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On a Domain Embedding Method for Flow around Moving Rigid Bodies

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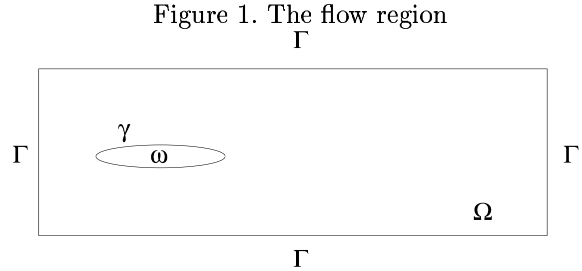
1 Introduction

Several applications lead to the numerical simulation of incompressible viscous flow around moving rigid bodies; let us mention for example blood flow around artificial heart valves. In this article we consider only the case where the rigid body motions are known a priori; the more complicated case where the rigid body motions are caused by hydrodynamical forces, among other forces, will be discussed in a forthcoming article. Following an approach advocated — to our knowledge — by Peskin [Pes72] we use a *domain embedding* method (also called *fictitious domain method* by some authors) which consists of filling the moving bodies by the surrounding fluid and taking into account the boundary conditions on these bodies by introducing a well chosen distribution of boundary forces. In the particular case of the *Dirichlet boundary* conditions considered in this article it is quite convenient to use a *Lagrange multiplier* method which is well suited to the variational formulations commonly used to study the Navier-Stokes equations and their approximation, by finite element methods for example. Another important component of the solution method is a time discretization by operator splitting which reduces the simulation to a sequence of subproblems for which efficient solution methods exist already.

Figure 1. The flow region



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2 A Model Problem and its Lagrange Multiplier/Domain Embedding Formulation

The geometrical situation is as in Figure 1. With $\omega = \omega(t)$ a moving rigid body ($\omega \subset \Omega \subset \boldsymbol{R}^d$, d =2, 3), we consider for $t > 0$ the solution of the *Navier-Stokes equations*

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \, in \, \Omega \setminus \overline{\omega(t)}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \, in \, \Omega \setminus \overline{\omega(t)}, \tag{2}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus \overline{\omega(0)}, (with \, \nabla \cdot \mathbf{u}_0 = 0), \tag{3}$$

$$\mathbf{u} = \mathbf{g}_0 \, on \, \Gamma, \tag{4}$$

$$\mathbf{u} = \mathbf{g}_1 \, on \, \gamma(t). \tag{5}$$

In (1)-(5) $\mathbf{u}$ and p denote, as usual, the *velocity* and *pressure*, respectively; $\nu(> 0)$ is the *viscosity*, $\mathbf{f}$ the density of external forces, $\mathbf{x}$ the generic point of $\boldsymbol{R}^d$ ($\mathbf{x} = \{x_i\}_{i=1}^d$), $\gamma(t) = \partial\omega(t)$ and $(\mathbf{u} \cdot \nabla)\mathbf{u} = \{\sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^d$. We suppose that $\mathbf{g}_1$ is the velocity on $\gamma(t)$ of the rigid body $\omega(t)$ which implies that $\int_{\gamma(t)} \mathbf{g}_1 \cdot \mathbf{n} \, d\gamma = 0$, and that $\int_\Gamma \mathbf{g}_0 \cdot \mathbf{n} \, d\Gamma = 0$. In the following, we shall use, if necessary, the notation $\phi(t)$ for the function $\mathbf{x} \to \phi(\mathbf{x}, t)$.

