CONSTRUCTING MEANINGS AND UTILITIES WITHIN ALGEBRAIC TASKS

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The Purposeful Algebraic Activity project aims to explore the potential of spreadsheets in the introduction to algebra and algebraic thinking. We discuss two sub-themes within the project: tracing the development of pupils' construction of meaning for variable from arithmetic-based activity, through use of spreadsheets, and into formal algebra, and tracing the ways in which children construct utilities for algebraic activity. Our analysis of pupils' activity suggests that tasks which offer opportunities to construct different utilities may also be associated with the construction of different meanings for variable.

INTRODUCTION

The *Purposeful Algebraic Activity* project[1] aims to explore the potential of spreadsheets in the introduction to algebra and algebraic thinking, making links to both the learning and teaching of arithmetic and the development of traditional school algebra. In this paper, we discuss two sub-themes within the project: tracing the development of pupils' construction of meaning for variable from arithmetic-based activity, through use of spreadsheets, and into formal algebra, and tracing the ways in which children construct *utilities* for algebraic activity: that is, an understanding of why and how this is useful (Ainley and Pratt, 2002). We focus on the key algebraic idea of *generational activity* (Kieran, 1996): expressing relationships in a general way through the use of a variable.

Through the focused use of the spreadsheet environment, and carefully designed pedagogic tasks which are purposeful for pupils, the project aims to create opportunities for pupils to not only develop the technical skills of working with formal notation to express relationships, and conceptual understanding of this activity, but also to construct utilities for algebraic activity. We identify two potential utilities: generating many examples (so that patterns can be seen more clearly), and showing structure. Our analysis of pupils' activity suggests that tasks which offer opportunities to construct these different utilities may also be associated with the construction of different meanings for formal notation and variable.

MEANINGS FOR VARIABLE IN THE SPREADSHEET ENVIRONMENT

The different meanings for variable which may be constructed by learners in the early stages of algebra have been explored and reported by many researchers. Limited

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space does not allow a lengthy discussion but we draw on Ursini and Trigueros' (2001) recent categorization as a way of articulating one distinction which has become apparent within our analysis.

In the algebra-like notation of the spreadsheet, the cell reference is used ambiguously to name both the physical location of a cell in a column and row, and the information that the cell may contain. The spreadsheet thus offers a strong visual image of the cell as a container into which numbers can be placed. The meaning for variable which this image seems likely to support is that of a *placeholder for general number* (Ursini and Trigueros, 2001), implying that pupils are able to:

> interpret a symbol as representing a general, indeterminate entity that can assume any value, and symbolise general statements, rules or methods (p. 336)

However, the image offered by the spreadsheet is ambiguous in another powerful way: when a formula is entered in a cell, it can be 'filled down' to operate on a range of cells in a column. The cell reference can then be seen as both specific (a particular number I may put in this cell) and general (all the values I may enter in this column). This image is likely to support the idea of variable as *a range of numbers in functional relationships*. Ursini and Trigueros (2001) associate this with (amongst others) the abilities to:

> determine the values of the dependent variable given the value of the independent one, and symbolise a functional relationship based on the analysis of the data of a problem. (p. 336-7)

Other features of the spreadsheet environment may offer opportunities for pupils to appreciate utilities of algebraic activity. The first is that its notation provides a 'language' which can mediate between pupils' natural language and formal algebraic notation (Sutherland, 1993). Meanings for 'spreadsheet language' develop during algebraic activity, alongside the use of natural language, to express ideas and relationships. The meaningful use of spreadsheet language may demonstrate the power of algebraic activity to allow the structure of a relationship to be easily seen. A second feature is that the use of spreadsheet notation has an immediate purpose in producing a result from which pupils can get meaningful feedback, such as a table of data which can be graphed. Pupils are thus able to appreciate the usefulness of algebraic activity to generate examples from which patterns can be seen.

These two features contrast sharply with more traditional approaches, where pupils may be required to translate their ideas into formal notation as the last stage of an activity. Here the purpose of expressing relationships in a formal notation may be unclear, and the only feedback accessible to the pupil may be the teacher's approval.

