STUDENTS' OVERRELIANCE ON PROPORTIONALITY: EVIDENCE FROM PRIMARY SCHOOL PUPILS SOLVING ARITHMETIC WORD PROBLEMS

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Building on previous research on the tendency in students of diverse ages to overrely on proportionality in different domains of mathematics (e.g., geometry, probability), this study shows that – when confronted with missing-value word problems – Flemish primary school pupils strongly tend to apply proportional solution strategies, also in cases where they are not applicable. The evolution of this tendency is also investigated. It appeared that the overreliance on proportionality already emerges in the 2nd grade, but it increases considerably up to the 5th grade.

THEORETICAL AND EMPIRICAL BACKGROUND

One of the most important goals in nowadays' reform documents and curricula on mathematics education is that students should acquire the ability to develop and use powerful models to make sense of everyday life situations and of the complex systems stemming from modern society (ICMI Study 14 – Discussion Document, 2002). Traditionally, the way of teaching mathematical modelling and applied problem solving in primary school is through the use of word problems. These word problems are assumed to offer an acceptably good substitute for "real" problems that the learners may encounter outside their mathematics lessons (Verschaffel, Greer, & De Corte, 2000).

Nevertheless, during the last 10-15 years, several investigations have shown that – due to the stereotyped diet of the word problems offered to students and to the way in which these problems are handled by teachers – students start to perceive word problem solving as a puzzle-like activity with little or no grounding in the real world, and as something quite far removed from the goal-directed, more authentic activity of mathematical modelling of "real" problems (for an extensive overview, see Verschaffel et al., 2000). Often, students can successfully use very superficial cues to decide which operations are required to solve a particular word problem in a traditional textbook or test. Arguably, such instruction does not lead to the ability to discriminate between cases where a certain arithmetical operation is required and where it is not appropriate, but rather to stereotyped, superficial coping behaviour.

One of the clearest examples of such a "corrupted" modelling process is students' tendency to overgeneralise the range of applicability of the proportional model. Because of its wide applicability in pure and applied mathematics and science, proportional reasoning is a major topic in primary and secondary mathematics education. Therefore, typically from grade 3 or 4 of primary school on, pupils are frequently confronted with "proportionality problems" such as: "*10 eggs cost 2 euro. What is the price of 30 eggs?*" There are studies, however, that indicate that at the beginning of secondary education pupils associate such "missing-value problems" (word problems in which three numbers are given and a fourth one is asked for) automatically with the scheme of proportionality, even when it does not appropriately model the problem situation (see, e.g., De Bock, Verschaffel, & Janssens, 1998, 2002). It seems as if these students develop the tendency to assume proportional relationships "anywhere". For example, several studies (Verschaffel et al., 2000) found that more than 90% of the pupils at the end of primary school answered "170 seconds" to the following "runner" item: "*John's best time to run 100 metres is 17 seconds. How long will it take him to run 1 kilometre?*" Another utterance of this excessive adherence to proportionality – observed in numerous studies – can be seen in students making graphs (e.g. drawing a straight line through the origin when representing the relation between the length and age of a person) (Leinhardt, Zaslavsky, & Stein, 1990). But also history provides several cases of unwarranted applications of proportionality (e.g., Aristotle believing that if an object is ten times as heavy as another object, it will reach the ground ten times as fast). The most systematically investigated case of the improper application of proportionality probably stems from geometry. In a series of experimental studies, De Bock and his colleagues have shown that there is a widespread and almost irresistible tendency among secondary school students to believe that if a figure enlarges *k* times, the area and volume of that figure are enlarged *k* times too (De Bock et al., 1998, 2002). Moreover, students were almost insensible to diverse types of help (drawings, metacognitive support, …), and systematic remedial teaching had only a limited positive effect (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2003).

Besides continuing the empirical research concerning this "proportionality illusion" in geometry, we recently set up a new line of research to explore the tendency towards unwarranted proportional reasoning in other mathematical domains. The first new domain in which we investigated this tendency is probabilistic reasoning (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). We found that 10th and 12th graders, and even university students, strongly tend to assume proportional relationships when comparing the probability of two events. For example, many of them believed that if one has 4 trials to roll a die, the probability of getting a six is double as large as if one gets only 2 trials.

