CHARACTERISTICS OF MATHEMATICAL PROBLEM SOLVING TUTORING IN AN INFORMAL SETTING

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The study was carried out within the framework of a project that provided after-school mathematics tutorial sessions to 10th-12th grade students by engineers in an informal setting. The participating students were selected from advanced level mathematics classes according to their need for additional support. The main goal of the study was to characterize the special learning environment that evolved within this project, and to identify distinctive elements that enhanced students learning. In our paper we present three main caharcteristics associated with the problem solving activities in which the tutor and students engaged, and discuss their contributions to students' learning.

BACKGROUND

The study was carried out within the framework of a project the goal of which was to increase the number of high school students who continue to higher education in science and high-tech engineering. In Israel the requirements for acceptance to these fields of higher education include a mathematics matriculation exam at an advanced level. Thus, the project provided opportunities for senior high school students who were conditionally enrolled in advanced level mathematics classes to receive additional support by attending weekly after-school tutorial lessons. For both practical and principled reasons, these lessons were conducted by high-tech engineers, who had a rather sound mathematical background as well as extensive experience in team work. None of the tutors had any formal pedagogical education. Similar to a teacher, the tutor had an overall responsibility for the management of students learning in the tutorial sessions. However, he was not responsible for the content and specific problems on which to focus. These were dictated indirectly by the regular mathematics teacher. Moreover, unlike the task of a teacher, the tutor did not have to evaluate students' performance.

THEORETICAL PERSPECTIVES

Currently, there is rather scarce literature dealing with informal learning settings in mathematics. Thus, we rely to a certain extent on literature addressing informal learning settings in science. In addition, in order to make sense of the data and characterize our specific learning environment, we also consider literature pertaining to relevant features of learning environments in mathematics in more formal settings. In particular, we draw on some elements of the mathematics classroom that raise concern of mathematics educators.

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The potential contributions of informal learning environments. We consider any learning environment that is operated outside of school (either in a different location or after school-hours) an informal one. In science education, there is much evidence of the contributing affects of informal environments (e.g., Hofstein & Rosenfeld, 1996). The main contributions attributed to informal environments are: attentiveness to a diversity of learning styles; provision of an extended variety of types of learning experiences (e.g., authentic problem solving, spontaneous collaboration); access to different communities and opportunities to meet and interact with experts. Given the above contributions, one of the main concerns of science educators is how to combine informal and formal settings productively (Hofstein & Rosenfeld, 1996). Thus, we look at the focal informal learning environment as complimenting the ordinary mathematics classroom in school.

Concerns regarding formal learning settings in mathematics. Much has been written about formal learning environments in mathematics. From the teacher's perspective, there is often a considerable amount of tension between the desire to be flexible and attentive to students' needs and ideas, and the constraints posed by the school mathematics curriculum (Leikin & Dinur, 2003; Sherin, 2002). Teachers' knowledge - particularly, their subject-matter, pedagogical, and curricular knowledge - has bearing on their practice, in terms of their effectiveness and flexibility (Simon, 1995; Sherin, 2002; Leikin & Dinur, 2003). Teachers' overall practice and choices have impact on students' beliefs about the nature of mathematics (Schoenfeld, 1992; Lampert, 1990). Thus, the common practice leads many students to believe that, for example, a mathematical problem always has one right answer, and one correct or preferable way to solve it (which is usually the procedure most commonly demonstrated by the teacher). "… Changing students' ideas about what it means to know and do mathematics was in part a matter of creating a social situation that worked according to rules different from those that ordinarily pertain in classrooms, and in part respectfully challenging their assumptions about what knowing mathematics entails" (p. 59, Lampert, 1990). One way to deal with the above concerns is by promoting students' meaningful interactions – with one another, with the learning material, and with the teacher (Leikin & Zaslavsky, 1997). In general, classroom norms and social processes are closely related to students' learning (e.g., Yackel & Cobb, 1996; Simon, 1995). Following this approach, our analysis examines the norms and processes that characterized the investigated informal learning environment.

