STUDENTS' IMPROPER PROPORTIONAL REASONING: THE CASE OF AREA AND VOLUME OF RECTANGULAR FIGURES

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In this paper, we investigate the predominance of the linear model in 12-13 year old Cypriot students, while solving non-proportional word problems involving area and volume of rectangular figures. Using three different kinds of tests related to the context of the word problems presented we attempt to identify a differentiation in students' responses. The results reveal students' tendency to apply proportional reasoning in problem situations in which this kind of reasoning is not suited. This tendency appears to decrease in the second phase of the study when the context of the problems was changed. The data also suggest that students are not able to detect the common non-linear character of area and volume tasks and therefore, deal with them in a different way.

INTRODUCTION

The rule of three, or in modern terminology the linear function, has been an important mathematical tool of explaining and mastering phenomena in different fields of human activity (Freudenthal, 1973; De Bock, Verschaffel, & Janssens, 1998). This suggests that proportionality, ratio preservation and linearity are universal models, a view that is reinforced by their frequent use. Moreover, the importance and significance of proportionality, as a mathematical tool, is also determined by the fact that its use dispenses one from rethinking a situation. Such a dispensation is usually gladly accepted.

The basic linguistic structure for problems involving proportionality includes i) four quantities *(a, b, c, d),* of which, in most cases, three are known and one unknown, and ii) an implication that the same relationship links *a* with *b* and *c* with *d.* In the case of true proportionality, this relationship is a fixed ratio (Behr, Harel, Post, & Lesh, 1992). However, if a problem matches this general structure, the tendency to evoke direct proportionality can be extremely strong even if this is not the case (Verschaffel, Greer & De Corte, 2000).

As Freudenthal (1983) points out, linearity is such a suggestive property of relations that one is readily seduced to dealing with each numerical relation as though it were linear. This phenomenon can be found even in traditional word problems where mathematical procedures, such as the rule of three, tended to be applied to problem situations without consideration of the realistic constraints (Verschaffel et al., 2000).

Students' tendency to apply proportional reasoning in problem situations for which it is not suited is partially caused by characteristics of the problem formulation, with which pupils learned to associate proportional reasoning throughout their school life. Multiplicative structures, notably those that on a superficial reading may create an

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illusion of proportionality, provide examples of inappropriate invocation of proportionality, as a result of an unconscious reaction to linguistic form (Greer, 1997). Thus, both at the level of individual students, and throughout history, there is a non-reflective link built up between the mathematical structure of proportional relationships and a stereotyped linguistic formulation (Verschaffel et al., 2000).

Freudenthal (1983) focuses on the appropriateness' of the linear relation as a phenomenal tool of description and indicates that there are cases in which this primitive phenomenology fails. One of these cases, which will be the focus of the present study, is the case of the non-linear behavior of area and volume under linear multiplication.

As De Bock, Verschaffel, & Janssens (2002b) point out, students' former real life practices with enlarging and reducing operations do not necessarily make them aware of the different growth rates of lengths, areas and volumes. Therefore, students strongly tend to see the relations between length and area or between length and volume as linear instead of quadratic and cubic. As a consequence, they apply the linear scale factor instead of its square or cube to determine the area or volume of an enlarged or reduced figure. This tendency is in line with the Intuitive Rule Theory (Stavy & Tirosh, 2000) which suggests that a change in a quantity A causes the same change in a quantity B.

Understanding that multiplication of lengths by d, of areas by d^2 and of volumes by d^3 is highly associated with the geometrical multiplication by d, is mathematically so fundamental, that, phenomenologically and didactically it should be put first and foremost (Freudenthal, 1983). Students should be able to distinguish that for instance, volume is proportional to length only when width and height are held constant; and similarly to width (or height) only when the other two variables are held constant. It is conceptually important and essential for students to understand the difference between the product of two variables in double proportion tasks, and the product of one variable by a constant in simple proportion problems (Vergnaud, 1997). Students have to break the pattern of linearity and become aware of the multi-dimensional impact of increase.