UBIQUITOUS APPLICATION OF PROPORTIONALITY IN SOLVING ARITHMETIC WORD PROBLEMS

The remarkably strong tendency towards unwarranted proportional reasoning in secondary school students – as observed in the studies mentioned above –, raises the question when and how the tendency to apply proportionality to missing-value problems actually originates and develops. Therefore we set up a new study with different age groups of primary school children. Because of the young age of the participants, we used rather simple arithmetic word problems (instead of the geometry or probability problems used in our previous studies). A review of the literature revealed that cases of unwarranted proportional reasoning may occur in reaction to two different kinds of arithmetic word problems: "unsolvable" and "solvable" ones.

First, there are studies in which "unsolvable" problems elicited proportional strategies. We already mentioned the studies reported in Verschaffel et al. (2000), in which proportional answers were observed for several items, e.g. the "runner" item cited above or items like: "*A shop sells 312 Christmas cards in December. About how many do you think it will sell altogether in January, February and March?"* There is, however, a problem with the interpretation of pupils' proportional answers here. There is no logico-mathematical relation between the givens in these items, so an exact answer cannot be given. Puchalska and Semadeni (1987) call this type of word problems "pseudoproportionality problems". As Reusser and Stebler (1997) demonstrated, pupils sometimes realise that a proportional model is inappropriate for the problem but give a proportional solution anyhow because they believe or feel it is necessary to give a numerical answer to every word problem.

Besides these "unsolvable" problems, there is a second category of non-proportional arithmetic problems, for which an exact numerical answer can be calculated. Nevertheless, unwarranted proportional answers were also observed for these problems. An example is the "ladder" problem used by Stacey (1989) with 9- to 13-year olds: "*With 8 matches, I can make a ladder with 2 rungs, as the one you see on the drawing*. [Figure 1] *How many matches do I need to make a ladder with 20 rungs?*" The most frequently observed erroneous answer was 10 × 8 = 80 matches. (for similar examples, see Linchevski, Olivier, Sasman, & Liebenberg, 1998). Another – even more striking – example comes from Cramer, Post and Currier (1993): 32 out of 33 pre-service elementary teachers answered proportionally to the following problem: "*Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?*" Similar results were recently obtained by Monteiro (2003). Finally, there is also an item used in a study by Van Lieshout, Verdwaald and Van Herk (1997). They found that many children answered the following "biker" item as if proportionality was appropriate: "*Joris and Pim live in the same house. They bike home together on 8 minutes. How many minutes must Joris bike when he bikes home alone?*"

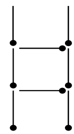


Figure 1

In the study that we will report in the rest of this paper, we only used non-proportional word problems from the second category (i.e. the "solvable" problems). The problems in our study were self-generated, but inspired by the above-mentioned studies.

RESEARCH QUESTIONS AND HYPOTHESES

The goal of the study was to search for the origins of students' tendency to apply the proportional model to solve non-proportional problems, and to describe how it evolves with age and educational experience. We hypothesized that – because of the wide applicability and intrinsically simple and intuitive character of the proportional model –

the overgeneralization of the proportional model would already be present at the beginning of primary education, more precisely when children begin to learn how to multiply and to divide and to recognize when to apply these operations in (standard) word problems (in Flanders, this is in grades 2 and 3). We also hypothesized that these errors would reach their peak when proportional relationships are systematically taught in classroom and when textbooks abound with "typical" missing-value proportional problems (i.e., in grades 4 and 5 in Flanders).

METHOD

729 primary school pupils participated in this study, belonging to three randomly selected Flemish schools. They were more or less equally divided over grades 2 to 6. Pupils from grade 3 to 6 received a paper-and-pencil test containing 10 experimental word problems in random order. The second graders only received the 5 easiest problems, and they were presented to them both in written and oral form. All problems were missing-value problems and were formulated as identical as possible.

The 10 experimental word problems were developed according to the design in Table 1. As can be seen in this table, the 10 word problems belonged to 5 categories. One category consisted of *proportional* problems (i.e. problems for which a proportional strategy leads to the correct answer) and the other 4 categories contained diverse types of *non-proportional* problems (i.e. problems for which another strategy must be applied to find the correct answer). These 4 types referred to different non-proportional mathematical models underlying the problem, i.e., additive, constant, linear (but not proportional), and a pattern. For each of the 5 problem categories an "easy" and a "difficult" version was designed, carefully controlling for the difficulty level within a category (and therefore the difference between the "easy" and "difficult" category) in several ways (number size, calculation complexity, provision of a drawing, verbal complexity).

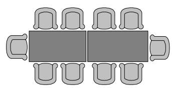
	Easy	Difficult
Proportional	1	6
Non-proportional		
Additive	2	7
Constant	3	8
Linear	4	9
Patterns	5	10

Table 1: Design of experimental items

By strictly controlling the formulation of the problem and manipulating the two experimental factors (category and difficulty level), differences in performance could very likely be attributed to these two experimental factors, rather than to uncontrolled differences in technical reading difficulty or complexity of calculations.