Teaching and learning mathematical problem solving. The activity in which students engaged in the after-school tutoring lessons concentrated mostly on mathematical problem solving. Schoenfeld points to one of the limitations and difficulties of teaching problem solving in school:

> "Part of the difficulty in teaching mathematical thinking skills is that we've gotten so good at them (especially when we teach elementary mathematics) that we don't have to think about them; we just do them, automatically. We know the right way to approach

most of the problems that will come up in class. But the students don't, and simply showing them the right way doesn't help them avoid all the wrong approaches they might try themselves. For that reason we have to unravel some of our thinking, so that they can follow it." (Schoenfeld, 1983, p. 8).

One of Schoenfeld's suggested ways to deal with the impediment of overly rehearsed teachers' problem solving strategies in the classroom is by creating genuine situations in which a teacher must solve a new problem "on the spot" (ibid). In the context of our study, this kind of situation was an integral part of the tutorial lessons, since the tutor was not an experienced mathematics teacher. His task was to help students solve mathematical problems which they found difficult, mostly without knowing in advance what the specific problems would be. Thus, this created an authentic context for investigation of possible ways of dealing with the need to unravel one's thinking.

THE STUDY

Goals: The main goal of the study was to identify the interplay between a number of distinctive features of our informal classroom learning environment and the nature of the problem solving activities that took place.

Participants: The participants in the study consisted of ten highly motivated 10^{th} grade students, who were provisionally placed in a class that studied mathematics at the most advanced level. In order to succeed in this top level mathematics class (that consisted of 30 students) they needed some extra support. Therefore, they attended the informal afternoon mathematics lessons. The principal participant was Dan, an engineer, who served as the students' mathematics tutor once a week in the afternoon throughout the school year. Galia was the participating students' regular mathematics teacher in the formal school setting, as well as the school coordinator of the project.

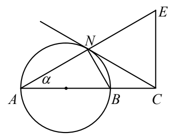
Data Collection and Analysis: Our research is an interpretive study of teaching that follows the qualitative research paradigm, based on thorough observational fieldwork, aiming to make sense and create meaning of a specific classroom culture (Erickson, 1986). In particular, we investigated "how the choices and actions of all the members constitute an enacted curriculum - a learning environment" (ibid, p. 129). Thirteen informal mathematics lessons with the above students were carefully observed and detailed protocols were written on-line for each of these lessons. Additionally, written feedback questionnaires were administered to the participating students at the end of the school year, and individual interviews were conducted with each student in the middle of the school year. The interviews focused on eliciting students' views of the characteristics and distinctive elements of the informal learning environment and the contribution they attributed to these elements in enhancing their mathematical knowledge and disposition. Two students were interviewed after the analysis was completed, in order to validate our interpretations. Dan was interviewed twice – once after his first tutorial lesson. We returned to Dan again after completing most of our analysis with a second interview, in order to validate our interpretations.

FINDINGS

Our perspectives developed in an inductive and iterative way. As the evidence and pieces of information accumulated, we began to notice several patterns that recurred, in terms of Dan's teaching style, his interactions with the students, and the classroom norms that developed. In this paper we focus on characteristics of Dan's teaching that were associated with the problem solving activities that constituted the core part of the lessons. Three main characteristics were identified: 1. Dan's ongoing efforts to attend to and understand students' ways of thinking; 2. Dan's readiness to expose himself to solving problems in real time; 3. Dan's tendency to attend to multiple approaches to solving a problem. We turn to a description of above three characteristics, interweaving some citations from observations and interviews with Dan and with his students. In addition, for the first characteristic we elaborate more by providing short excerpts from a lesson with Dan.

Dan continuously encouraged students to think and to express their ideas. When a student raised a suggestion, Dan went along with it, prompting the student to explain his thinking. As Dan gained understanding to the student's ideas, he made them accessible to the other students, by elaborating on the student's reasoning. We illustrate this characteristic in the excerpt below, taken from a lesson dealing with the following (textbook) problem:

> AB in the figure is the diameter of the circle; CN is a tangent, and $AC \perp CE$. Prove that □CEN is an isosceles triangle.
>
> Hint: construct the chord NB and mark □NAB by α.