We introduce first the functional spaces $\mathbf{V}_{\mathbf{g}_0(t)} = \{\mathbf{v} | \mathbf{v} \in (H^1(\Omega))^d, \mathbf{v} = \mathbf{g}_0(t) \, on \, \Gamma\}$, $\mathbf{V}_0 = (H_0^1(\Omega))^d$, $L_0^2(\Omega) = \{q | q \in L^2(\Omega); \int_\Omega q \, d\mathbf{x} = 0\}$ and $\Lambda(t) = (\mathbf{H}^{-1/2}(\gamma(t)))^d$. With $\tilde{\mathbf{f}}$ an L^2-lifting of $\mathbf{f}$ in Ω (we can take $\tilde{\mathbf{f}}|_{\overline{\omega(t)}} = \mathbf{0}$) and $\nabla \cdot \mathbf{U}_0 = 0$ ($\mathbf{U}_0|_{\Omega \setminus \overline{\omega(0)}} = \mathbf{u}_0$), it can be shown — at least formally — that problem (1)-(5) is *equivalent* to

For $t \geq 0$, *find* $\{\mathbf{U}(t), P(t), \lambda(t)\} \in \mathbf{V}_{\mathbf{g}_0(t)} \times L_0^2(\Omega) \times \Lambda(t)$ *such that*

$$\int_\Omega \frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{v} \, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{U} \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega (\mathbf{U} \cdot \nabla)\mathbf{U} \cdot \mathbf{v} \, d\mathbf{x} - \int_\Omega P \nabla \cdot \mathbf{v} \, d\mathbf{x}$$
$$= \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\gamma(t)} \lambda \cdot \mathbf{v} \, d\gamma, \; \forall \mathbf{v} \in \mathbf{V}_0, \tag{6}$$

$$\int_\Omega q \nabla \cdot \mathbf{U}(t) \, d\mathbf{x} = 0, \;\; \forall q \in L^2(\Omega), \tag{7}$$

$$\int_{\gamma(t)} (\mathbf{U}(t) - \mathbf{g}_1(t)) \cdot \mu \, d\gamma = 0, \;\; \forall \mu \in \Lambda(t), \tag{8}$$

$$\mathbf{U}(0) = \mathbf{U}_0 \;\; in \;\; \Omega, \quad \mathbf{U} = \mathbf{g}_0 \;\; on \;\; \Gamma, \tag{9}$$

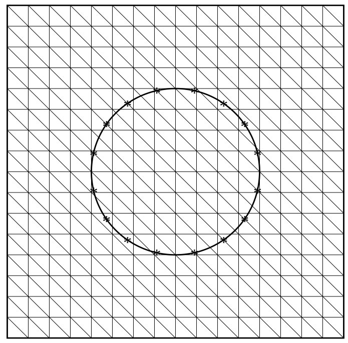
in the sense that $\mathbf{U}(t)|_{\Omega \setminus \overline{\omega(t)}} = \mathbf{u}(t)$ and $P(t)|_{\Omega \setminus \overline{\omega(t)}} = p(t)$. We can easily show that $\lambda = [\nu \partial \mathbf{U}/\partial \mathbf{n} - \mathbf{n}P]_\gamma$, where $[\;]_\gamma$ denotes the *jump* at γ.

Remark 2.1: The mathematical analysis of flow problems such as (1)-(5) is addressed in, e.g., [AG93] (see also the references therein).

Remark 2.2: We observe that the *actual geometry*, i.e., $\omega(t)$ and $\gamma(t)$ occurs "only" in the $\gamma(t)$-integral in (6) and in (8); this is a justification of the domain embedding approach.

3 Finite Element Approximation of Problem (6)-(9)

Figure 2. Part of the triangulation of Ω with mesh points indicated by "$*$" on the disk boundary



We suppose that $\Omega \subset \boldsymbol{R}^2$ $(d = 2)$. With h a space discretization step we introduce a *finite element* triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ and then $\mathcal{T}_{h/2}$ a triangulation twice finer obtained by joining the midpoints of the edges of $\mathcal{T}_h$. We define then the following finite dimensional spaces which approximate $\mathbf{V}_{\mathbf{g}_0}$, $\mathbf{V}_0$, $L^2(\Omega)$, $L^2_0(\Omega)$ respectively

$$\mathbf{V}_{\mathbf{g}_{0h}} = \{\mathbf{v}_h | \mathbf{v}_h \in C^0(\bar{\Omega})^2,\ \mathbf{v}_h|_T \in P_1 \times P_1,\ \forall T \in \mathcal{T}_h,\ \mathbf{v}_h|_\Gamma = \mathbf{g}_{0h}\}, \tag{10}$$

