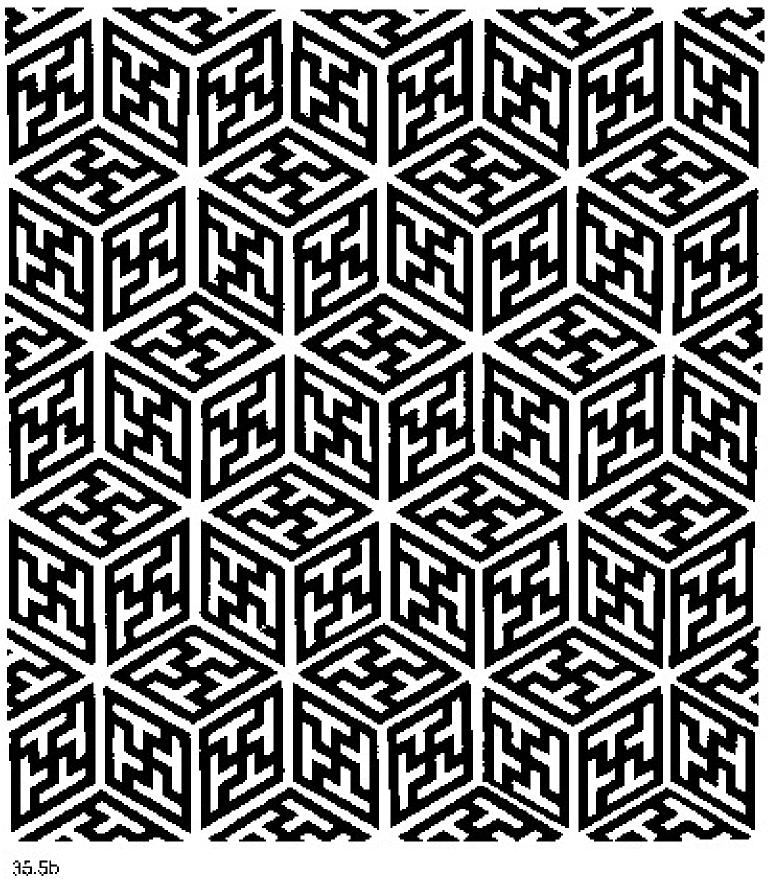


Fig. 4.65

The **T**-like figures inside the rhombuses have turned them from $\mathbf{D_2}$ sets into homostrophic $\mathbf{C_2}$ sets, destroying all possibilities for (glide) reflection, and yet preserving the rotations: the lattice of centers from figure 4.5 remains intact, with twofold, threefold, and sixfold centers placed at the vertices of 90^0-60^0-30^0 triangles (on which you may read more in 6.16.1 and 7.5.4). Such multi-rotational, rotation-only patterns are denoted by **p6**.

4.14.2 Hexagons with 'blades'. One can get a **p6** pattern directly from the beehive by cleverly turning the hexagons from $\mathbf{D_6}$ sets into homostrophic $\mathbf{C_6}$ sets; here is one of many ways to do that, turning three out of every four old sixfold centers into twofold centers (and eliminating three quarters of the old threefold centers as well):

