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A NOTE ON THE BOUNDEDNESS OF OPERATORS ON WEIGHTED BERGMAN SPACES

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Dedicated to the memory of José Guadalupe

Abstract. Let ρ be a weight function, let X be a complex Banach space and let B_ρ denote the space of analytic functions in the disc $\mathbb{D}$ such that $\int_0^1 \rho(1-r)M_1(f',r)\,dr < \infty$, we prove that, under certain assumptions on the weight, the space of bounded operators $L(B_\rho, X)$ is isometrically isomorphic to the space $\Lambda_\rho(X)$ of X-valued analytic functions such that $\|F'(z)\| = O(\frac{\rho(1-|z|)}{1-|z|})$. Several applications are presented.

1. Introduction and preliminaries

The study of the spaces that will be considered in this paper goes back to my stay at the University of Zaragoza, where I had the chance to know José Guadalupe, better known by "Chicho", who always encouraged me to do research and was always willing to discuss about mathematics. He liked very much Hardy spaces and weighted inequalities. The subject I will be dealing with, actually comes from both topics.

In [3] it was shown that the boundedness of operators between weighted Bergman spaces and a general Banach space X can be characterized by the fact that certain associated operator-valued function belongs to certain Lipschizt space. Similar results were extended to more general weighted spaces $B_p(\rho)$ for $0 < p \leq 1$ in [4]. As application of these results new results on multipliers, Carleson measures and composition operators were achieved. In this note we use similar arguments to deal with generalized Lipschizt classes introduced by Janson (see [15]) and the considered by different authors (see [9], [2], [8]).

The starting point of the story is the consideration of the Banach envelope of the Hardy spaces H^p for $0 < p < 1$ denoted in [11] by B^p. In fact they give a definition for the spaces but these have implicitely appeared in an inequality due to Hardy and Littlewood (see [14], or [10] page 87) which establishes that for $0 < p < 1$ and $F \in H^p$ then

$$(1) \qquad \int_0^1 (1-r)^{1/p-2} M_1(F,r)\,dr \leq C\|F\|_{H^p}$$

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where $M_p(F,r) = (\int_0^{2\pi} |F(re^{it})|^p \frac{dt}{2\pi})^{1/p}$ and $\|F\|_{H^p} = \sup_{0<r<1} M_p(F,r)$.

This inequality was a crucial point in proving the duality between the Hardy classes for $0 < p < 1$ and the Lipschitz classes $(H^p)^* = \Lambda_\alpha$ for $\alpha = 1/p - 1$ achieved in [11]. Inspired by (1) they gave the definition of B_p as the space of analytic functions in the disc such that

$$\|F\|_{B^p} = \int_0^1 (1-r)^{1/p-2} M_1(F,r)\,dr < \infty.$$

For the limiting case in $p = 1$ one first needs to observe that $F \in B^p$ if and only if

$$\int_0^1 (1-r)^{1/p-1} M_1(F',r)\,dr < \infty.$$

Now let us define B^1 as the space of analytic functions such that

$$\|F\|_{B^1} = |F(0)| + \int_0^1 M_1(F',r)\,dr < \infty.$$

This space was shown to be the predual of the Bloch space (see [1] or [17]).

In this paper we consider a weighted version of these spaces.

Definition 1.1. *Let $\rho(t)$ denote a non-negative, non-decreasing integrable function on $(0,1)$. An analytic function F is said to belong to B_ρ if*

$$\int_0^1 \rho(1-r) M_1(F',r)\,dr < \infty.$$

It is easy to see that B_ρ becomes a Banach space under the norm

$$\|F\|_{B_\rho} = |F(0)| + \int_0^1 \rho(1-r) M_1(F',r)\,dr.$$

Note that the integrability condition on ρ allows to say that the polynomials belong to B_ρ. It is not difficult to see, using the density of the polynomials in the Hardy space H^1, that actually the polynomials are dense in B_ρ.

Of course the cases $\rho(t) = t^{1/p-1}$ for $0 < p \le 1$ give the just mentioned B^p-spaces.

Let us now introduce the generalized Lipschitz classes, already in the vector-valued setting.