THE TEACHING PROGRAMME

Within the *Purposeful Algebraic Activity* Project we have developed a teaching programme of six tasks (used as three pairs) which incorporate different uses of the spreadsheet, and different algebraic ideas, within settings designed to have clear purposes for pupils. The tasks have been developed by the project team, in collaboration with a group of primary and secondary teachers. The primary teachers trialled initial versions of the tasks with their pupils (aged 10-11 years), and this experience was fed into the development of a more polished set of materials to be used by the secondary teachers. These were used as part of the normal curriculum with five classes in the first year of secondary schooling (aged 11-12 years, covering a range of attainment, in two schools).

The data presented in this paper is taken from work on two tasks: *Hundred Square* (task 2 in the programme) and *Mobile Phones* (task 4).

DATA COLLECTION

The secondary classes in schools worked on each task for 2-3 lessons. Unfortunately, the time available for these lessons was limited by timetabling and curriculum constraints, and many classes did not have enough time to complete the tasks as they were originally designed. All lessons were observed by a researcher, and four sources of data were collected:

- researcher's field notes, which included observations of pupils' working
- audio-recordings from a radio microphone worn by the teacher
- video and screen recordings from a targeted pair of pupils in each lesson
- examples of pupils' written work and spreadsheet files.

The audio and video recordings were transcribed. The transcripts of the pairs of children were annotated with observations from the video and screen recordings, and examples of the pupils' files saved during the lessons. All sources of data were then coded to identify examples of different kinds of generational, transformational and meta-level activity (Kieran, 1996), including different meanings for variable, and use of natural language, spreadsheet and formal notation to generate expressions. Codings were cross-checked amongst the project team, and sub-themes emerged and developed during the coding process.

ACTIVITY IN THE *HUNDRED SQUARE* TASK

Hundred Square involves exploring patterns with a 100 square (created on the spreadsheet) by taking 3x3 cross-shapes from within the square, and comparing the sums of values on the two arms of the cross (see example in Fig. 1). To make the exploration easier, pupils are asked to set up a 'testing cross' on their spreadsheet, and use formulae to create the complete cross once the middle number is entered. They are asked to explain the patterns that they find in their results, and then set a final challenge: to design some new shapes on the Hundred Square with interesting patterns.

In Judith’s class, most pupils managed to find some patterns in their results when adding the numbers on the horizontal and vertical arms of the cross, noticing that each gave the same total, and that this total was three times the number at the center of the cross. However, Louise was the only pupil prepared to try to explain this in the discussion at the start of the second lesson.

13	14	15	16
23	24	25	26
33	34	35	36
43	44	45	46
53	54	55	56

Figure 1

> Louise said that 16+10=26 and 16-10=6, explaining why the column total was three times the middle number. (*Field notes*)

With her partner Harriet, Louise was then comfortable with writing formulae to express this relationship on the spreadsheet, using *=B14-10* and *=B14+10* as the formulae above and below the middle cell (B14) in their ‘testing cross’. Later they went on to make a new shape, a cross covering five rows and columns. They confidently used the cell reference of the center cell as the starting point for their formulae, but had difficulty in deciding how this related to the cells at the far ends of their cross.

Harriet	M13 “minus a hundred
Louise	No ten
Harriet	No minus, minus a hundred
Louise	Minus, no minus two and then minus ten (*Pair transcript*)

They continued in this way for some time before producing the cross shown in Figure 2. This was correct on the horizontal arm, but the numbers which would appear in the vertical arm of the cross (4, 12, 14, 16, 24) clearly could not appear in those positions in the Hundred Square. Louise explained to Harriet that the vertical arm was correct, on the basis of the symmetry of the formulae. This cross does, indeed, ‘work’ in so far as the sums for the two arms are equal, and these are 5 times the central number.

		=M13-10		
		=M13-2		
=M13-2	*=M13-1*	14	*=M13+1*	*=M13+2*
		=M13+2		
		=M13+10		

Figure 2

Louise and Harriet’s conversation was all in terms of operations on the unknown central number in the cross, and they seem to be comfortable with using the cell reference as a placeholder for a general number. Other pupils tended to talk about particular values, but used similar arguments to explain the structure.

In Ann’s class, Elizabeth and Shannon also worked confidently, producing appropriate formulae, and explaining their findings in general terms.