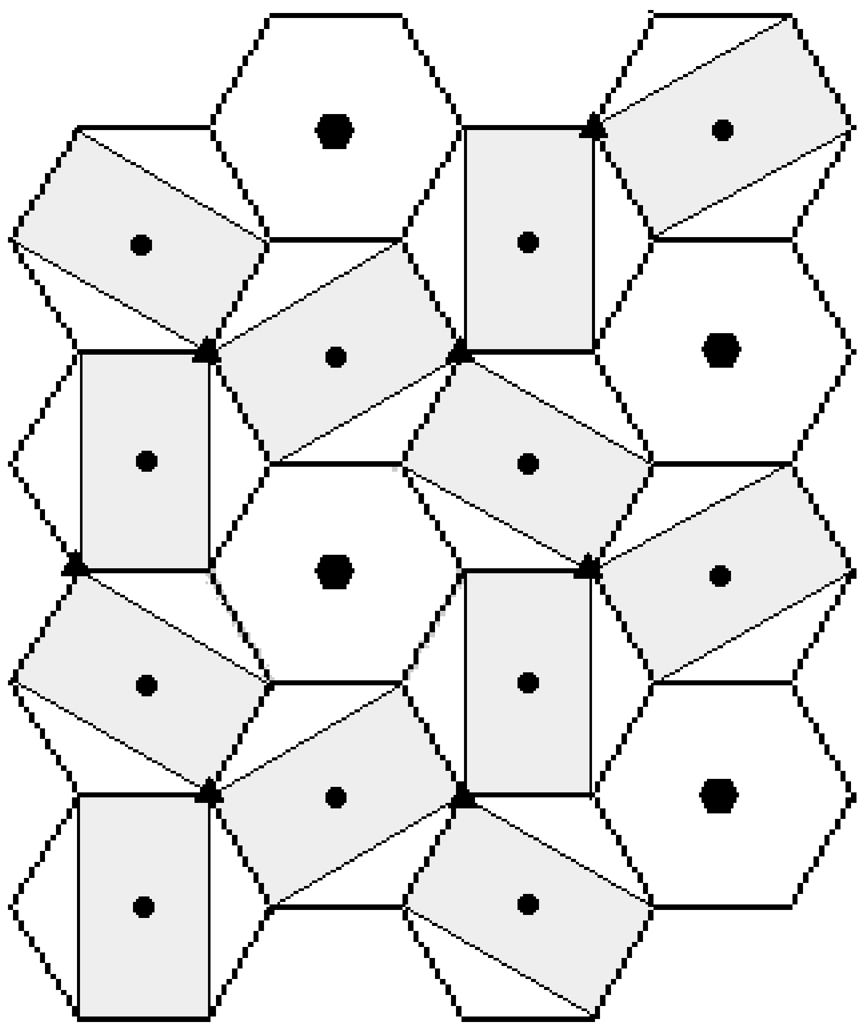


Fig. 4.66

4.15 120^0, translations only (p3)

4.15.1 Further rhombus 'ornamentation'. Let's have a look at the following Arabic design from ***Stevens*** (p. 260), similar in spirit to the **p6** pattern of figure 4.65:

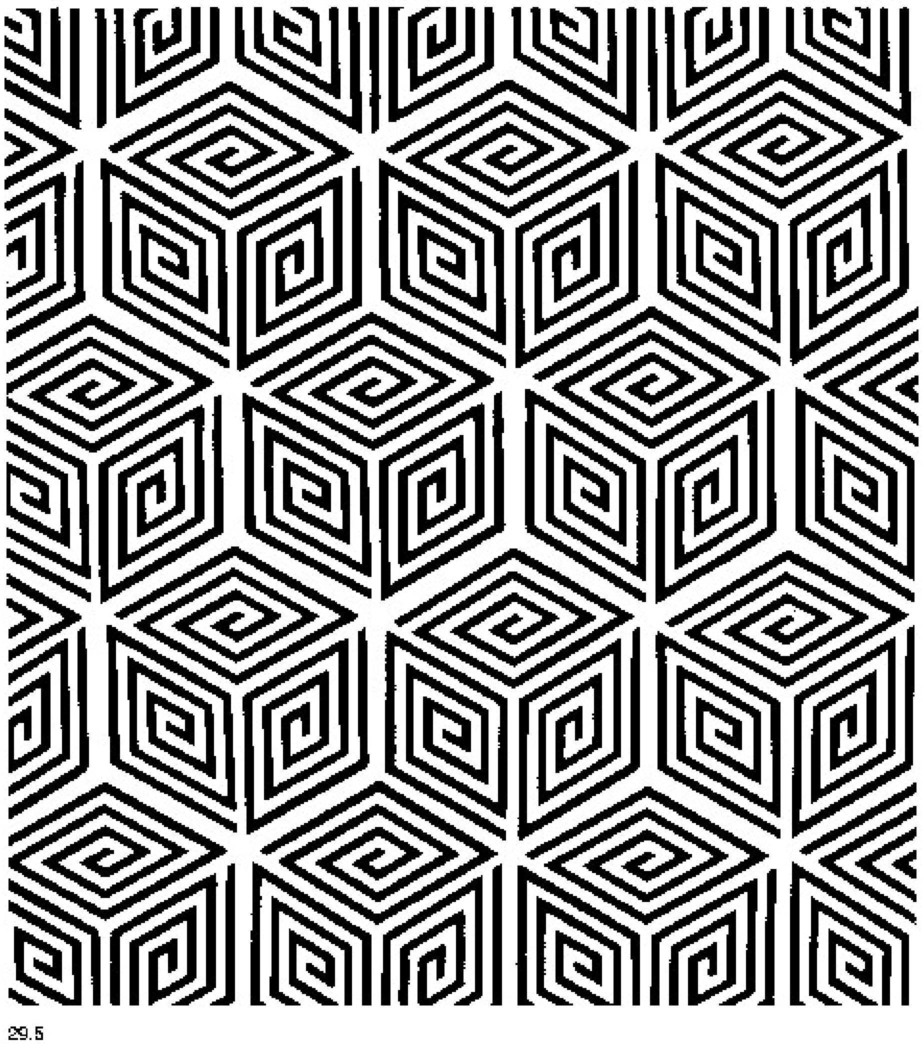


Fig. 4.67

Both patterns create a **three-dimensional** feeling, consisting of **cube-like** hexagons split into three rhombuses rotated to each other via 120^0 rotation, but that's where their similarities end. Indeed, while the rhombuses in figure 4.65 are homostrophic $\mathbf{C_2}$ sets (allowing for **two rotations** between any two adjacent rhombuses, one by 60^0 and one by 120^0), the rhombuses in figure 4.67 are homostrophic $\mathbf{C_1}$ sets allowing for **only one rotation** between any two adjacent ones, the 120^0 rotation already mentioned: there goes our 60^0 rotation, with the old 60^0 centers **reduced** to 120^0 centers!

It seems that we will have to settle for a wallpaper pattern having no other isometries than 120^0 rotations and translations: such patterns are known as **p3**.

4.15.2 Three kinds of rotation centers. There is a little bit of compensation for this reduction of symmetry: unlike **p6m** or **p6** wallpaper patterns, every **p3** pattern has **three kinds** of 120^0 rotation centers; this is perhaps easier to see in the following direct modification of the beehive than in the pattern of figure 4.67:

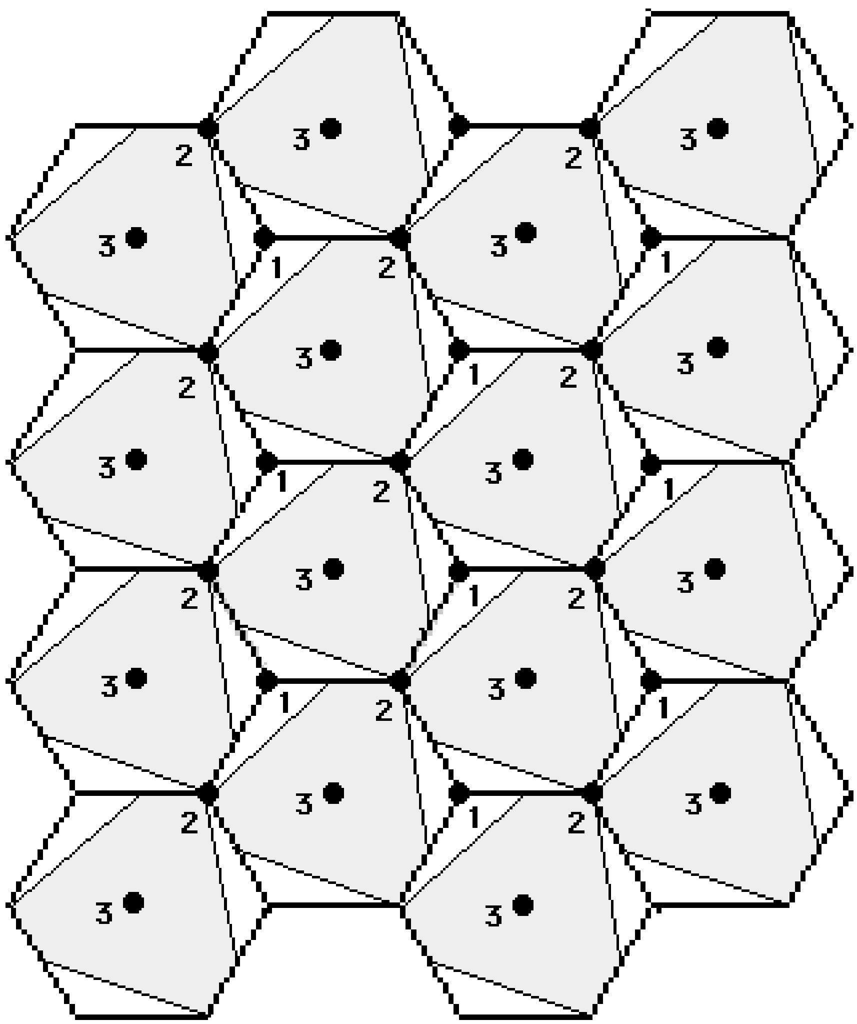


Fig. 4.68

4.16 120^0, three reflections, three glide reflections, some rotation centers off reflection axes (p31m)