Definition 1.2. *Let $\rho(t)$ denote a non-negative, non-decreasing integrable function on $(0,1)$ such that $\frac{\rho(t)}{t} \ge C > 0$ and let X be a complex Banach space. An analytic function from the disc $\mathbb{D}$ into X, $F(z) = \sum_{n=0}^\infty x_n z^n$ where $x_n \in X$, is said to belong to $\Lambda_\rho(X)$ if there exists a constant $C > 0$ such that for all $z \in \mathbb{D}$*

$$\|F'(z)\| \le C\,\frac{\rho(1-|z|)}{1-|z|}.$$

It is easy to see that this becomes a Banach space under the norm

$$\|F\|_{\Lambda_\rho(X)} = \|F(0)\| + \sup_{|z|<1} \Big\{ \frac{1-|z|}{\rho(1-|z|)} \|F'(z)\| \Big\}.$$

Observe that the assumption $\frac{\rho(t)}{t} \geq C > 0$ is now needed to have the vector-valued polynomials in the space.

Of course the case $\rho(t) = 1$ corresponds to the Bloch space:

$$\mathrm{Bloch}(X) = \Big\{ F : \mathbb{D} \to X \text{ analytic } : \sup_{|z|<1} (1 - |z|^2)\|F'(z)\| < \infty \Big\}.$$

The reader is referred to Theorem 5.1 in [10], to see that the scalar-valued proof which shows that the space Λ_α, corresponding to $\rho(t) = t^\alpha$, coincides with the Lipschizt class defined in terms of the modulus of continuity goes over the vector-valued case, and then we have

$$\Lambda_\alpha(X) = \Big\{ f \in C_X(\mathbb{T}) : w(f,t) = \sup_{s\in\mathbb{T}} \|f(e^{i(t+s)} - f(e^{is})\| = O(t^\alpha) \Big\}. \tag{2}$$

The reader is referred to [16, 15, 9, 8] for results concerning these and related spaces in the scalar-valued case and to [3, 4, 7] for results in the vector-valued situation.

Let us now indicate how to construct examples of functions in the generalized Lipschitz classes in the vector-valued case (see [3, 7] for more examples).

Example 1.1. *Let us assume that ρ satisfies*

$$\sum_{n=0}^{\infty} \rho\Big(\frac{1}{n+1}\Big) r^n = O\Big(\frac{\rho(1-r)}{1-r}\Big).$$

Then

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \rho\Big(\frac{1}{n+1}\Big) e_{n+1} z^n \in \Lambda_\rho(l^1), \tag{3}$$

$$F(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \rho\Big(\frac{1}{n+1}\Big) a_{n+1} z^n \in \Lambda_\rho(c_0), \tag{4}$$

where $\{e_n\}$ stands for the canonical basis of l^1 and $a_n = \sum_{k=1}^n e_k$.

Example 1.2. *Let $\phi_z(w) = \frac{z-w}{1-\bar{w}z}$ be the Moebious transformation in the disc for $z \in \mathbb{D}$. If $\rho(t) = t^{2/p}$ for $2 < p$ then*

$$F(z) = \phi_z \in \Lambda_\rho(L^p(\mathbb{D}, dA)) \tag{5}$$

where dA stands for the normalized Lebesgue measure in the disc $\mathbb{D}$.

Proof. It suffices to use the following well-known estimates for $q > 1$ and $\gamma > \alpha + 1$

$$\int_0^{2\pi} \frac{dt}{|1 - ze^{it}|^q} \leq C \frac{1}{(1 - |z|)^{q-1}} \quad \text{(see [10], page 65)}, \tag{6}$$

$$\int_0^{1} \frac{(1-r)^\alpha}{(1 - rs)^\gamma} dr \leq C \frac{1}{(1 - s)^{\gamma-\alpha-1}} \quad \text{(see [16], page 291)}. \tag{7}$$

□

Example 1.3. *Given $z \in \mathbb{D}$ let us denote the Cauchy kernel by $C_z(w) = \frac{1}{1-wz}$. Then a measure μ on $\mathbb{D}$ is a Carleson measure if and only if $F(z) = C_z$ is a $L^1(\mu)$-valued Bloch function.*

Proof. It suffices to recall the following characterization of Carleson measures

$$\sup\Big\{\int_{\mathbb{D}} \frac{1-|z|^2}{|1-z\bar{w}|^2}\, d\mu(w) : z \in \mathbb{D}\Big\} < \infty \quad \text{(see [13], page 239).} \tag{8}$$

□

There are some natural conditions on the weight ρ which allow to extend several results from the case $p < 1$ to a more general context. We recall the following notions which turned out to be relevant for different purposes (see [15, 2, 8, 9]).