$$\mathbf{V}_{0h} = \{\mathbf{v}_h | \mathbf{v}_h \in C^0(\bar{\Omega})^2,\ \mathbf{v}_h|_T \in P_1 \times P_1,\ \forall T \in \mathcal{T}_h,\ \mathbf{v}_h|_\Gamma = \mathbf{0}\}, \tag{11}$$

$$L^2_h = \{q_h | q_h \in C^0(\bar{\Omega}),\ q_h|_T \in P_1,\ \forall T \in \mathcal{T}_{2h}\},\ L^2_{0h} = \{q_h | q_h \in L^2_h,\ \textstyle\int_\Omega q_h \, d\mathbf{x} = 0\}; \tag{12}$$

in (10)-(12), P_1 is the space of the polynomials in x_1, x_2 of degree ≤ 1 and $\mathbf{g}_{0h}$ is an approximation of $\mathbf{g}_0$ such that $\int_\Gamma \mathbf{g}_{0h} \cdot \mathbf{n} \, d\Gamma = 0$. Concerning the space $\Lambda_h(t)$ approximating $\Lambda(t)$, we define it by

$$\Lambda_h(t) = \{\mu_h | \mu_h \in (L^\infty(\gamma(t)))^2, \mu_h \text{ is constant on the arc joining } 2 \text{ consecutive mesh points on } \gamma(t)\}. \tag{13}$$

A particular choice for the mesh points on γ is visualized on Figure 2, where ω is a disk. Let us resist any requirement that the mesh points on γ have to be at the intersection of γ with the triangle edges of $\mathcal{T}_{h/2}$; (see [GG95] for more details and the relations between h_Ω and h_γ). This kind of decoupling between the Ω and γ meshes makes the domain embedding approach attractive for problems with moving boundaries like those discussed in this note. With the above spaces it is natural to approximate problem (6)-(9) by (with obvious notation)

$$\int_\Omega \frac{\partial \mathbf{U}_h}{\partial t} \cdot \mathbf{v} \, d\mathbf{x} + \nu \int_\Omega \nabla \mathbf{U}_h \cdot \nabla \mathbf{v} \, d\mathbf{x} + \int_\Omega (\mathbf{U}_h \cdot \nabla)\mathbf{U}_h \cdot \mathbf{v} \, d\mathbf{x} - \int_\Omega P_h \nabla \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \tilde{\mathbf{f}}_h \cdot \mathbf{v} \, d\mathbf{x} + \int_{\gamma(t)} \lambda_h \cdot \mathbf{v} \, d\gamma,\ \forall \mathbf{v} \in \mathbf{V}_{0h},\ \mathbf{U}_h(t) \in \mathbf{V}_{\mathbf{g}_0(t)h}, \tag{14}$$

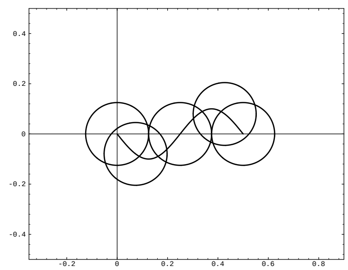
$$\int_{\Omega} q\nabla \cdot \mathbf{U}_h(t)\, d\mathbf{x} = 0,\ \forall q \in L_h^2,\ P_h(t) \in L_{0h}^2, \tag{15}$$

$$\int_{\gamma(t)} (\mathbf{U}_h(t) - \mathbf{g}_1(t)) \cdot \mu\, d\gamma = 0,\ \forall \mu \in \Lambda_h(t),\ \lambda_h(t) \in \Lambda_h(t), \tag{16}$$

$$\mathbf{U}_h(0) = \mathbf{U}_{0h}; \tag{17}$$

in (17), $\mathbf{U}_{0h}$ is an approximation of $\mathbf{U}_0$, approximately divergence-free.

Figure 3.