4.16.1 Regaining the reflection. All the **p6** and **p3** patterns we have seen so far may be viewed as modifications of the beehive, with $\mathbf{D_6}$ sets (hexagons) replaced by homostrophic $\mathbf{C_6}$ and $\mathbf{C_3}$ sets, respectively: twelve (or just six) rhombuses 'build' a $\mathbf{C_6}$ set in figure 4.65, while only three suffice for a $\mathbf{C_3}$ set in figure 4.67. What happens when $\mathbf{D_6}$ sets turn into $\mathbf{D_3}$ sets? Here is an answer:

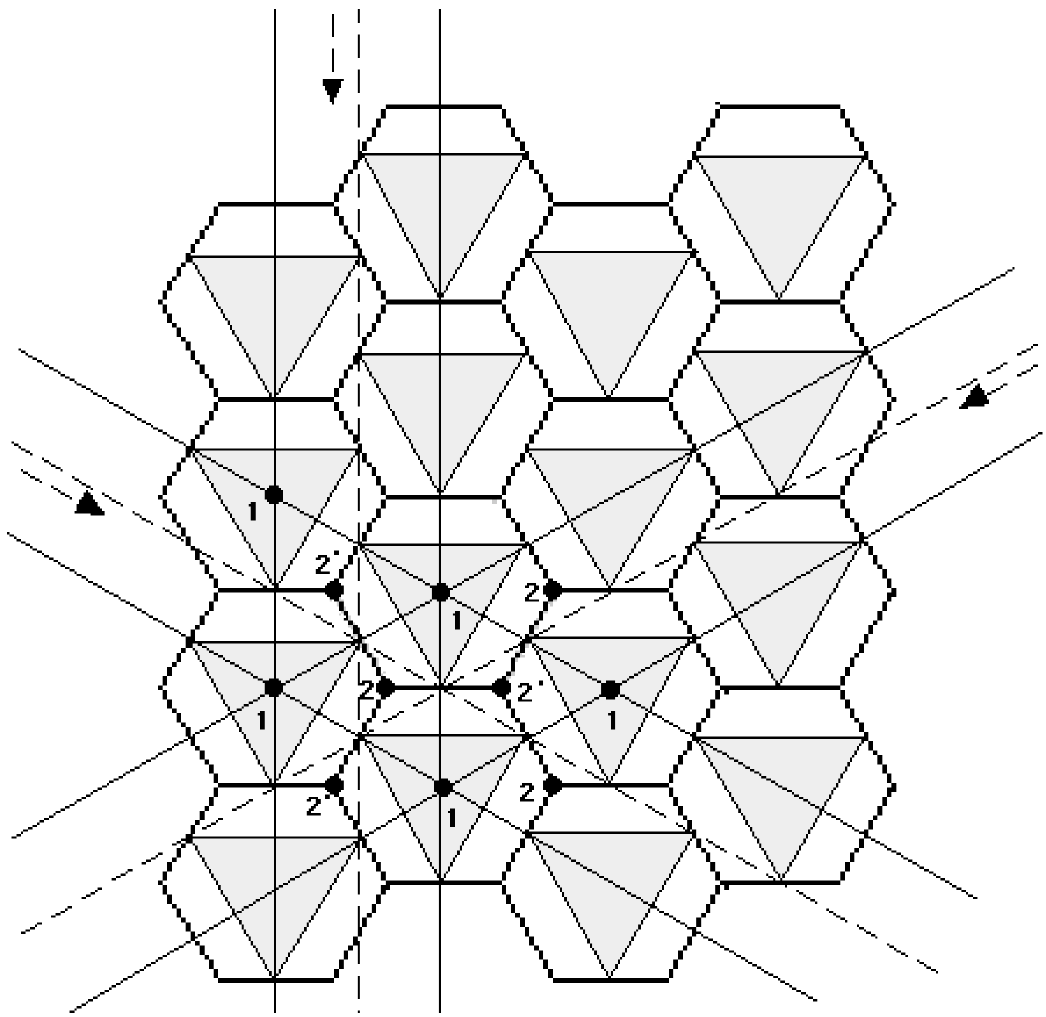


Fig. 4.69

Rather luckily, the reflections of the $\mathbf{D_3}$ sets (hexagons with an **inscribed equilateral triangle**) have survived, producing a wallpaper pattern with reflection and in-between glide reflection in three directions; notice that these reflections are precisely the **type 1** reflections of the original **p6m** pattern, passing through its sixfold and twofold, but not threefold, centers (4.13.1).

Which other isometries of the original **p6m** pattern (beehive) have survived, and how? Well, all sixfold centers have turned into threefold centers, and all threefold centers have remained intact! One may say that there are **two kinds** of 120^0 rotation centers: those -- denoted by **1** in figure 4.69 and always mappable to each other by translation -- **at the intersections of three reflection axes** (old sixfold centers); and those -- denoted by **2** or **2′** in figure 4.69 and mappable to each other by either (glide) reflection (**2** to **2′**) or translation/rotation (**2** to **2** or **2′** to **2′**) -- **on no reflection axis** (old threefold centers). All **type 2** reflections and glide reflections are gone -- in this example at least (see also 4.17.4). Wallpaper patterns of this type are known as **p31m**.

4.16.2 Japanese triangles. In our next example from ***Stevens*** (p. 274) the off-axis 120^0 centers are hidden inside curvy triangles:

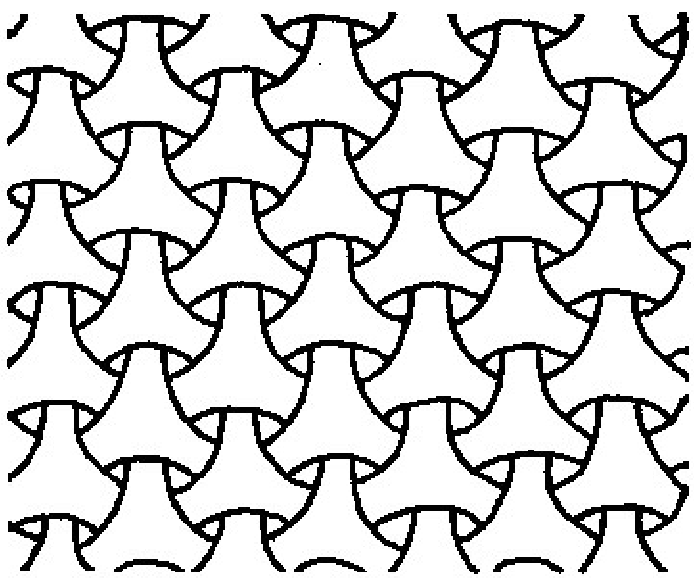


Fig. 4.70

4.17 120^0, three reflections, three glide reflections, all rotation centers on reflection axes (p3m1)