Definition 1.3. *Let $\rho(t)$ denote a non-negative, integrable function on $(0,1)$. It is said to be a (b_1)-weight if there exists a constant $C > 0$ such that*

$$\int_s^1 \frac{\rho(t)}{t^2}\, dt \le C\, \frac{\rho(s)}{s}. \tag{9}$$

It is said to be a (d_1)-weight (or to satisfy Dini condition) if there exists a constant $C > 0$ such that

$$\int_0^s \frac{\rho(t)}{t}\, dt \le C\rho(s). \tag{10}$$

Observe that ρ being non-decreasing gives always

$$\frac{\rho(s)}{s} \le C \int_s^1 \frac{\rho(t)}{t^2}\, dt$$

and that the assumption $\frac{\rho(t)}{t} \ge C > 0$ together with (b_1) give

$$\log\Big(\frac{1}{1-s}\Big) \le C\, \frac{\rho(1-s)}{1-s}. \tag{11}$$

We refer the reader to the mentioned papers for examples and properties of these weights.

2. The theorem and its proof

Before stating the theorem we shall need a lemma which has its own interest. Let us give some notation to be used in the sequel. We write $u_n(z) = z^n$, $C_z(w) = \frac{1}{1-wz}$ and $K_z(w) = \frac{w}{(1-wz)^2}$.

Notice that $C_z = \sum_{n=0}^{\infty} u_n z^n$ and $K_z = \sum_{n=1}^{\infty} n u_n z^{n-1}$ and $\|u_n\|_\rho \le Cn$. Therefore both series are absolutely convergent in B_ρ.

Lemma 2.1. *If ρ is a non-decreasing (b_1)-weight then*

$$\sum_{n=0}^{\infty} \|u_n\|_{B_\rho} |z|^n = O\Big(\frac{\rho(1-|z|)}{1-|z|}\Big), \tag{12}$$

$$\|K_z\|_{B_\rho} = O\Big(\frac{\rho(1-|z|)}{1-|z|}\Big). \tag{13}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\|u_n\|_{B_\rho}|z|^n &= \sum_{n=1}^{\infty}\int_0^1 \rho(1-r)nr^{n-1}|z|^n\,dr\\ &= \int_0^1 \rho(1-r)\Big(\sum_{n=1}^{\infty} nr^{n-1}|z|^n\Big)\,dr\\ &= \int_0^1 \frac{\rho(1-r)}{(1-r|z|)^2}\,dr.\end{aligned}$$

On the other hand, using (6)

$$\begin{aligned}\|K_z\|_{B_\rho} &\le C\int_0^1 \rho(1-r)\Big(\int_0^{2\pi}\frac{dt}{|1-zre^{it}|^3}\Big)\,dr\\ &= \int_0^1 \frac{\rho(1-r)}{(1-r|z|)^2}\,dr.\end{aligned}$$

Therefore both results follow from the following property of (b_1)-weights:

$$\int_0^1 \frac{\rho(1-r)}{(1-rs)^2}\,dr = O\Big(\frac{\rho(1-s)}{1-s}\Big). \tag{14}$$

Indeed

$$\begin{aligned}\int_0^1 \frac{\rho(1-r)}{(1-rs)^2}\,dr &= \int_0^s \frac{\rho(1-r)}{(1-rs)^2}\,dr + \int_s^1 \frac{\rho(1-r)}{(1-rs)^2}\,dr\\ &\le \int_0^s \frac{\rho(1-r)}{(1-r)^2}\,dr + \int_s^1 \frac{\rho(1-r)}{(1-s)^2}\,dr\\ &\le \int_{1-s}^1 \frac{\rho(t)}{t^2}\,dt + \frac{1}{(1-s)^2}\int_s^1 \rho(1-r)\,dr\\ &\le C\,\frac{\rho(1-s)}{1-s}.\end{aligned}$$

□

Although the following elementary lemma is not needed in its full generality, we include here a proof to exhibit how Dini condition is used to make classical facts to remain valid in the weighted situation.

