DIFFERENTIATION OF STUDENTS' REASONING ON LINEAR AND QUADRATIC GEOMETRIC NUMBER PATTERNS

Fou-Lai Lin & Kai-Lin Yang

National Taiwan Normal University

There are two purposes in this study. One is to compare how 7th and 8th graders reason on linear and quadratic geometric number patterns when they have not learnt this kind of tasks in school. The other is to explore the hierarchical relations among the four components of reasoning on geometric number patterns: understanding, generalizing, symbolizing, and checking, and to differentiate them between linear and quadratic geometric number patterns. From the national survey results, we argue that reasoning on geometric number patterns is a proper initial activity for learning algebraic thinking in Grade 7, and the relations between the checking component and the other components appear to be different between linear and quadratic patterns. Therefore, we propose that checking can play two kinds of role in reasoning on geometric number patterns. One is to induce a strategy for generalizing, and the other is to initiate the development of symbolizing after it is integrated with generalizing.

INTRODUCTION

During recent years, more emphasis has shifted from computational skills to effective reasoning about quantitative and qualitative relationships in school mathematics curricula (Thompson & Thompson, 1995; NCTM, 2000). The change in emphasis has contributed to a renewed interest in the teaching and learning of algebra. Pattern generalization is just one principal trend of current research and curriculum development of school algebra. Many studies have also suggested that recognizing, experiencing, expressing, generalizing and symbolizing of functional relationships establish a foundation for algebraic thinking and a precursor to formal algebra (Bednarz, Kieran & Lee, 1996; Orton, 1999).

However, algebra in Taiwan curriculum mainly demonstrated the function of generalized arithmetic and provided a vehicle for solving word problems. Students learnt patterns from number series and the rule of judging whether a number is a given multiple and learnt algebra from solving equations or word problems. Therefore, exploration of geometric patterns does not always stand its own as a curricular topic or activity in Taiwan. Although there is currently a significant mathematics curriculum innovation under way in Taiwan, 'A draft plan of nigh-year joint mathematics curriculum guidelines (Taiwan Ministry of Education, 2003)' also highlights recognition of regulations, algorithms of number series and symbolic expressions of relationships between patterns. The processes of generalization and symbolization, which incorporate exploring and searching for geometric number patterns, and explaining patterns verbally or diagrammatically still remain neglected.

On the other hand, Bishop had proposed a developmental sequence from the concrete, the recursive, to the functional category; however, the status of proportional

Proceedings of the 28th Conference of the International
Group for the Psychology of Mathematics Education, 2004   Vol 4 pp 457–464

Vol 4–30

category and reasoning on non-linear geometric patterns still required further research (Bishop, 2000). Taking into account that reasoning strategies may be influenced by different components (generalization or symbolization) and structures (linear or non-linear) which compose of different learning activities, we converted into exploring hierarchical relations among different components of reasoning on geometric number patterns: understanding, generalizing, symbolizing, and checking. These relations may be essential and illuminating when we investigate how to improve and evaluate children's learning in this area.

In summary, there are two purposes in this study. One is to compare how 7th and 8th graders reason on linear and quadratic geometric number patterns when they have not learnt this kind of tasks in school. The other is to explore hierarchical relations among the four components of reasoning on geometric number patterns.

DESIGN OF THIS STUDY

Ongoing projects on the development of mathematical argumentation in England and in Taiwan are conducted bilaterally. In Taiwan, the instruments were adapted from England and modified based on Taiwanese students' responses. In addition, some new items were included. The six booklets comprised questions in two domains of mathematics — Algebra and Geometry with respect to grade 7, 8 and 9. Not only the coding systems but also some conjectures as to the relations among the four components were formulated from a pilot study. Herein, we mainly report students' reasoning on geometric number patterns in this paper, part in algebra domain, and their reasoning on statements about number patterns will be written in another article.

Number Pattern Items

Table 1 showed our components of reasoning on number patterns. The patterns are labeled linear or quadratic because their nth terms can be expressed as $an+b$ or an^2+bn+c (Stacey, 1989). When presented with a sequence of configurations of dots or a figural pattern, students were expected to predict the number of dots or a sub-figural pattern for the fifth, twentieth and nth picture and to check if a given number can represent some term in the sequence or sub-figure.

