PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' SOLUTION STRATEGIES AND REASONING FOR A MISSING VALUE PROPORTION TASK

Jane-Jane Lo

Western Michigan University

This paper adds to the research studies on prospective elementary school teachers' mathematics knowledge with findings on proportional reasoning. In addition to analyzing prospective elementary school teachers' solution strategies for solving a missing value proportion task, it also examines the nature of the drawings and writing they used to explain their strategies.

Previous studies revealed that both prospective and practicing U.S. elementary school teachers possessed limited knowledge of mathematics, including the mathematics they taught (Ball, 1990; Ma, 1999; Post, Harel, Behr & Lesh, 1991; Simon, 1993, Simon & Blume, 1994). There is a growing consensus that the development of the "profound understanding of fundamental mathematics" (Ma, 1991) takes a long time and should be considered as a part of professional continuum that begins with the mathematics courses specifically designed for prospective teachers.

Even though conventional wisdom suggests that the mathematics knowledge possessed by the teachers plays a major role in shaping their instructional practices, there has been very little evidence that support a strong link between the number of mathematics courses teachers took and their student achievement (National Research Council [NRC], 2001). This result suggests the need to pay careful attention to both the content and instructional approach of these courses.

To address this issue, the American Mathematical Society and the Mathematical Society of American jointly put forth recommendations for the mathematical preparation of teachers in a book titled "The Mathematical Education of Teachers" (Conference Board of the Mathematical Sciences [CBMS], 2001). Furthermore, the book's authors point out the need for more detailed descriptions of prospective teachers' mathematical thinking, including their alternative conceptions, problem solving approaches and reasoning. Such information is needed to enable mathematics faculty who teach mathematics courses for prospective teachers to facilitate their students' development of mathematical proficiency more effectively. The current paper helps to meet this need by describing prospective elementary school teacher's reasoning and justification within the context of a missing value proportion task.

Proportional reasoning plays an important role in school mathematics (NCTM, 2000). When solving ratio, rate and proportion problems, students draw upon their concepts of multiplication and division of whole and rational numbers which are all part of the multiplicative conceptual field that is fundamental to upper elementary and middle

Proceedings of the 28th Conference of the International
Group for the Psychology of Mathematics Education, 2004 Vol 3 pp 265–272

Vol 3–18

school mathematics (Thompson & Saldanha, 2003).

Prior research studies on proportional reasoning have identified k-12 students' errors and difficulties, as well as the kinds of informal strategies they use in solving proportion tasks. Kaput & West (1994) identified three informal reasoning strategies commonly observed in solving missing value proportional tasks: 1) coordinated build-up/build down processes, 2) abbreviated build-up/build down processes, and 3) a unit factor approach. They suggested that the informal approaches were based on relationship that emerged naturally from given situations. The strategies defined in these previous studies might not translate to problem-solving methods used by adults. This paper seeks to address the following questions: Will prospective elementary school teachers be able to come up with reasoning strategies similar to those identified by the previous studies when they were encouraged to do so? What types of strategies will they develop? These are questions that the current paper seeks to address.

Furthermore, this paper analyzes the drawings and writings prospective elementary school teachers used to explain their solution strategies. Many researchers studied children's mathematical thinking have observed the spontaneous use of pictures or diagrams by students encountering problematic situations embedded in rich contexts. However, Diezmann and English (2001) also noted several difficulties students have in using pictures or diagrams. They point out the need for teachers to actively facilitate the development of "diagram literacy" among their students, which suggests that college mathematics courses should provide prospective elementary school teachers with opportunities to develop such literacy themselves. Another major question this paper seeks to address is: Given a missing value proportion task embedded in rich context, what types of drawings will prospective elementary school teachers develop to explain and justify the meaning behind their computation steps?

METHODOLOGY

The data reported in this paper is part of a research project that aimed at identifying the types of solution strategies, mathematical reasoning and justification prospective elementary school teachers developed as they participated in a specially-designed mathematics course focusing on numbers and operations. In this course, students were provided with opportunities to develop multiple methods to solve problems, to explain their approaches and reasoning to other students, worked toward understanding reasoning different from their own, to evaluate the efficiency of different strategies, and to make connections between different representations.

The concepts of ratio and proportion were discussed at the middle of the course through comparative ratio problems and a distance-time-speed problem, after instructional units on meanings and models of multiplication and division, and the concepts of factors, LCM and GCD. The technique of cross-and-multiply was strongly discouraged because students tend to carry out this method as a rote procedure that requires little proportional reasoning (Kaput & West, 1994).

Following the unit on ratio and proportion, students worked with problems on fractions, decimals and percents. The connections between these concepts and proportionality were highlighted continually throughout the units.