Lemma 2.2. *If ρ is a (d_1)-weight and ϕ is a bounded analytic function with $\|\phi\|_\infty = 1$, then the multiplication operator $M_\phi(f)(z) = \phi(z)(f(z)-f(0))$ defines a bounded operator from B_ρ into itself.*

Proof. Let us denote $g(z) = M_\phi(f)(z)$. Since $|\phi'(z)| \le \frac{C}{1-|z|}$ then

$$\begin{aligned}|g'(z)| &\le C\Big(|f'(z)| + \frac{|f(z)-f(0)|}{1-|z|}\Big)\\ &\le C\Big(|f'(z)| + \frac{1}{1-|z|}\int_0^{|z|}\Big|f'\Big(\frac{sz}{|z|}\Big)\Big|\,ds\Big).\end{aligned}$$

Hence

$$M_1(g',r) \leq C\Big(M_1(f',r) + \frac{1}{1-r}\int_0^r M_1(f',s)\,ds\Big).$$

Using now (d_1)-condition one gets

$$\begin{aligned}\int_0^1 \rho(1-r)M_1(g',r)\,dr &\leq C\Big(\|f\|_{B_\rho} + \int_0^1 \frac{\rho(1-r)}{1-r}\Big(\int_0^r M_1(f',s)\,ds\Big)\,dr\Big)\\ &\leq C\Big(\|f\|_{B_\rho} + \int_0^1 \Big(\int_s^1 \frac{\rho(1-r)}{1-r}\,dr\Big)M_1(f',s)\,ds\Big)\\ &\leq C'\Big(\|f\|_{B_\rho} + \int_0^1 \Big(\int_0^{1-s} \frac{\rho(t)}{t}\,dr\Big)M_1(f',s)\,ds\Big)\\ &\leq C'\Big(\|f\|_{B_\rho} + \int_0^1 \rho(1-s)M_1(f',s)\,ds\Big) = 2C'\|f\|_{B_\rho}.\end{aligned}$$

□

Arguments as above but simpler give the following

Remark 2.1. *If ϕ is an analytic function with bounded derivative, then the multiplication operator $M_\phi(f)(z) = \phi(z)(f(z) - f(0))$ defines a bounded operator from B_ρ into itself.*

Let us now give a natural correspondence between operators and vector-valued analytic functions, that will allow us to identify the bounded operators from B_ρ into X with $\Lambda_\rho(X)$ with equivalent norms. This idea has been used by the author several times with slight modifications (see [4] , [3] or [6]).

Given an analytic function $F(z) = \sum_{n=0}^\infty x_n z^n$ where $x_n \in X$ we can define a linear operator T_F which acts on polynomials as follows:

$$T_F\Big(\sum_{k=0}^n \alpha_k z^k\Big) = \sum_{k=0}^n \alpha_k x_k. \tag{15}$$

Conversely, given a linear operator T defined on the subspace of polynomials and with range in X one can define the vector-valued analytic function F_T given by

$$F_T(z) = \sum_{n=0}^\infty T(u_n)z^n. \tag{16}$$

Theorem 2.3. *Let ρ be a non-decreasing (b_1)-weight such that $\rho(t) \geq Ct$, X any complex Banach space and T a linear operator on the polynomials with range in X. Then T extends to a bounded operator from B_ρ into X if and only if F_T belongs to $\Lambda_\rho(X)$. Moreover $\|T\| \approx \|F_T\|_{\Lambda_\rho(X)}$.*

Proof. Assume T is bounded from B_ρ. Since $\sum_{n=0}^\infty u_n z^n$ is convergent in B_ρ then

$$F_T(z) = \sum_{n=0}^\infty T(u_n)z^n = T\Big(\sum_{n=0}^\infty u_n z^n\Big) = T(C_z).$$

Same argument gives

$$F_T'(z) = T(K_z).$$

Now use (13) to get

$$\|F_T'(z)\| \leq \|T\| \cdot \|K_z\|_{B_\rho} \leq C\,\frac{\rho(1-|z|)}{1-|z|}.$$