4.17.1 Thinning things out a bit. Consider the following diluted version of the wallpaper pattern in figure 4.69:

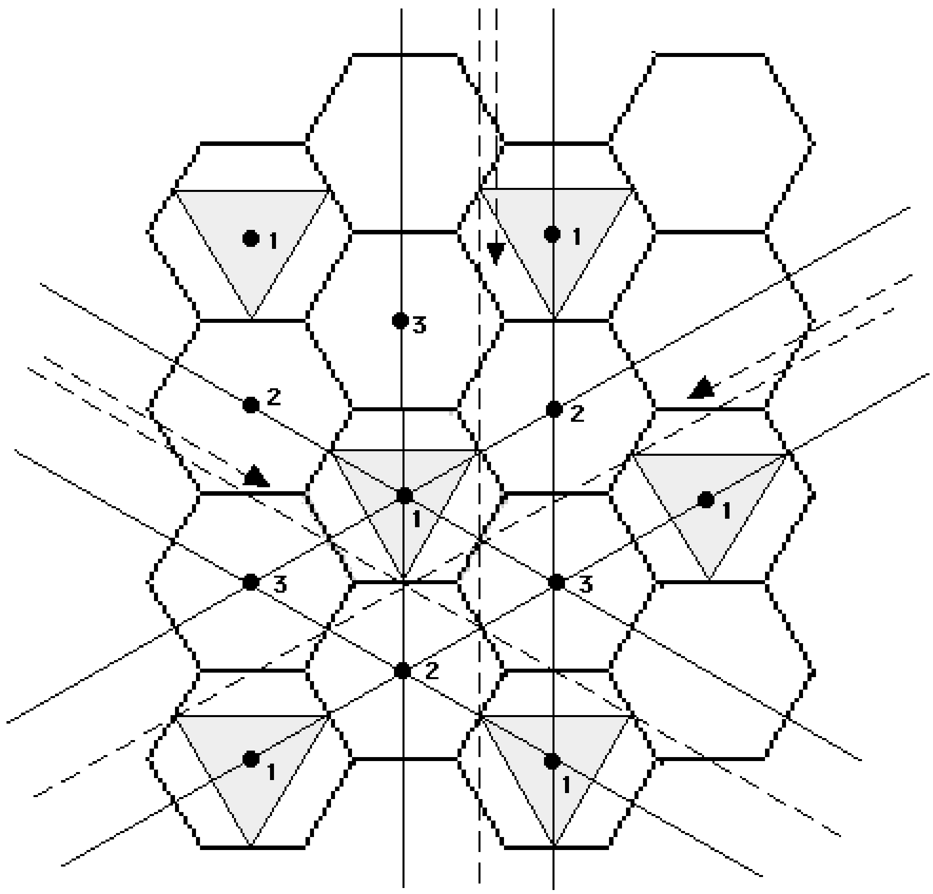


Fig. 4.71

This new pattern is of course very similar to that of figure 4.69: they both have rotation by 120^0 only, and they both have reflection and in-between glide reflection in three directions. What, if anything, makes them different in that case? To simply say that the one in figure 4.69 is 'denser' than the one in figure 4.71 is certainly not that precise or acceptable mathematically! (See also 4.17.4 below.) Well, a closer look reveals that, unlike in the case of the **p31m** type, **all** the **120^0** centers in the new pattern were **60^0**

centers in the beehive and lie at the intersection of three reflection axes: such patterns are known as **p3m1**, our 'last' type.

4.17.2 How many kinds of threefold centers? In a visual sense, the pattern in figure 4.71 has three kinds of 120^0 rotation centers: one at the center of a triangle (**1**), one between three vertices (**2**), and one between three sides (**3**). From another perspective, all rotation centers are the same: they are all old sixfold centers, lying on reflection axes, and **at the same distance from the closest glide reflection axis**. More significantly though, and in the spirit of 4.11.2, the three kinds of centers are distinct because no isometry maps centers of any kind to centers of another kind. Either way, **p3m1** patterns (**three or one** kinds of centers, depending on how you look at it) are distinguishable from **p31m** patterns (**two** kinds of centers)!

4.17.3 Persian stars. In the following **p3m1** example from ***Stevens*** (p. 267), six-pointed stars and hexagons give the illusion of a **p6m** pattern, but you already know too much to be fooled (and miss the 'tripods' that turn the $\mathbf{D_6}$ sets into $\mathbf{D_3}$ sets):