Elizabeth	We found out that the formula was seventy six times three equals the column and the row number (looks at Researcher) (.). Well not seventy six, the middle number times three equals the column and the row number …And that is because if you take ten from the, one of the column numbers (points to the bottom cell) and put it on the other column number (points to the top cell) then they both equal seventy six (*Pair transcript*)

In Graham's class, some pupils offered similar descriptions of why the row and column totals are the same, for example:

Pupil	Basically they cancel each other out because
Graham	What cancels each other out?
Pupil	Well the top, the number above would be minus ten and the number below would be plus ten so they'd cancel each other out and then that one would be minus one and that would be plus one (*Teacher transcript*)

Although the language here is imprecise, it is clear that the pupils were describing the structure of a general pattern, which might apply to any 'middle number', just as Elizabeth recognised that her pattern would not just apply to seventy six.

The examples offered here of pupils' activity in the *Hundred Square* task suggest strongly that they are constructing a meaning for variable as a placeholder for any 'middle number' in the cross, and using this generalized number to symbolize general rules, whether expressed as spreadsheet formulae or in natural language.

We suggest that they are also constructing a utility for the use of generalized expressions of relationships, however they are expressed: that of showing structure. In this task we see pupils moving between explanations which are strongly rooted in the arithmetic and physical structure of the Hundred Square, generalized descriptions, and spreadsheet formulae which reflect the symmetry of that structure, even when in the case of Harriet and Louise, the formulae they produce do not actually match that arithmetic structure.

ACTIVITY IN THE *MOBILE PHONES* TASK

In *Mobile Phones*, pupils are presented with information about two different tariffs offered by a mobile phone company, together with the calltime someone uses each month for half a year. They are asked to set up a spreadsheet so that they can investigate which tariff offers the best value.

Tariff	Monthly Rental	Calls
Tariff A	£12.95	20p per minute
Tariff B	£14.50	15p per minute

November	15 minutes
December	48 minutes
January	80 minutes
February	44 minutes
March	113 minutes
April	63 minutes

They are then asked to investigate three more possible tariffs, and write a brief guide to say which tariffs would be most suitable for different types of users.

At the beginning of the *Mobile Phones* task Harriet and Louise realized quite quickly what they needed to do to calculate the cost for November for tariff A.

Harriet	No 'cause that's "monthly (points to £12.95 on the table on the worksheet), you have to pay that anyway and then … that, times, where's the times (enters '=A2')
Louise	Why do you need "times?

Harriet	You need times ‘cause you need to that (points to 15 minutes) “times twenty (*Pair transcript*)

After some difficulties with syntax, Harriet typed in the formula correctly. Although she typed *A2* for the calltime for November, she actually talked about the calculation as ‘fifteen times nought point 2’. Louise realized that this would not achieve what they wanted.

Louise	No, no, Harriet listen. If you do fifteen times twenty (she means 0.20) then it’ll only work for fifteen, it won’t work for forty-eight (the calltime for December)
Harriet	No, no I put A, A”2 (points to cell A2 then looks directly at Louise as if checking that she understands) times twenty
Louise	It will only work for A2
Harriet	No it “won’t, it will work for A3 as well (points to A3) (*Pair transcript*)

In this exchange it is clear that both girls realized that they needed to give a general expression for the total cost, which would work for every month. Throughout their discussion, Harriet seemed to be using the specific number of minutes (15) and the cell reference (A2) interchangeably, suggesting that she is thinking of this quantity as a variable. Louise knew that using 15 would not give them an expression which would work for all the cases, but did not immediately recognize the power of using the cell reference. However, she was quickly convinced when they filled the formula down and she saw the costs for each month calculated.

In Ann’s class, Max and Peter also struggled with the syntax involved in entering amounts of money, and despite having a sense of the structure of the calculation, made several errors in working out the cost for November. When they began on the calculation for December (48 minutes) Peter was clearly getting more confused.

Peter	I don’t get you
Max	(faces Peter and points to the worksheet throughout) That’s like your monthly line rental right, so you have to do, you have to pay that every month …And then say you used forty-eight minutes, that’s twenty p per minute so you have to times them to there plus twelve pounds ninety-five
Peter	Go on then tell me (sits back in chair)
Max	↑Do a ↑”formula [loudly] (palms face upwards then holds hands against forehead) (*Pair transcript*)

Max’s frustration continued for some time as they struggled with the syntax, getting answers which they recognized were incorrect. Eventually they tried to attract the teacher’s attention, but before she arrived, Peter looked across at the screen of the pupils sitting next to them.