4 Time Discretization of (14)-(17) by Operator Splitting Methods

From an abstract point of view problem (14)-(17) is a particular case of the following class of initial value problems

$$\frac{d\phi}{dt} + A_1(\phi) + A_2(\phi) + A_3(\phi) = f,\ \phi(0) = \phi_0, \tag{18}$$

where the operators A_i can be *multivalued.* Among many operator splittings which can be employed to solve (18) we advocate the very simple one below (analyzed in, e.g., [Mar90]); it is only first-order accurate but its low order accuracy is compensated by good stability and robustness properties.

A fractional step scheme à la Marchuk-Yanenko: With $\triangle t$ a time discretization step and the initial guess, $\phi^0 = \phi_0$, the scheme is defined as follows:

For $n \geq 0$, we obtain ϕ^{n+1} from ϕ^n via the solution of

$$(\phi^{n+j/3} - \phi^{n+(j-1)/3})/\triangle t + A_j(\phi^{n+j/3}) = f_j^{n+1}, \tag{19}$$

with $j = 1, 2, 3$ and $\sum_{j=1}^{3} f_j^{n+1} = f^{n+1}$. Applying scheme (19) to problem (14)-(17) we obtain (with $0 \leq \alpha, \beta \leq 1$, $\alpha + \beta = 1$, and after dropping some of the subscripts h):

$$\mathbf{U}^0 = \mathbf{U}_{0h}; \tag{20}$$

for $n \geq 0$, *we compute* $\{\mathbf{U}^{n+1/3}, P^{n+1/3}\}$, $\mathbf{U}^{n+2/3}$, $\{\mathbf{U}^{n+1}, \lambda^{n+1}\}$ *via the solution of*

$$\begin{cases} \displaystyle\int_\Omega \frac{\mathbf{U}^{n+1/3} - \mathbf{U}^n}{\triangle t} \cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega P^{n+1/3} \nabla \cdot \mathbf{v}\, d\mathbf{x} = 0,\ \forall \mathbf{v} \in \mathbf{V}_{0h}, \\ \displaystyle\int_\Omega q \nabla \cdot \mathbf{U}^{n+1/3}\, d\mathbf{x} = 0,\ \forall q \in L_h^2;\ \mathbf{U}^{n+1/3} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1},\ P^{n+1/3} \in L_{0h}^2, \end{cases} \tag{21}$$

$$\begin{cases} \displaystyle\int_\Omega \frac{\mathbf{U}^{n+2/3} - \mathbf{U}^{n+1/3}}{\triangle t} \cdot \mathbf{v}\, d\mathbf{x} + \alpha\nu \int_\Omega \nabla \mathbf{U}^{n+2/3} \cdot \nabla \mathbf{v}\, d\mathbf{x} \\ \quad \displaystyle + \int_\Omega (\mathbf{U}^{n+1/3} \cdot \nabla) \mathbf{U}^{n+2/3} \cdot \mathbf{v}\, d\mathbf{x} = \alpha \int_\Omega \tilde{\mathbf{f}}^{n+1} \cdot \mathbf{v}\, d\mathbf{x},\ \forall \mathbf{v} \in \mathbf{V}_{0h}; \\ \mathbf{U}^{n+2/3} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1}, \end{cases} \tag{22}$$

$$\begin{cases} \displaystyle\int_\Omega \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n+2/3}}{\triangle t} \cdot \mathbf{v}\, d\mathbf{x} + \beta\nu \int_\Omega \nabla \mathbf{U}^{n+1} \cdot \nabla \mathbf{v}\, d\mathbf{x} \\ \quad \displaystyle = \beta \int_\Omega \tilde{\mathbf{f}}^{n+1} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\gamma^{n+1}} \lambda^{n+1} \cdot \mathbf{v}\, d\gamma,\ \forall \mathbf{v} \in \mathbf{V}_{0h}, \\ \displaystyle\int_{\gamma^{n+1}} (\mathbf{U}^{n+1} - \mathbf{g}_{1h}^{n+1}) \cdot \mu\, d\gamma = 0,\ \forall \mu \in \Lambda_h^{n+1}; \\ \mathbf{U}^{n+1} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1} (= \mathbf{V}_{\mathbf{g}_0((n+1)\triangle t)h}), \lambda^{n+1} \in \Lambda_h^{n+1} (= \Lambda_h((n+1)\triangle t)). \end{cases} \tag{23}$$