After sketching the problem givens (including the chord NB) on the board, Dan turned to Omer who wanted to present his solution (of which he had thought at home). Omer's suggestion was not the simplest one, yet Dan went ahead with it all the way.

Omer: The angle α is equal to the angle □BNC, according to the theorem that an angle between a tangent and a chord is equal to the inscribed angle resting on the same arc.

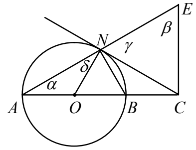
Smadar: Omer invented a new theorem!

Dan addressed Smadar's surprise regarding the theorem that Omer stated by dealing with the theorem separately, reminding the students the theorem's meaning and illustrating it. Then Dan turned back to Omer, who suggested connecting the center of the circle *O* with the tangent point *N*. At first, Dan did not understand why Omer's construction was necessary, thus, asked Omer to continue his explanation. Even though Omer's approach was more complicated than needed, Dan followed his reasoning attentively, throughout Omer's struggle to refine his approach (by modifying some calculations), until he finally reached a correct proof.

Omer: We got 180º-2α

Dan: How did you get 2α?

Omer: The radiuses are equal [AO=NO], so we get an isosceles triangle [thus in □AON α=δ]. So the adjacent angle [□ONB] is 90º-α, not 180º-α. [sketches an auxiliary figure with the relevant part for this stage]



We get …

[dictates to Dan an expression, while Dan writes it simultaneously on the board].

$\gamma = 180^{\circ}-\alpha- (90^{\circ}-\alpha) -\alpha$

Dan's attentiveness to students' thinking was a deliberate action he took. He attributed this tendency of his to the experience he had with tutoring his two daughters. He did not believe in disclosing the right answer, nor did he feel the need to "prove" that he knew mathematics. He preferred "to check if they [the students] were going in the right direction, [and if not, check] if it's a technical problem or a conceptual one". As he maintained: "I ask many questions, 'why did you solve it this way?', 'how did you solve it?', 'what did you or did you not understand?' in order to try to follow their thinking as much as possible". He continued: "There were cases in which all the students took part in solving a problem, and sometimes did it in a special way of which I had not thought".

Dan's students appreciated this approach, and brought it as an example of Dan's typical helpful behavior: "Dan lets us think for ourselves and reach a solution on our own or with his help"; "Dan follows our head, even if [we suggest] a long and complicated strategy, he puts himself in our place and when necessary corrects us"; "Thus, we learn to follow and check each others' ways of solving".

The second characteristic of Dan's teaching was exhibited by the way he felt comfortable to solve the mathematical problems in real time together with his students. The students brought to the lesson the problems with which they wanted to deal. These were homework problems with which they had encountered difficulties. When dealing with an unfamiliar and non-trivial problem, Dan made his thinking transparent to his students. He unraveled his thought processes by sharing with them each step, including his doubts and barriers in an apprenticeship like manner. He felt comfortable to turn to them for help when he felt stuck. This in return allowed them to become true contributing participants, leading to the emergence of a community engaging in collaborative problem solving activities.

We noticed that Dan used some key words to express his state when dealing with a challenging unfamiliar problem. The following words and phrases recurred: "I don't know", "wait, we need to think", "I think that this may be a direction", "let's try", "there must be another fact that I can't see here", "I'm beginning to see it", "please help me solve it, it's a tough exercise". In a stimulated recall interview with Dan he was asked to react to two excerpts in which we identified this mode of his. His reaction was: "Yes, that was a difficult problem. I didn't know how to solve it at first."; "Here I recall that the problem involved hyperbolas, which I did not remember, and they [the students] caught me unprepared. I didn't know I would have to solve with them problems in this topic. So I had to learn on my feet. This happened to me more than once throughout the year"….

In Dan's interview he was also asked to react to the way we analyzed his teaching, and particularly to what we referred to as his transparent problem solving. We also presented him with some excerpts from his tutorial lessons. At one point he said:

> "… I tried to be as transparent as I could, because this is a way to learn lots of things, such as, how to decompose a complex question into smaller parts, how to choose the right solution path among various alternative paths, and how to plan the different steps of a solution. Yet, it's impossible to explain exactly how you are thinking.… Overall I tried to convey to them the message that it is ok sometimes not to be able to solve a problem, and that it is perfectly ok to improvise. I have no regrets for acting in this way."