After identifying the four types of reasoning on number patterns, exemplary items are presented as Fig 1. and Fig 2. Question A1 and A2 are concerned with generalization in a setting (tile patterns) familiar to English students (Kuchemann & Hoyles, 2001) but unfamiliar to Taiwan students. An approach to seeing a pattern is suggested only in quadratic geometric number patterns (e.g. A2-a), and students are required to predict the number of dots for the forth (e.g. A2-ai) or fifth picture in the understanding task. The approach provides a hint that the relation between the number of terms and the number of dots within each pattern is the focus. It is no doubt that understanding the meaning of the task is necessary before generalizing, symbolizing or checking the sequence of patterns. In particular, we provide the checking items, A1(c) and A2(d), in addition to the items in English study. Students' responses to the generalizing and checking items were respectively coded into 6 categories (Table 2).

The coding system is similar to the English system, but we are interested in whether patterns students *see* or *use* are *improper*, *useful but incomplete*, or *complete*. Therefore, we would be able to find the differences among seeing, recording or using a pattern for students respectively via the generalizing, symbolizing or checking items.

Number Pattern Subject	Linear Geometric				Quadratic Geometric			
	U	G	S	C	U	G	S	C
Grade 7	—	A1(a)	—	A1(c)	A2(a)	A2(b)	—	—
Grade 8	—	A1(a)	A1(b)	A1(c)	A2(ai)	A2(aii)	A2(c)	A2(d)

U, G, S and C denote understanding, generalizing, symbolizing and checking respectively.

Table 1. Four components of reasoning on number patterns.

(A1) Larry has some white rectangular tiles and some gray square tiles. The white tiles are twice as long as the gray tiles but have the same width.

He makes a row of white tiles, like this:

He then builds a '⊓' frame of gray tiles over

the white tiles, like this
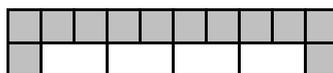
(a) How many gray tiles does he need to build a '⊓' frame over a row of 40 white tiles? Explain your answer.

(b) Write an expression for the number of gray tiles needed for a row of n white tiles.

(c) Can 195 gray tiles be built a '⊓' frame over a row of some white tiles?

Fig. 1. Question A1 in Grade 8

(A2) Karen and Josie are looking at these first four patterns in a sequence of dot patterns:

(a) Karen wants to calculate the number of dots in the 4th and 20th pattern. She says each pattern looks like a square with lacking one corner.

(i) the 4th pattern

(ii) the 20th pattern

1st pattern   2nd pattern   3rd pattern   4th pattern

(c) Write an expression for the number of sots in the nth pattern, using

(i) Karen's way of looking at the pattern.

(d) Do 9999 dots fit into this pattern?

Question (b) is similar to question (a) but provides another approach which did not present in this paper.

Fig. 2. Question A2 in Grade 8

Code	Key character of response	Exemplary response (Item)
1	Seeing(Using) an improper pattern	12x10(A1-a); 24x5(A2-aii); It can be divided by 3(A1-c).
2	Seeing(Using) some useful but incomplete pattern or only with correct result	40x2(A1-a); 20x20-1(A2-aii); odds numbers are impossible (A1-c).
3	Seeing(Using) a complete pattern only with correct arithmetic or photo-picture(manipulation)	84(A1-a); 21x21-1(A2-aii); No matter how you arrange, However you arrange, 195 is impossible (A1-c).
4	Seeing(Using) a complete pattern with correct result and verbal explanation	The dot number of length is 1 more than the corresponding term (A2-aii); 191(195-4) can not be divided into two equal parts(A1-c).
5	Seeing(Using) a complete pattern towards correct algebraic strategies	$(n+1)^2-1$(A2-aii); $(n+1)^2-1=9999$(A2-d)
6	Not showing to see(use) any pattern	4+20(A2-aii); Misunderstand 9999 as the 9999th figure (A2-d); or no response.

Table 2. Response code for the generalizing and the checking tasks.

Sample and Administration

This survey, which was to be completed in 45 minutes, was administered to 1,181 seventh graders, 1,105 eighth graders and 1,059 ninth graders. The subjects were nationally sampled by means of a two-stage sampling. The first stage was to divide our nation into six regions, and to randomly sample schools from each region. The second stage was to equally distribute these classes of sampled schools into 13 groups. Two of the thirteen groups were used as samples in our project. Half the sampled students in each class answered the booklet in Algebra(A), and the others answered the booklet in Geometry(G) according to their grade.