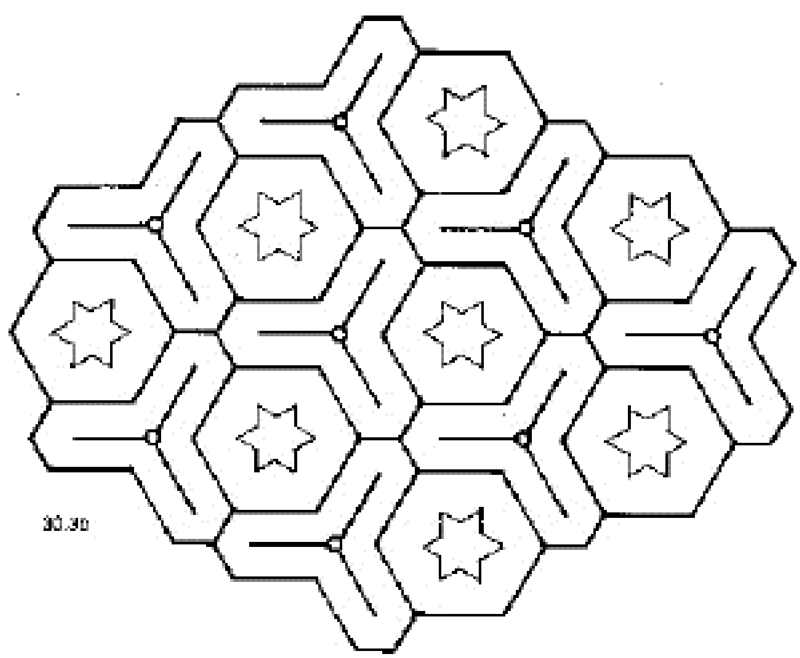


Fig. 4.72

4.17.4 More on (glide) reflection. The reflections and in-between glide reflections in both figures 4.69 (**p31m**) and 4.71 (**p3m1**) are none other than those **type 1** (glide) reflections inherited from the beehive pattern in figure 4.62. This may give you the impression that the beehive's **type 2** (glide) reflections can never survive in a 120^0 pattern. But as figure 4.73 demonstrates, it is possible to 'build' a **p3m1** pattern 'around' **type 2** (glide) reflection; and we leave it to you to demonstrate the same for **p31m** patterns -- a simple way to do that would be to modify figure 4.71 so that the vertices of the triangles would be each hexagon's vertices rather than edge midpoints!