The third characteristic of Dan's teaching was reflected in his tendency to expose his students to multiple solution strategies and approaches to most of the problems with which they dealt. This was done in several ways. One way was to prompt students to think of additional ways, by words such as: "more ideas?", "is there another way?". These prompts encouraged students to suggest numerous directions, which Dan always carried further, even if they were incomplete or led to an impasse. Students felt comfortable to share there thoughts and make suggestions, even when they were not sure about their way. There were also several cases in which Dan suggested on his own an alternative way to solve a problem. This tendency led to discussions that focused on issues such as what is a (relatively) simple or complicated solution as well as preferences regarding aesthetic features of the suggested solutions.

In Dan's interview, he said: "I want to show them [the students] that usually there really is more than one solution and that they shouldn't worry if someone else in the class solved it differently. This doesn't necessarily mean that they are wrong. I want to reinforce their confidence that it is possible to solve in more than one way."

CONCLUDING REMARKS

The findings reported herein point to three typical complimenting and interrelated characteristics of Dan's teaching style with respect to problem solving activities. Interestingly, although Dan did not have any formal pedagogical education, these three characteristics are consistent with current trends in the mathematics education community. The first characteristic has to do with sensitivity to students and openness to their ideas, which is one of the three components of Jaworski's teaching triad (1992). Apparently, Dan developed this sensitivity through his experience with his daughters and colleagues at work. His attentiveness to students' needs was enhanced by his flexible state of mind, allowing the learning environment to evolve "as a result of interaction among the teacher and students as the engage in the mathematical content" (Simon, 1995, p. 133). This characteristic of Dan, together with the fact that he did not have to assess student's achievement, reduced their anxiety and allowed them to express their ideas more openly. The second characteristic is related to Schoenfeld's (1983) concern that teachers hardly ever unravel their authentic thinking processes. In Dan's case, he intentionally and confidently chose to model transparent problem solving. This often involved situations that experienced teachers would probably consider embarrassing (Leikin & Dinur, 2003). Although the special conditions of the tutorial lessons encouraged his tendency to unravel his thinking, Dan could have avoided such situations by maintaining direct contact with Galia, the students' math teacher, and preparing in advance for each lesson. However, borrowing from his experience at work, Dan was accustomed to facing problems that he could not solve instantly and felt confident enough to face similar problems with his students. Students accepted and respected this position of the tutor, without lessening their appreciation of him. We assume that their expectations of their math teachers would be quite different. Nonetheless, the positive impact of this characteristic on the learning that took place reinforces Schoenfeld's (1983) assertion that teachers should occasionally deliberately exhibit genuine problem solving situations in the classroom. The third characteristic, namely, exposure of students to numerous approaches to problem solving, is significant to developing mathematical competence (Ma, 1999; NCTM, 2000). Dan's students appreciated this opportunity, maintaining that their teachers do not have the "luxury" of devoting so much time to each problem. According to the students, their math teacher has the responsibility to cover the curriculum, and needs to do it "efficiently" within the many constraints she faces. As mentioned above, the three characteristics are interrelated. We believe that by personally encountering difficulties in solving some of the mathematical problems, Dan became more aware and sensitive to students' similar experiences. By thinking on his feet and sharing his difficulties with his students, Dan contributed to the evolving classroom norms that included collaborative efforts, the legitimacy of the students to err, and their mutual responsibility to check their own and each others ideas and (Sfard, 1998; Bransford et al., 2000). Attending to students' diverse ways of thinking naturally lends itself to multiple problem solving perspectives and approaches. The fact that both the tutor and the students thought on the spot and suggested half baked ideas, contributed to the search for alternative solution methods that were simpler, more efficient, or better understood.

Clearly, Dan's lessons complimented the regular math lessons but could not replace them. The different kinds of learning experiences that were fostered in the (informal) tutorial lessons were possible only because of their connection to the learning that took place in the (formal) classroom setting. In this sense, our study provides evidence of the possible fruitful interplay between formal and informal learning environments.

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