In recent years, there has been a considerable effort from researchers (De Bock et al., 1998; De Bock, Van Dooren, Janssens, & Verschaffel, 2002a; De Bock et al., 2002b; De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2003) to examine and overcome students' tendency to deal with non-proportional tasks concerning area as if they were proportional. In particular, De Bock et al., (1998) revealed an alarmingly strong tendency among 12-13 year old students to apply proportional reasoning in problem situations concerning area for which it was not suited. However, even the use of a number of different experimental scaffolds like the increase of the authenticity of the problem context (De Bock et al., 2003) and the use of metacognitive and visual scaffolds (De Bock et al., 2002b), did not yield the expected result. Only the rephrasing of the usual missing value problems into comparison problems proved to substantially help many students to overcome the “illusion of linearity” (De Bock et al., 2002b).

The actual processes and the mechanisms used by students while solving non-proportional problems were unraveled by means of interviews. It appears that not only is the "illusion of linearity" responsible for inappropriate proportional responses, but also intuitive reasoning, shortcomings in geometrical knowledge, and inadequate habits and beliefs about solving word problems (De Bock et al., 2002a; De Bock, Van Dooren, Janssens, & Verschaffel, 2001). All these aspects appear to favor a superficial and deficient mathematical modeling process, which leads to the unwarranted application of linearity.

From the literature review it becomes evident that the “illusion of linearity” is not a result of a particular experimental setting. In contrast, it is a recurrent phenomenon that seems to be quite universal and resistant to a variety of forms of support aimed at overcoming it (De Bock et al., 2003). Proportions appear to be deeply rooted in students’ intuitive knowledge and are used in a spontaneous and even unconscious way, which makes the linear approach quite natural, unquestionable and to a certain extend inaccessible for introspection or reflection (De Bock et al., 2002a). Therefore, as Verschaffel et al. (2000) illustrate, it takes a radical conceptual shift to move from the uncritical application of this simple and neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described.

In order to achieve this radical conceptual shift in the present study, we asked students to confront two different experimental settings with a common aim; to investigate the predominance of the linear model in 12-13 year old Cypriot students and to create a cognitive conflict in order to differentiate students’ behavior while solving non-proportional word problems involving area and volume of rectangular figures.

METHODS

Participants in the study were 307 students of grades 7 and 8 of 6 different gymnasiums of Cyprus. Specifically, the sample of the study consisted of 163 students of grade 7 (12 year olds) and 144 students of grade 8 (13 year olds).

The study was completed in three different phases in each of which a different written test was administered. All three tests were administered separately in the span of 15 days for about 30 minutes each. The first test (Test A) was administered to all 307 students of both grades and consisted of 5 different word problems, two of which concerned volume (pr.1, 4), two area (pr. 3, 5) and the fifth one length (pr.2). All problems were in a multiplicative comparison form where the area or volume of the rectangular figure was given as well as the relation that connected it with the area or volume of the new figure, respectively. The purpose of this test was to examine the extend to which Cypriot students apply proportional reasoning while solving non-proportional word problems involving area and volume of rectangular figures.

The second test (Test B) was administered to only 157 of the students that participated in Test A and consisted of the same 5 word problems of the first test, but with a differentiation in the amount of data given for solving each problem. In particular, in each problem in addition to the area or volume of the rectangular figure and the relation that connected it with the area or volume of the new figure, respectively, the dimensions of the first rectangular figure were also given. The purpose of this test was to examine whether the inclusion of the dimensions of the figures would influence students in such a degree as to execute the multiplicative comparison first with the dimensions of each figure and then find out the area or volume of the new figure, instead of applying direct proportionality between length and area or volume respectively.

The third test (Test C) was administered to the remaining 150 students that were not administered Test B and consisted of the same 5 word problems of Test A, but with a different presentation. In particular, each item of the test was accompanied by two alternative answers of two fictitious peers. One expressed the dominant misconception that the area and volume are directly proportional to length whereas the other expressed the correct answer. Students were then asked to find the solution strategy each peer used to find the answer given and then to choose the correct reasoning justifying their choice. Purpose of this scaffold was to create a cognitive conflict in students' minds in order to question the appropriateness of the direct proportionality between length and area and length and volume.

It is worth mentioning that all three tests included the formulas for finding the area and volume of rectangular figures. As for the grading of the tests, each correct answer was assigned the score 1, each wrong answer the score 0, whereas in the cases that the mathematical expression for the problem was correct but not the answer, the score 0.5 was given.