This paper focused on the analyses of thirty-six written solutions to one proportional task from students' final exam papers. The statement of the problem, which we call *Painting Walls*, is as follows:

> *If it took Jane 3/4 hour to paint a wall that was 12 ft by 12 ft, how long will it take to paint another wall that is 15 ft by 16 ft?*
>
> *a) Draw pictures to illustrate the situation in the problem.*
>
> *b) Explain how you can use your pictures to solve this problem.*

At a very basic level, *Painting Walls* can be classified as a missing value proportion. However, the fraction quantities, time conversion, and length and area measurements added layers of complexity for students.–Although students used the area model frequently throughout the semester to convey the meaning of multiplication and as a setting to explore LCM and GCD, it was not used when discussing proportionality in this course. Also, students had not worked on a missing value proportion task involving fractions. Therefore, *Painting Walls* had the potential to assess prospective elementary school students' ability to reason and to make connections among various mathematics topics they had discussed throughout the course.

The analysis of the data was guided initially by the following questions.

1. What types of solution strategies and reasoning did prospective elementary school teachers use to solve this particular missing-value proportional problem?

2. What kinds of drawings did prospective elementary school teachers use to justify their computation steps? Were they able to provide drawings and written explanations that supported their computation steps? If not, what seemed to be the difficulties?

Strategies with conceptual structures similar to what Kaput and West described were identified first. New categories and subcategories were formed to account for additional variations. Next, each strategy was classified based on whether or not the drawings supported the computation steps or not. Finally, descriptions were created to illustrate different types of drawings and to highlight the descrepencies between the computation steps and the drawings/written explanations.

SUMMARY OF RESULTS

In all, twenty-two students (61%) solved the problem correctly and nine additional students (25%) had a valid strategy to solve the problem but mistakenly stated the additional time instead of the total time or made other minor computation errors. From the subset of thirty-one correct or partially correct papers, six different solution strategies were identified (Table 1).

Strategy 1a: Equivalent ratio/multiplicative factor approach (Using 144 sq. ft. to 45 min. as unit ratio)	4 (13%)

Strategy 1b: Equivalent ratio/multiplicative factor approach (Using 48 sq. ft. to 15 min. as unit ratio)	11(35%)
Strategy 1c: Equivalent ratio/multiplicative factor approach (Using 12 sq. ft. to 3.75 min. as unit ratio)	3(10%)
Strategy 2: Per min. approach	8(26%)
Strategy 3: Per sq. ft. approach	2(6%)
Strategy 4: Ratio table approach	3(10%)

Table 1: Percentage of students used each strategy

Conceptually, strategy 1a, 1b and 1c are all similar to what Kaput and West (1994) identified as "abbreviated build-up/build-down process." Yet, there were three different "unit ratios" students used to build-up. The following is one example of student drawing and reasoning using Strategy 1b.
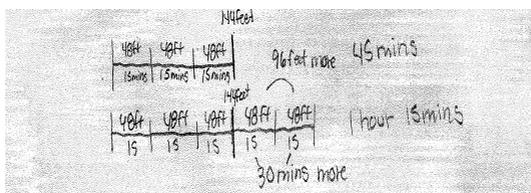
We know that each of the walls have 144 feet. The second has 144 +96 =240 feet. We divide 144 by 3 cause it takes 3/4 of an hour to paint. So it takes 15 mins for every 48 ft. So 96 feet more on the second wall is 48+48 = 96. So 30 more min. was spent on the second wall.

Figure 1. An example of Strategy 1b.

This student first reasoned proportionally to figure out the unit ratio 48 sq. ft. for 15 min. Then she figured out the amount of time needed for painting the extra area, the 96 sq. ft., was two of 48 sq. ft. Thus, she would need two 15 min. more to paint the second wall.

Students who used Strategy 2 (Per Min.) and Strategy 3 (Per Sq. Ft.) approaches formed a new rate quantity, either 3.2 ft per min. or 0. 3125 min. per ft. West and Kaput considered these two strategies conceptually similar and called them both "unit factor approach." However, to justify them with pictures required totally different drawings. For example, a 12 by 12 grid embeded in a 15 by 16 grid could only be used to justify the Per Sq. Ft. approach but not the Per Min. approach. Yet three students who used the Per Min. approach drew this picture and referred to each 1by1 cell as one min.

The ratio table approach was used by three students and was well-grounded in the context and the numerical relationship of the problem. All three students created

drawing similar to the following one (Figure 2). These students subdivided the extra area into three regions: 4 by 12, 3 by 12 and 3 by 4. Then they used the proportional reasoning to figure out the amount of time needed for smaller regions based on "45 min. for 12 by 12" For example, one student reasoned that 4 by 12 would require 15 min. because 4 is 1/3 of 12 (and 15 is 1/3 of 45). She then noticed that 3 by 12 and 3 by 4 would add up to the total area of 48 sq. ft., the same as the 4 by 12. Thus another 15 min. will be needed to paint the remaining region. The students' use of several proportionalities involving fraction relationships and their flexibly switching between the linear and area measurement made this a unique and rich approach worth noting.
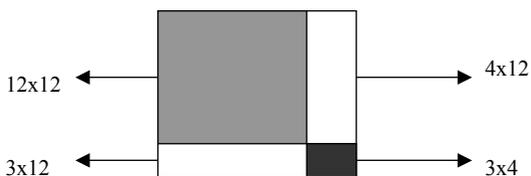

Figure 2. An example of a drawing accompanying the Ratio Table approach.