5 Solution of the Subproblems (21), (22) and (23)

By inspection of (21) it is clear that $\mathbf{U}^{n+1/3}$ is the $L^2(\Omega)^2$-projection of $\mathbf{U}^n$ on the (affine) subset of the functions $\mathbf{v} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1}$ such that $\int_\Omega q \nabla \cdot \mathbf{v}\, d\mathbf{x} = 0$, $\forall q \in L_h^2$, $P^{n+1/3}$ being the corresponding Lagrange multiplier in L_{0h}^2. The pair $\{\mathbf{U}^{n+1/3}, P^{n+1/3}\}$ is *unique* and to compute it we can use an Uzawa/conjugate gradient algorithm operating in L_{0h}^2 equipped with the scalar product $\{q, q'\} \to \int_\Omega \nabla q \cdot \nabla q'\, d\mathbf{x}$. We obtain thus an algorithm preconditioned by the discrete equivalent of $-\Delta$ for the homogeneous Neumann boundary condition. Such an algorithm is *very* easy to implement and is described in [GPP96]; it seems to have excellent convergence properties.

If $\alpha > 0$, problem (22) is a classical one; it can be easily solved, for example, by a least squares/conjugate gradient algorithm like those discussed in [Glo84].

If $\beta > 0$ the solution of problem (23) has been discussed in [GPP94]. In the particular case where $\beta = 0$, problem (23) reduces to an $L^2(\Omega)^2$-projection over the subspace of $\mathbf{V}_{\mathbf{g}_{0h}}^{n+1}$ of the functions $\mathbf{v}$ satisfying the condition $\int_{\gamma^{n+1}} (\mathbf{v} - \mathbf{g}_{1h}^{n+1}) \cdot \mu\, d\gamma = 0$, $\forall \mu \in \Lambda_h^{n+1}$. It follows from the above observation that if $\beta = 0$, problem (23) can be solved by an Uzawa/conjugate gradient algorithm operating in Λ_h^{n+1}, which has many similarities with the algorithm used to solve problem (21). If one uses the trapezoidal rule to compute the various $L^2(\Omega)$-integrals in (23), taking $\beta = 0$ brings further simplification since in that particular case $\mathbf{U}^{n+1}$ will coincide with $\mathbf{U}^{n+2/3}$ at those vertices of $\mathcal{T}_{h/2}$ such that the support of the related shape function does not intersect γ^{n+1}; from the above observation it follows that to obtain $\mathbf{U}^{n+1}$ and λ^{n+1} we have to solve a linear system of the following form

$$A\mathbf{x} + B^t\mathbf{y} = \mathbf{b},\ \ B\mathbf{x} = \mathbf{c}. \tag{24}$$

For the numerical simulations presented in Section 6 we have used $\alpha = 1$ and $\beta = 0$ in (22), (23).

6 Numerical Experiments

We simulate a two-dimensional flow with $\Omega = (-0.35, 0.9) \times (-0.5, 0.5)$ (see Figure 3) and ω a moving disk of radius 0.125. The center of the disk is moving between $(0,0)$ and and $(0.5,0)$ along a prescribed trajectory $(x(t), y(t)) = (0.25(1 - \cos(\frac{\pi t}{2})), -0.1 \sin(\pi(1-\cos(\frac{\pi t}{2}))))$ (see Figure 3) of period 4. Several different positions of the disk have been shown on Figure 3. The boundary conditions are $\mathbf{u} = \mathbf{0}$ on Γ and $\mathbf{u}$ on $\partial\omega(t)$ coinciding with the disk velocity. We suppose that the disk rotates counterclockwise at angular velocity 2π. Since we are taking $\nu = 0.005$, the maximum Reynolds number based on the disk diameter as characteristic length is 102.336. On Ω we have used a regular triangulation $\mathcal{T}_{h/2}$ to approximate the velocity, like the one in Figure 2, the pressure grid $\mathcal{T}_h$ being twice coarser. Concerning $\Lambda_h(t)$, $\gamma(t)$ has been divided into M subarcs of equal length. We have done two simulations: For the first one we have taken $h = 1/128$, $\triangle t = 0.00125$ and $M = 80$. For the second we have taken $h = 1/256$, $\triangle t = 0.00125$ and $M = 160$. With stopping criteria of the order of 10^{-12} we need around 10 iterations at most to have convergence of the conjugate gradient algorithms used to solve the problems at each step of the scheme (20)-(23). On Figure 4, we show the isobar lines, the vorticity density and the streamlines obtained at t =5, 6, 7, 8 for $h = 1/256$, $\triangle t = 0.00125$ and $M = 160$. There is a good agreement between the results obtained from these two simulations.