Due to space restrictions, we only give one exemplary item here, namely the (non-proportional) linear item 4. Several other items will be given in the results section.

In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now the teacher puts 6 tables in a line. How many chairs fit around these tables?



This word problem is non-proportional, since there is no proportional relationship between the number of tables and the number of chairs fitting around. Instead, there is a linear function of the form $f(x) = ax + b$ (graphically represented by a straight line but not going through the origin) underlying the problem situation. In Figure 2 the known and unknown elements in the problem – and the relations between them – are represented. The correct reasoning (see Figure 2a) is that there are 4 chairs around each table plus 2 chairs at both heads of the table line (thus (6 × 4) + 2 = 26 chairs around 6 tables). Incorrect proportional solutions (see Figure 2b) could consist of reasoning that there are 3 times as many tables (6 instead of 2), so 3 times as many chairs (3 × 10 = 30) fit around (i.e. using the internal ratio), or that there are 10 chairs for 2 tables, meaning 5 chairs per table, so 30 chairs for 6 tables (i.e. applying the "rule of three"). For each of the other non-proportional items, a similar schematic representation was made, distinguishing the correct reasoning for that item from the incorrect proportional one(s).

2a: correct solution *2b: proportional solution*

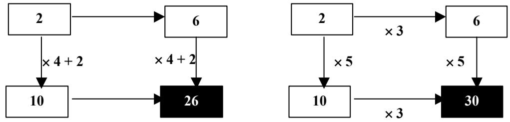


Figure 2 : Representation of the correct (fig 2a)

Pupils' answers to the problems were classified as either "*correct*" (= the correct answer was given), a "*proportional error*" (= a proportional strategy applied to a non-proportional item) or an "*other error*" (= another solution procedure was followed, or the item was not answered). When a purely technical calculation error was made, the answer was still categorised as either *correct* or *proportional* (depending on the reasoning that was made). Due to space restrictions, we will limit ourselves to the frequency of correct answers (for proportional problems) and the frequency of proportional answers (for non-proportional problems).

MAIN RESULTS

Table 2 presents the performances for the *proportional items* of each grade. The data clearly show that the ability to solve the proportional problems gradually increased from grade 3 until grade 6, where proportional reasoning was nearly perfectly mastered. The greatest progress was made from grade 3 to 4 and from grade 4 to 5.

Grade→	**2**	**3**	**4**	**5**	**6**	**Total**
Item 1 (easy)	68.2	68.5	83.7	93.9	96.4	82.1
Item 6 (diff)		39.2	63.7	86.7	93.4	73.0
Total		53.9	73.7	90.3	94.9	79.9

Table 2: % correct answers on the proportional problems

Table 3 presents the percentages of proportional answers for the eight *non-proportional items*. For all non-proportional items, 38.5% proportional solution methods were found. Nevertheless, large differences exist between the age groups and between the different types of word problems. Whereas the tendency to apply proportions was already present in the 2nd grade, it strongly increased over grades 3, 4 and 5, before slightly decreasing in 6th grade. Since there are major differences in the number of proportional answers for the different categories of non-proportional items (and between the two versions of each category), we will now consider the results for each problem category in detail.

Grade→	2	3	4	5	6	Total
Additive						
Item 2 (easy)	1.5	1.5	1.5	11.5	12.6	5.7
Item 7 (diff)		4.6	17.0	35.8	50.3	28.8
Constant						
Item 3 (easy)	41.7	59.2	72.6	82.4	70.1	71.7
Item 8 (diff)		35.4	49.6	72.1	41.3	50.4
Linear						
Item 4 (easy)	37.1	46.1	57.8	64.9	61.1	53.4
Item 9 (diff)		32.3	37.0	39.4	31.7	35.2
Patterns						
Item 5 (easy)	15.9	33.9	34.1	33.3	23.3	28.1
Item 10 (diff)		21.5	36.4	29.1	22.2	25.8
Total	24.1	29.3	37.5	46.1	39.1	38.5

Table 3: % proportional answers on the non-proportional problems

With respect to the *additive* items, there was a sharp difference between the number of proportional answers to the easy and the difficult variant. The easy additive problem (i.e., item 2) ("*Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?*") elicited only a small percentage of unwarranted proportional solutions in grades 2, 3 and 4 (1.5%), but remarkably, for this item, proportional reasoning suddenly showed up in more than 10% of the 5th and 6th graders (e.g. thinking that initially, Lies is 3 times as old as Bert, and that this should also be the case when Bert is 12 years old). A parallel, but much more pronounced evolution was observed for the difficult additive problem (item 7): "*Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 rounds. When Ellen has run 30 rounds, how many has Kim run?*" While in 3rd grade incorrect proportional solutions were practically absent, this type of error strongly increased with grade, so that by grade 6 more than half of the pupils made the proportional error (e.g. thinking that Kim initially has run 3 times as many rounds as Ellen, and that this ratio also holds at the second moment)!