For the analysis and processing of the data collected the statistical package of SPSS was used as well as an implicative statistical analysis by using the computer software CHIC (Bodin, Coutourier, & Gras, 2000). The statistical package CHIC produces three diagrams: (a) the similarity diagram which represents groups of variables which are based on the similarity of students' responses, (b) the implication graph which shows implications $A \Rightarrow B$. This means that success in question A implies success in question B and (c) the hierarchical tree which shows the implication between sets of variables. In this study we use only the similarity diagram.

RESULTS

The analysis of the data collected revealed the tendency of 12-13 year old Cypriot students to apply proportional reasoning in problem situations, concerning area and volume of rectangular figures, for which it was not suited. From Table 1, one can detect the great difference in students' achievement at the non-linear tasks of area and volume in relation to the linear tasks of length. This difference, even though more prominent in Test A, is statistically significant in all three tests (tA=40,9, p=0,00, tB=15,29, p=0,00, tC=21,92, p=0,00). Students’ achievement at the non-proportional tasks of area and volume is also differentiated, at a statistically significant level, in the experimental settings of Tests B and C. More specifically, students have greater success while dealing with the non-linear tasks of Tests B and C compared with the respective tasks of Test A. As far as the difference in students’ achievement at the non-linear tasks of area and volume is concerned, this is statistically significant and in favour of the area tasks, only in Test A (t =-2,99, p=0,003) and in none of the other two tests. Therefore, it seems that both interventions assisted the diminution of this difference.

	Problem	Test A	Test B	Test C	Test	t	p
Area	3	7%	32%	19%	A-B	-7,38	0,00*
					A-C	-3,51	0,01*
	5	13%	34%	24%	A-B	-4,86	0,00*
					A-C	-3,07	0,00*
Volume	1	6%	30%	19%	A-B	-6,78	0,00*
					A-C	-4,17	0,00*
	4	7%	31%	21%	A-B	-6,43	0,00*
					A-C	-4,51	0,00*
Lenght	2	90%	90%	93%	A-B	-0,43	0,67
					A-C	-1,32	0,19

*p< 0, 01

Table 1: Students’ percentages at Tests A, B and C

Figure 1 illustrates the similarity diagram of all variables (tasks) of Tests A and B. Students’ responses to the tasks are responsible for the formation of three clusters (i.e., groups of variables) of similarity. The two first groups consist of the same tasks (1, 3, 4 & 5), which represent the non-linear problems of area and volume of Tests A and B, respectively. The third group consists of the linear problems of Tests A and B, something quite natural since both tasks are the same.

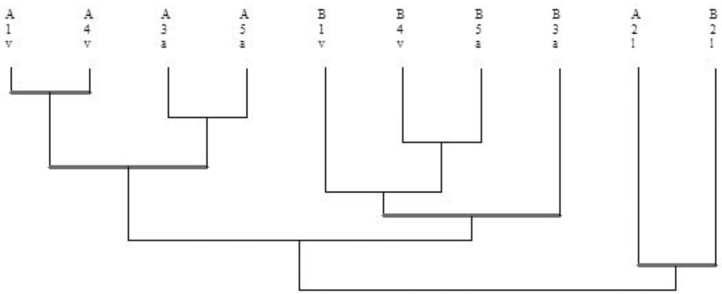

Figure 1: Similarity diagram of the variables of Tests A and B

Note: Similarities presented with bold lines are important at significant level 99%.

The first similarity cluster is formed by two distinct sub-groups of tasks that correspond to the problems of volume (1 & 4) and area (3 & 5) of Test A. The fact that the area and volume tasks are separated in Test A indicates that, prior to the intervention, students deal with these problems in a different way, without taking into consideration their common non-linear character.

The above tendency does not seem to apply to the non-proportional tasks of Test B, that constitute the second similarity cluster, since at this intervention all the problems of area and volume are mingled together. Consequently, students seem to realize that the tasks that are asked to deal with are not just problems of different mathematical content, but also problems that are characterised by the same phenomenon. That is, they realise the common non-linear character of the tasks despite the differentiation of the dimension number of the rectangular figures.