For the converse we shall use the following equality for all $n \geq 1$

$$2n(n+1)\int_0^1 (1-s^2)s^{2n-1}\,ds = 1. \tag{17}$$

Then, given a polynomial $p(z) = \sum_{k=0}^n \alpha_k z^k$ we have

$$\begin{aligned}
T\Big(\sum_{k=0}^n \alpha_k z^k\Big) &= \alpha_0 x_0 + \sum_{k=1}^n \alpha_k x_k \\
&= \alpha_0 x_0 + 2\int_0^1 \sum_{k=1}^n k(k+1)\alpha_k x_k s^{k-1} s^k (1-s^2)\,ds \\
&= p(0)F_T(0) + 2\int_0^1 (1-s^2)\Big(\int_0^{2\pi} p_1'(se^{-it})F_T'(se^{it})e^{-it}\,\frac{dt}{2\pi}\Big)\,ds
\end{aligned}$$

where $p_1(z) = z(p(z) - p(0))$.

Using that $F_T \in \Lambda_\rho(X)$ and remark 2.1 we have

$$\begin{aligned}
\|T(p)\| &\leq |p(0)|\|F_T(0)\| + 2\int_0^1 (1-s^2)\Big(\int_0^{2\pi} |p_1'(se^{-it})| \cdot \|F_T'(se^{it})\|\,\frac{dt}{2\pi}\Big)\,ds \\
&\leq C\|F_T\|_{\Lambda_\rho(X)}\Big(|p(0)| + \int_0^1 \rho(1-s)\Big(\int_0^{2\pi} |p_1'(se^{-it})|\,\frac{dt}{2\pi}\Big)\,ds\Big) \\
&\leq C\|F_T\|_{\Lambda_\rho(X)}\|p_1\|_{B_\rho} \leq C\|F_T\|_{\Lambda_\rho(X)}\|p\|_{B_\rho}.
\end{aligned}$$

Now extend to B_ρ by density of the polynomials. □

Remark 2.2. *Taking $X = \mathbb{C}$ we obtain the duality $(B_\rho)^* = \Lambda_\rho$. This gives a unified proof of the duality results for B^p and B^1 due to Duren, Romberg and Shields (see* [11]*) and to Anderson, Clunie and Pommerenke (see* [1]*). Nevertheless our proof follows the ideas in those papers, but our arguments seem to be a bit simpler. The duality can also be obtained from results in* [9] *but requiring stronger assumptions in the weight.*

It is rather usual in operator theory that if the "sup" condition expresses the boundedness then the "lim = 0" should express the compactness. Let us clarify this in our setting.

Definition 2.4. *Let $\rho(t)$ denote a non-negative, non-decreasing integrable function on $(0,1)$ such that $\frac{\rho(t)}{t} \geq C > 0$ and let X be a complex Banach space. An analytic function F from the disc $\mathbb{D}$ into X is said to belong to $\lambda_\rho(X)$ if*

$$\lim_{|z|\to 1} \frac{1-|z|}{\rho(1-|z|)}\,\|F'(z)\| = 0.$$

It is easy to see that $\lambda_\rho(X)$ is a closed subspace of $\Lambda_\rho(X)$.

Note that (11) shows that for (b_1)-weights one has that the X-valued polynomials are actually contained in $\lambda_\rho(X)$. In fact the closure of X-valued polynomials gives $\lambda_\rho(X)$ under the condition (b_1) in the weight as the following proposition shows.

Proposition 2.5. *Let ρ be a non-decreasing (b_1)-weight with $\rho(t) \geq Ct$ and $F \in \Lambda_\rho(X)$. The following are equivalent:*

(i) $F \in \lambda_\rho(X)$.
(ii) $\lim_{r\to 1} F_r = F$ *in* $\Lambda_\rho(X)$, *where* $F_r(z) = F(rz)$.
(iii) F *can be approached by polynomials in* $\Lambda_\rho(X)$.

Proof. Assume (i), that is $\lim_{\delta\to 1} \sup_{|z|=\delta} \left\{\frac{1-|z|}{\rho(1-|z|)}\|F'(z)\|\right\} = 0$. Since $F_r'(z)$ converges to $F'(z)$ uniformly on compact sets and the fact

$$\sup_{|z|=\delta} \{\|F_r'(z) - F(z)\|\} \leq 2 \sup_{|z|=\delta} \{\|F'(z)\|\}$$

we easily deduce (ii).