Peter	(looking at the computer of pupils sitting next to them) Cool (.) ‘ere we didn’t get that answer, actually we didn’t get that answer. … What formula did you get for that (points) ‘cause we couldn’t get a formula (..) A2

Max	Oh yeah it's supposed to be A2 times
Peter	(looks at Max) You "stupid "[thing] you
Max	Yes I am stupid, we put twenty times fifteen (*Pair transcript*)

Although they had used cell references confidently in earlier tasks in the programme, Max and Peter had not immediately recognized the power of using them here. Max's explanation suggests that his attempts to enter a formula were based on a clear image of the general structure of the calculation. Once he realized that he had been 'stupid', Max insisted that he could quickly complete the rest of the task, saying '*I can get all these answers down in two seconds*'.

Again in this task we see pupils moving from arithmetic calculations, to generalized calculations of the dependent variable (the total monthly cost), which are often expressed in informal language. These functional relationships are then formally expressed as spreadsheet formulae which are replicated to generate sets of data, suggesting that pupils are constructing meanings for variable as a range of values.

Towards the end of the lesson, Peter and Max discussed their progress with the researcher.

Researcher	How you getting on?
Peter	Okay
Max	We've now realised what we're doing (laughs)
Researcher	Why did you decide to write a formula?
Peter	'Cause it's easier
Max	'Cause it's quicker (*Pair transcript*)

Like Harriet and Louise, Peter and Max could now clearly see the utility of expressing the calculation of monthly costs in a general way, so that they could generate a lot of data for each tariff quickly. Pupils across all five classes went on to use line graphs to compare Tariffs A and B, and most were able to make some links between the way in which these costs changed as calltime increased, and the cross-over points on the graph. Comparing the costs of further tariffs proved to be more challenging, and for all but a few pupils, time ran out before they were able to complete their users' guide to all the tariffs. However, many did make some recommendations like the following in their reports, indicating that they had appreciated the value of comparing sets of data rather than individual values.

> We have found out that if you spend a lot of time on your phone (1 and a half hours to 2 hours) you should go with Tariff C ... If you spend about (30 minutes to 1 and a half hours) on your phone then you should go with Tariff B ... if you spend a little amount of time on your phone (0 to 30 minutes) you should go with Tariff A.
>
> Tariff A is cheeper (*sic*) up to 30 minutes but when you get to 31 minutes Tariff A and B are equal, after that Tariff B is cheeper. (*Written reports*)

DISCUSSION

The snapshots of data presented here give an indication of the complex interactions between different elements in shaping the ways in which meanings are constructed in what may superficially appear to be similar activity on the part of pupils. In both tasks, pupils have a problem to solve involving generational activity, which takes the form of writing spreadsheet formulae. In the examples presented in this paper, we see the overall purpose of the task (understanding and explaining an intriguing pattern in order to design a new one, and comparing patterns of data produced by similar functions) as crucial in influencing the way in which the spreadsheet is used, and thus the meanings and utilities which may be constructed.

The teaching programme was designed around the ideas of different kinds of algebraic activity (generational, transformational and meta-level), opportunities for exploiting different features of the spreadsheet environment, and possibilities for pupils to move between arithmetic and algebraic structures, using natural language and informal notations, spreadsheet notation and formal algebraic notation. The sequence of tasks in the programme aims to combine these elements with different foci, in progressively more complex algebraic activity. However, as we have observed the use of the tasks by different teachers it has become apparent that subtle changes in emphasis by the teacher can lead to changes in the way that the purpose of the task is perceived. For example, an attempt to simplify the introduction to *Mobile Phones* may lead pupils to focus on calculating the cost for each month separately, rather than seeing the functional relationship between minutes used and total cost, so that the need for constructing a general expression using a variable is lost. As our longitudinal analysis of the teaching programme data continues, the ways in which the focus of attention within tasks may be shaped by teachers' and pupils' perceptions of purpose, and of the role of the spreadsheet, will be a significant theme.

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