The largest number of proportional answers was undoubtedly elicited by the most "a-typical" word problems, i.e. the *constant* items. Already in 2nd grade, 41.7% of the pupils solved the "easy" item 3 ("*Mama put 3 towels on the clothesline. After 12 hours they were dry. The neighbour woman put 6 towels on the clothesline. How long did it take them to dry?*") proportionally, and the percentage of pupils making this error went up to 82.4% in the 5th grade! A parallel evolution was found for the "difficult" item 8 ("*A group of 5 musicians plays a piece of music in 10 minutes. Another group of 35 musicians will play the same piece of music. How long will it take this group to play it?*"): 35.4% of the 3rd graders applied a proportional strategy and this raised to 72.1% of the 5th graders.

The same increase in the number of proportional answers was observed for the "easy" *linear* item 4 (the "tables" item already cited in the methods section). Whereas already more than one third of the 2nd graders solved this item proportionally, this raised to almost two thirds of the 5th graders. For the difficult variant (item 9: "*The locomotive of* *a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. If there would be 8 carriages connected to the locomotive, how long would the train be?*") the number of proportional errors was lower, but again, the peak was observed in the 5th grade.

The *patterns* problems generally elicited less proportional answers than the other problem categories. Our analysis has shown that pupils made a lot of other errors here, which is not surprising because these problems required a rather complex reasoning. For example, item 10 was: "*Jan participates in a quiz. Each time he wins a round, his points are doubled. After the second round, he has 8 points. When Jan wins all rounds up to round 6, how many points does he have?*" The correct answer to this problem (128 points) was found by only about 15% of the 4th, 5th and 6th graders, whereas more than 60% of them made another error than the proportional one. Nevertheless, the general trend also seems to hold for these "patterns" items: some of the 2nd graders already applied proportional solution strategies, and the number of pupils making this error increased up to grade 5 and then slightly decreased in grade 6.

CONCLUSIONS AND DISCUSSION

This study essentially combined two lines of research. On the one hand, there are the studies showing that when solving word problems, primary school pupils tend to apply very superficial solution strategies, to exclude their real-world knowledge and to believe that all mathematics problems can be solved by applying some simple operations on the given numbers. On the other hand, there are studies revealing that students of diverse ages persistently apply proportionality "anywhere", in different mathematical domains. Our study has shown that primary school pupils strongly tend to apply proportional solution strategies when confronted with non-proportional missing-value word problems. The tendency already emerged in the 2nd grade, but it increased considerably up to 5th grade. By then, pupils have had increasing training in solving proportionality problems. Despite important differences between the different item categories, this trend seems general. Currently, we are using the same research instrument to collect data with secondary school students (in grades 7 and 8), in order to investigate the further evolution of the tendency to give proportional answers to the diverse non-proportional word problems.

Keeping in mind the goal that students should be able to develop and apply mathematical models to make sense of everyday life situations, the data from our study are quite alarming. Pupils use superficial cues (e.g. linguistic hints in a word problem) to decide upon a solution scheme. In this respect, the proportional model seems to have a special status. Because of its intrinsic simplicity and intuitiveness, and because of the attention it gets in (elementary) mathematics education, the proportional model is prominently present in pupils' minds, and they strongly overrely on it.

Our study stresses the importance of approaching mathematics from a genuine modelling perspective from the very beginning of primary education on. In treating some basic mathematical concepts (e.g., multiplication, division, direct and inverse proportionality) immediate attention should also be paid at these concepts' capacity of describing, interpreting, predicting and explaining situations, with a strong emphasis on the "fitness" of models for specific situations (Lesh & Lehrer, 2003; Verschaffel et al., 2000). So, instructional moments wherein word problems are used mainly to create strong links between mathematical operations and prototypically "clean" model situations should be alternated with other lessons wherein applied problems are used primarily as exercises in relating real-world situations to mathematical models and in reflecting upon that complex relationship between reality and mathematics.

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