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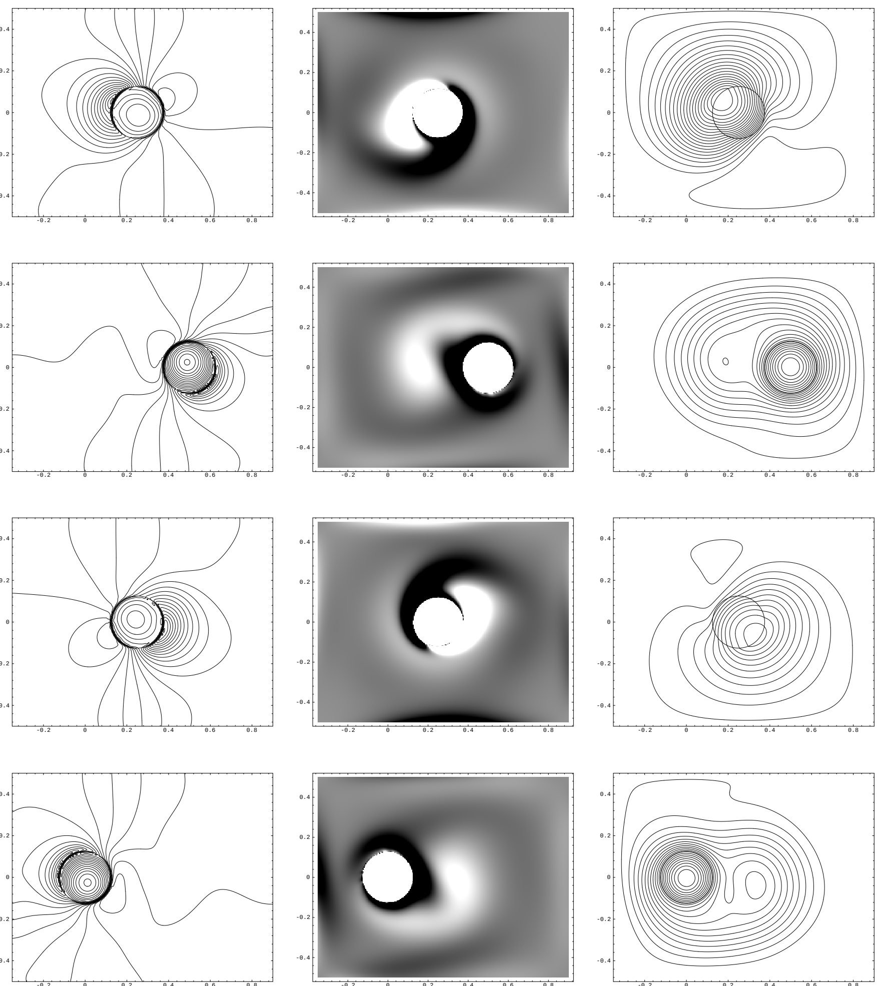
Figure 4. Isobar lines (at left), vorticity density (at middle) and streamlines (at right) at time $t = 5, 6, 7, 8$ in one period of disk motion. The disk moves from the left to the right, then to the left. The mesh size for velocity (resp., pressure) is $h = 1/256$ (resp., $h = 1/128$).



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