RESULTS WITH DISCUSSION

7th and 8th Graders' Reasoning on Geometric Number Patterns

We first compare 7th and 8th graders' spontaneous reasoning on geometric number patterns with respect to generalizing and checking, linear and quadratic. Table 3 shows the distribution of their responses to the generalizing items. While *generalizing* the *linear* geometric number patterns, 35.4% of Grade 7 and 52.7% of Grade 8 could answer correctly. But 30.3% of Grade 7 and 14.3% of Grade 8 incorrectly answered with the proportional reasoning strategy as English students did (Kuchemann & Hoyles, 2001). While *generalizing* the *quadratic* geometric number patterns, 36.3% of Grade 7 and 64.3% of Grade 8 could answer correctly. In particular, 8.9% of Grade 7 seeing an improper pattern with focusing on one dimension only, e.g. the number of

rows, columns or diagonal dots and misused partial information. However, reasoning on geometric number patterns is suggested to be the initial activity for learning algebraic thinking in Grade 7 because above one third of the 7th or 8th graders could correctly generalize linear and quadratic geometric number patterns unfamiliar to them. More 7th or 8th graders gave the correct answer while generalizing in quadratic than in linear geometric number pattern. It may result from that we provided an approach in quadratic geometric number pattern or that more students were attracted to the proportional relation between gray and white tiles in linear geometric number pattern.

Grade / Structure of Geometric Pattern (Percentage) / Generalizing	Grade 7 (N=1181)		Grade 8 (N=1105)	
	Linear	Quadratic	Linear	Quadratic
Seeing an improper pattern	30.3	11.4	14.3	1.4
Seeing some useful but incomplete pattern or only with correct result	7.2	10.3	7.1	13.2
Seeing a complete pattern only with correct arithmetic or photo-picture	34.9	29.8	45.0	50.1
Seeing a complete pattern with correct result and verbal explanation	0.4	6.4	7.1	14.1
Seeing a complete pattern towards correct algebraic strategies	0.1	0.1	0.6	0.1
Not showing to see any pattern	27.0	42.1	26.0	21.1

Table 3. Distribution of students' responses to the generalizing items

Grade / Structure of Geometric Pattern (Percentage) / Checking	Grade 7 (N=1181)	Grade 8 (N=1105)	
	Linear	Linear	Quadratic
Using an improper pattern	38.4	6.9	1.4
Using some useful but incomplete pattern or only with correct result	11.9	19.3	19.2
Using a complete pattern only with correct arithmetic or photo-picture	15.8	22.9	23.0
Using a complete pattern with correct result and verbal explanation	3.2	1.8	2.1
Using a complete pattern towards correct algebraic strategies	1.4	11.3	11.2
Not showing to use any pattern	29.2	37.9	43.0

Table 4. Distribution of students' responses to the checking items

Table 4 shows the distribution of 7th and 8th graders' responses to the checking items. While *checking* the *linear* geometric number patterns, 32.3% of Grade 7 and 55.3% of Grade 8 could at least use some useful but incomplete pattern to check. But there were 24.6% of Grade 7 and 5.6% of Grade 8 who used an improper pattern with the proportional reasoning strategy (e.g. It can be divided by 3.). While *checking* the *quadratic* geometric number patterns, 36.3% of Grade 8 could answer correctly. Although 8th graders better generalized and checked than 7th graders in general, we draw attention to that the percentage of the response of not showing to see or use any pattern in linear geometric number patterns did not decrease as the grade. Therefore, it is needed to study on whether parts of students do or do not progress after one year, and further on why they do or do not progress.

Hierarchical Relations among the Four Components

In the following, we further investigated the hierarchical relations among the four components of reasoning on number patterns and differentiated them between linear and quadratic geometric number patterns.

Understanding / Generalizing	Grade 7 Correct	Grade 7 Incorrect	Grade 8 Correct	Grade 8 Incorrect
Correct	33.9%	2.5%	57.9%	0.7%
Incorrect	35.4%	28.3%	12.1%	29.3%

Table 5. Understanding and generalizing the quadratic geometric number patterns

(Grade 8) Symbolizing	Understanding Correct	Understanding Incorrect	Generalizing Correct	Generalizing Incorrect
Correct	36.2%	1.3%	35.3%	2.2%
Incorrect	33.8%	28.7%	23.3%	39.2%

Table 6. Symbolizing, understanding and generalizing the quadratic geometric number patterns

Generalizing / Symbolizing	Correct	Incorrect
Correct	46.5%	2.9%
Incorrect	6.5%	44.1%

Table 7. Symbolizing and generalizing the linear geometric number pattern

In the *quadratic* geometric number patterns, Table 5 shows that most 7th and 8th graders who correctly generalized this pattern also correctly understood it. Table 6 shows that most 8th graders who correctly symbolized this pattern also correctly understood (or generalized) it. In the *linear* geometric number patterns, Table 7 shows that most 8th graders who correctly symbolized this pattern also correctly generalized

it. The MacNemar's test result ($\chi^2 = 14.6$, N=1105, p<0.001) suggests that the frequencies of different responses between generalizing and symbolizing linear geometric number patterns (Table 7) are significantly different. The results of Table 5 to Table 7 seem to sustain that a hierarchy proceeds from understanding, generalizing to symbolizing linear or geometric number patterns.