In Figure 2 all the similarity relations of the tasks of Tests' A and C are illustrated. As in Figure 1, three distinct similarity groups are formed. The first two groups consist of the same tasks (1, 3, 4 & 5), which represent the non-linear problems of area and volume of Tests A and C respectively, whereas the third group consists of the linear problems of Tests A and C.

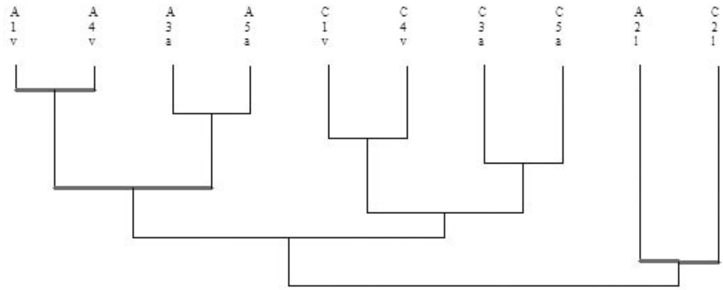

Figure 2: Similarity diagram of the variables of Tests A and C

In this figure, in contrast with the previous one, the first and the second similarity cluster, are formed by two distinct sub-groups of tasks that correspond to the non-proportional problems of volume (1 & 4) and area (3 & 5) of Tests A and C, respectively. The existence of this task separation indicates that students, prior and after the intervention, do not take into consideration the common non-linear character of the tasks and deal with them in a different way.

DISCUSSION

The results of our study reveal the great discrepancy in students' performance while dealing with linear and non-linear geometrical tasks. The existence of this difference was due to the mathematical errors students' made, while dealing with the non-linear tasks of area and volume, because of their tendency to see the linear function everywhere (Gagatsis & Kyriakides, 2000). One explanation could be that students, especially in Test A, failed to realise the multidimensional increase of the rectangular figure's size, since proportional reasoning appeared to be deeply rooted in students' intuitive knowledge. It could also be asserted that students' weak performance in non-proportional items may be merely the result of a superficial reading of the tasks that spontaneously connected them with a stereotype problem formulation, which is linked with proportional reasoning (De Bock et al., 2002b). Therefore, it seems that students matched the area and volume tasks with the primitive linear model, since the words double, triple etc., that were used in the problems, triggered the operation of a linear multiplication. Consequently, the operation of linear multiplication was applied to the two numbers embedded in the problem text and the result of the calculation was found, without referring back to the problem text to check for reasonableness (Greer, 1997). Another explanation could be their tendency to respond in line with the intuitive rule Same A – Same B, according to which the same change that occurs to the length will also occur in the area and volume.

Students' quite natural and spontaneous use of proportional reasoning appears to be questioned when Tests B and C were administrated. In particular, the cognitive conflict that both experimental settings promote seems to impel an examination of the appropriateness of the linear model for the solution of all tasks. Therefore, both interventions yield significant positive effects on students' performance on non-proportional items. However, these effects, as with other experimental manipulations, were disappointingly small and thus did not suffice to make the illusion of linearity disappear (De Bock et al., 2003).

Students' performance on non-proportional tasks was differentiated, depending on whether the task concerned the area or the volume of a rectangular figure, only at Test A. The experimental design of Test B succeeded in making students realize the common non-linear character of the area and volume tasks, despite the differentiation of the dimension number of the rectangular figures. The main reason for this is the fact that in Test B, all the students that overcame the illusion of linearity used the dimensions of the figures, which were given in the problems, in order to find the area or the volume of the figure. In particular, they first performed the multiplicative comparison with the dimensions of the figures and then used the formula for finding either the area or the volume of the figure. It must be noted that this was students' favoured method in most non-proportional tasks of all three tests, even if the dimensions of the figures were not included in Tests A and C. However, students in Tests A and C treated volume and area tasks differently, without understanding their uniformity.

In a subsequent research, the cognitive factors that prevent students from realising the common non-linear character of area and volume tasks can be investigated. Moreover, the results of a more systematic didactical intervention, concerning the non-proportional nature of geometrical tasks, can be evaluated with regard to their sufficiency to modify Cyprus's mathematics curriculum, so that this illusion of linearity be diminished.

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