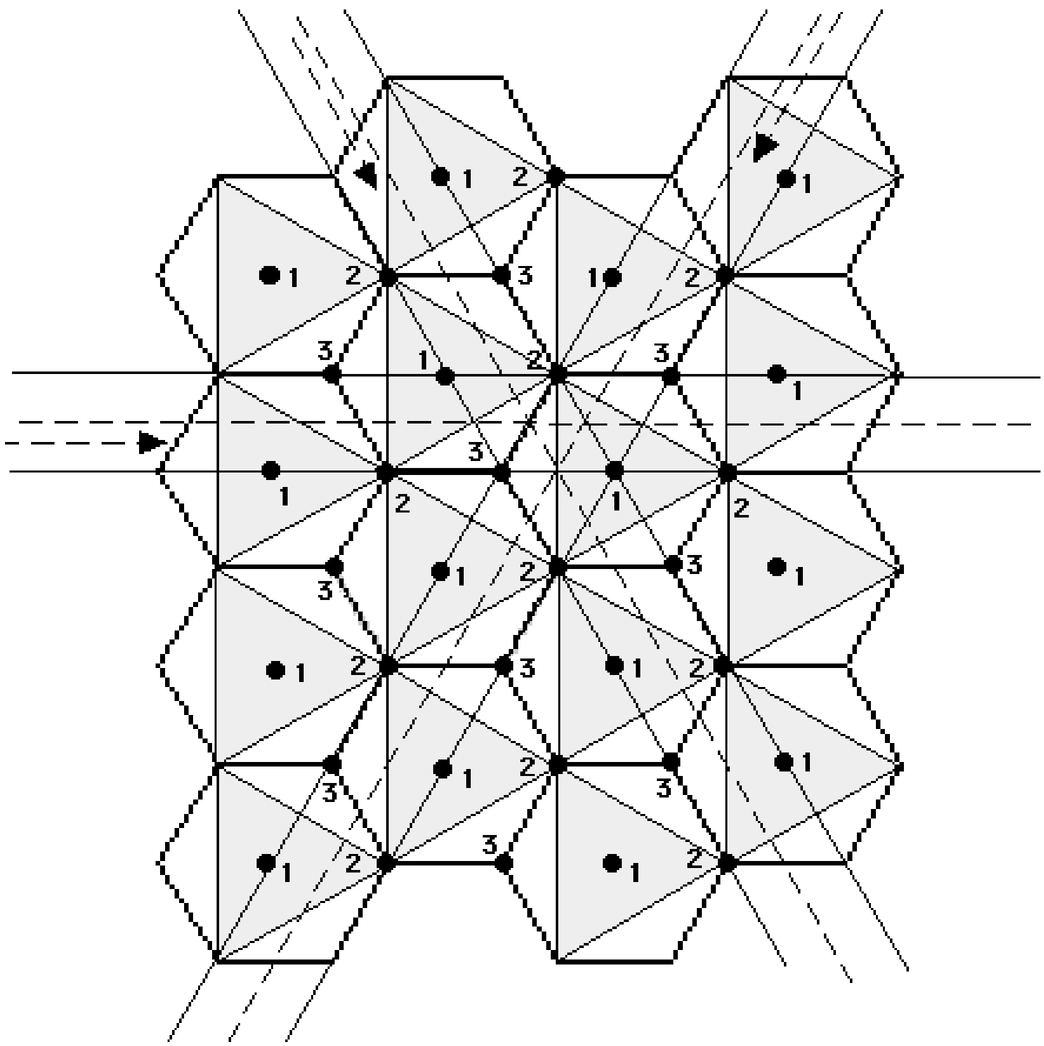


Fig. 4.73

As we indicate in 8.4.3 (and figure 8.41 in particular), and as you may verify in figures 4.69-4.73, the real difference between **p3m1** and **p31m** has to do with the **placement** of their (glide) reflection axes with respect to their lattice of rotation centers; for example the **ratio** of the glide reflection vector's length to the distance between two nearest rotation centers equals $\sqrt{3}/2$ in the case of the **p31m** as opposed to 3/2 in the case of the **p3m1**.

A medieval design very similar to the pattern in figure 4.73 was actually at the center of a famous controversy regarding whether or not **all seventeen** types of wallpaper patterns (and the **p3m1** in particular) appear in the Moorish **Alhambra Palace** in Spain: indeed the pattern in figure 4.73 may be viewed as a **two-colored** pattern of **p6m** (rather than **p3m1**) type, more specifically a **p6′mm′** (described, among other two-colored **p6m** types, in section 6.17)! This can be avoided simply by starting with a **'sparse'** beehive (i.e., one from which two thirds of the hexagons have been removed, in such a way that no two hexagons touch each other): see figures 6.132 & 6.133, as well as the 60^0 & 120^0 examples in ***Crystallography Now*** (http://www.oswego.edu/~baloglou/103/seventeen.html, a web page devoted to a geometrical classification of wallpaper patterns in the spirit of chapters 7 and 8).

4.18 The seventeen wallpaper patterns in brief

(I) Patterns with no rotation (360^0)

p1 : nothing but translation (common to all seventeen types)

pg : glide reflection in one direction; no reflection

pm : reflection in one direction; no in-between glide reflection

cm : reflection in one direction, in-between glide reflection

(II) Patterns with smallest rotation of 180^0

p2 : 180^0 rotation only

pgg : glide reflection in two perpendicular directions, no reflection; no rotation centers on glide reflection axes

pmm : reflection in two perpendicular directions, no in-between glide reflection; all rotation centers at the intersection of two perpendicular reflection axes