Checking \ Generalizing	Grade 7		Grade 8	
	Correct	Incorrect	Correct	Incorrect
Correct	18.7%	8.4%	38.5%	10.3%
Incorrect	16.5%	56.4%	14.2%	37.0%

Table 8. Checking and generalizing the linear geometric number pattern

In the *linear* geometric number patterns, Table 8 shows that the percentage of students who correctly checked and incorrectly generalized the same pattern was more than 8%. After further analyzing the responses, most of them used a useful but incomplete pattern to get the correct answer, but this strategy is insufficient to generalize correctly. After combining the above results and the result of Table 8, we diagram a hierarchy as follows and conjecture that using a pattern to check may be helpful to inducing a strategy for seeing this pattern.
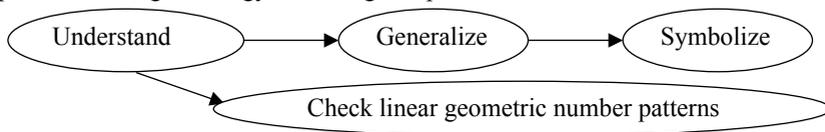
Symbolizing \ Checking	Correct	Incorrect
Correct	21.4%	16.0%
Incorrect	14.8%	47.8%

Table 9. Symbolizing and checking the quadratic geometric number pattern

Checking \ Generalizing	Correct	Incorrect
Correct	30.7%	5.6%
Incorrect	28.0%	35.8%

Table 10. Checking and generalizing the quadratic geometric number pattern

In the *quadratic* geometric number patterns, Table 9 shows that above 40% (16.0/37.4) of 8th graders who correctly symbolized the pattern were unable to correctly check it. However, Table 10 shows that about 85% (30.7/36.3) of 8th graders who correctly checked the pattern also correctly generalized it. After combining the above results and the results of Table 9 and Table 10, we diagram the

hierarchical relation the four components as follows and conjecture that using a pattern, while integrated with seeing this pattern, may initiate the development of recording it.
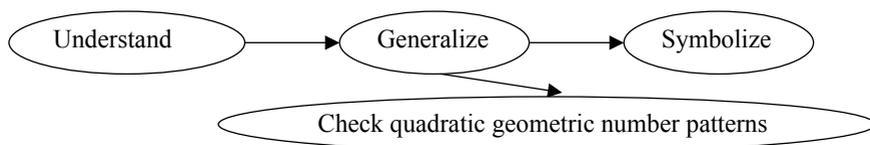
SUMMARY

From the national survey, we argue that reasoning on geometric number patterns is a proper initial activity for learning algebraic thinking in Grade 7. The checking component appears to be different between linear and quadratic patterns. Therefore, we propose that checking can play two kinds of role in reasoning on geometric number patterns. One is to find out a strategy for generalizing, and the other is to initiate the development of symbolizing after it is integrated with generalizing.

References:

Bednarz, N., Kieran, C. & Lee, L. (1996). *Approaches to Algebra*. Dordrecht/Boston/London: Kluwer Academic Publishers.

Bishop, J. (2000). Linear geometric number patterns: Middle school students' strategies. *Mathematical Education Research journal, 12(2)*, 107-126.

Kuchemann, D. & Hoyles, C. (2001). Investigating factors that influence students' mathematical reasoning. *PME25, 3,* 85-92.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Orton,A.(1999). *Pattern in the teaching and learning of mathematics.* London & New York: Cassell Wellington House.

Stacey, K. (1989). Finding and using patterns in linear generalizing problems, *Educational Studies in Mathematics, 20*, 147-164.

Taiwan Ministry of Education (2003). *A draft plan of nine-year-joint mathematics curriculum guidelines.* Taiwan: Author. (In Chinese)

Thompson, A. G. & Thompson, P. W. (1995). A cognitive perspective on the mathematical preparation of teachers: The case of algebra. In C. B. Lacampagne, W. Blair & J. Kaput (Eds.). *The algebra initiative colloquium, 1,* 95-116. Washington, DC: US Department of Education.