Let us assume (ii) and write $F(z) = \sum_{n=0}^\infty x_n z^n$. Given $\varepsilon > 0$ we find $r < 1$ such that $\|F - F_r\|_{\Lambda_\rho(X)} < \varepsilon/2$. Now, since the series $\sum_{n=0}^\infty x_n r^n u_n$ is absolutely convergent in $\Lambda_\rho(X)$, then we can consider $P_k(z) = \sum_{n=0}^k x_n r^n z^n$ for k large enough so that $\|F_r - P_k\|_{\Lambda_\rho(X)} < \varepsilon/2$. This shows (iii).

Let us now assume (iii). Given $\varepsilon > 0$ we choose a polynomial P so that $\|F - P\|_{\Lambda_\rho(X)} < \varepsilon/2$. Now

$$\frac{1-|z|}{\rho(1-|z|)}\|F'(z)\| \leq \frac{1-|z|}{\rho(1-|z|)}\|P'(z)\| + \|F - P\|_{\Lambda_\rho(X)}.$$

Using that P' is bounded and (11) one gets (i). □

Proposition 2.6. *Let ρ be a (b_1)-weight and X a Banach space. If $F \in \lambda_\rho(X)$ then T_F is a compact operator from B_ρ to X.*

Proof. Using Proposition 2.5 we have a sequence of polynomials P_n with values in X which approaches F in $\Lambda_\rho(X)$. Note that the associated operators T_{P_n} are finite rank operators and that T_{P_n} converges to T in norm. Therefore T is compact. □

Remark 2.3. *The converse of Proposition 2.6 is not true. Take $F \in \Lambda_\rho \setminus \lambda_\rho$ and $T = T_F$ the corresponding operator for $X = \mathbb{C}$. This would be compact but $F_T = F \notin \lambda_\rho$.*

3. Applications

Corollary 3.1. *Let $1/2 < p < 1, \alpha = 1/p - 1$ and X a Banach space. Then the following are equivalent:*

(i) $T : B_p \to X$ *is bounded.*
(ii) $T : H^p \to X$ *is bounded.*
(iii) $F_T \in \Lambda_\alpha(X)$.

Proof. The only implication is needed to be shown is that (ii) gives (iii). As in the proof of Theorem 2.3 one has to estimate $\|K_z\|_{H^p}$. In order to do that, note that (6) gives

$$\|K_z\|_{H^p} \leq \Big(\int_0^{2\pi} \frac{dt}{|1-ze^{it}|^{2p}}\Big)^{1/p} \leq C\frac{1}{(1-|z|)^{2-1/p}} = C\frac{1}{(1-|z|)^{1-\alpha}}.$$

□

The reader is referred to [4] for applications to multipliers, Carleson measures and composition operators.

Given an analytic function g on the unit disc we denote by $M_g(f)(z) = f(z)g(z)$ the multiplication operator with symbol g. Since $F'_{M_g}(z) = g \cdot K_z$ then condition (iii) in Corollary 3.1 gives the following application.

Corollary 3.2. *Let $1/2 < p < 1$ and an analytic function g. The following are equivalent:*

(i) $M_g : B^p \to B^p$ *is bounded.*
(ii) $M_g : H^p \to B^p$ *is bounded.*
(iii) $\int_{\mathbb{D}} |g(w)| \frac{(1-|w|)^{1/p-2}}{|1-zw|^2}\, dA(w) \leq \frac{C}{(1-|z|)^{2-1/p}}$.

Given $\phi : \mathbb{D} \to \mathbb{D}$ analytic we denote by $C_\phi(f)(z) = f(\phi(z))$ the composition operator. Since $F'_{C_\phi}(z) = C\phi(K_z)$ then Theorem 2.3 gives the following corollary.

Corollary 3.3. *Let ρ be a (b_1)-weight and ϕ an analytic function from the disc into itself. Then $C_\phi : B_\rho \to B_\rho$ is bounded if and only if*

$$\int_{\mathbb{D}} \rho(1-|w|)| \frac{|\phi(w)|}{|1-z\phi(w)|^2}\, dA(w) \leq C\, \frac{\rho(1-|z|)}{1-|z|}.$$

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