cmm : reflection in two perpendicular directions, in-between glide reflection; all rotation centers either at the intersection of two perpendicular reflection axes or at the intersection of two perpendicular glide reflection axes

pmg : reflection in one direction (with no in-between glide reflection), glide reflection in a direction perpendicular to that of the reflection; all rotation centers on glide reflection axes, none of them on a reflection axis

(III) Patterns with smallest rotation of 90^0

p4 : 90^0 rotation only; distinct 180^0 rotation, too

p4m : reflection in four directions; in-between glide reflection in two out of those four directions; all 90^0 rotation centers at the intersection of four reflection axes; all 180^0 rotation centers at the intersection of two reflection axes and two glide reflection axes

p4g : reflection in two directions; in-between glide reflection in both of those directions; additional glide reflection in two more (diagonal) directions; all 90^0 rotation centers at the intersection of two perpendicular (vertical and horizontal) glide reflection axes, none of them on a reflection axis or a diagonal glide reflection axis; all 180^0 rotation centers at the intersection of two perpendicular reflection axes

(IV) Patterns with smallest rotation of 120^0

p3 : 120^0 rotation only

p3m1: reflection in three directions with in-between glide reflection; all rotation centers at the intersection of three reflection axes; no rotation center on a glide reflection axis

p31m: reflection in three directions with in-between glide reflection; some rotation centers at the intersection of three reflection axes; some rotation centers on no reflection axis; no rotation center on a glide reflection axis

(V) Patterns with smallest rotation of 60^0

p6 : 60^0 rotation only; distinct 120^0 and 180^0 rotations, too

p6m : reflection in six directions with in-between glide reflection; all 60^0 (120^0) rotation centers at the intersection of six (three) reflection axes, none of them on a glide reflection axis; all 180^0 rotation centers at the intersection of two reflection axes and four glide reflection axes

first draft: fall 1998