Even though a relatively high percentage of the students were able to come up with valid reasoning strategies to find the numerical answers to the *Painting Wall* task, only sixteen students were able to draw meaningful pictures to explain their computation. The following table shows the percent of students <u>within</u> each strategy who were able to justify with their reasoning with pictures successfully.

Strategy 1a	Strategy 1b	Strategy 1c	Strategy 2	Strategy 3	Strategy 4
0%	82%	67%	13%	50%	100%

Table 2: Percentages of successful pictorial justification within each strategy

The most common drawing that failed to explain the computational steps looked like the one in Figure 3. Two separate rectangles were drawn with relevant information noted but with no additional mark on the two rectangles to help connect the problem context to the computation steps. This kind of picture illustrated the initial static status of the given problem, but did not show how this particular image could facilitate the conceptual process needed to solve the problem.
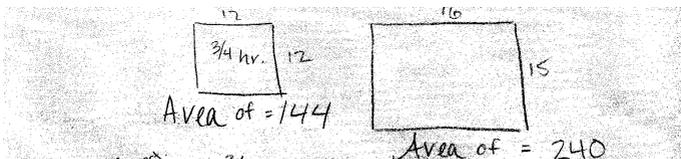

Figure 3. The most common unsuccessful drawing.

Finally, among incorrect solutions, the most common error showed the inability to coordinate the linear and area measures as noted by Simon and Blume (1994). The following is such an example (Figrue 4). Instead of conceiving 12 by 12 multliplicatively, this student came to the conclusion that 4 by 4 would be 1/3 as large as 12 by 12, thus requiring 1/3 the amount of time, or 1/4 hour. She then reasoned that 15 by 16 was very close to 16 by 16, so it would require approximately one hour to paint a 15 by 16 wall.
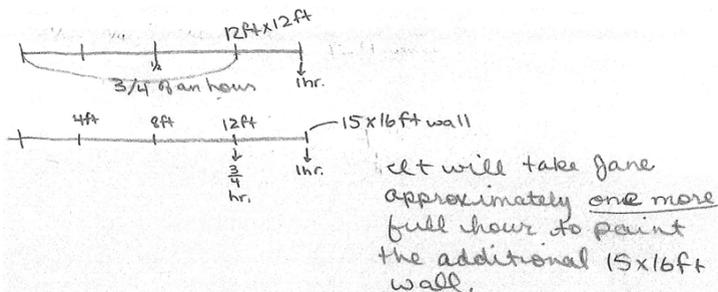
Figure 4. An incorrect solution.

SUMMARY OF CONCLUSIONS

As indicated at the beginning of this paper, instructors of prospective elementary school teachers could benefit from having information about the conceptions and strategies these students generally employ when solving mathematics problems. The current study identified six different solution methods prospective elementary school teachers used to solve a missing value proportion task. Three of these methods (using 144 sq. ft. to 45 min. as the unit ratio, Per Min. and Per Sq. Ft. approaches) have been identified by other researchers in prior studies with k-12 students, while the other three were more unique because of their tie to the given numerical relationship and the area context.

Even though a high percentage of the prospective elementary school teachers could solve the missing value problem with approaches other than the traditional cross-multiply method, a much lower percentage of these student could draw appropriate pictures to explain the meaning behind their computations. Many students drew static pictures to illustrate the given problem situation and did not use those pictures to explain the computation steps. Others attempted to use their pictures but were unable to use them successfully. Interestingly, students who used the three previouslyidentified methods had the most difficulty in drawing pictures and providing written explanations to justify the connection between the computation steps and pictures.

Why did students have difficulty in drawing pictures to explain these particular methods? Results from this and previous studies suggested two possible explanations.

First, many prospective elementary school teachers might have used these methods because they recognized the given task as a typical missing value task. Having identified the problem-type, they then used a method with which they had prior success in solving other missing value tasks without thinking more deeply about the meaning of the method itself in the given context. As Thompson and Thompson (1994, 1996) noted, having strong conceptual understanding of a particular mathematics concept alone might not be sufficient for a teacher to help his or her students understand that particular concept. Considering the level of mathematical abstraction at the elementary school levels, it's vitally important to provide opportunities for prospective elementary teachers to ground their computations in some image-rich context so that they will be better equipped to facilitate the conceptual development of their own students.

Second, many students decided to draw rectangles to illustrate the problem context. But as illustrated by Figure 2, it was not an easy task to figure out the answer to this question strictly based on the rectangular drawings with the given dimensions of the walls. It was also very challenging to explain certain approaches such as Per Min. approach using rectangular drawings. Some students recognized these difficulties and chose to use line models to represent the "quantities (area vs. time)" of the problem, rather than the physical dimensions (length by width vs. time) and used linear models similar to Figure 1 to support their computations. There is a clear need for prospective elementary school teachers to continue developing their "diagram literacy," including knowing when to use what types of drawings (Diezmann & English, 2001.)

Providing future elementary school teachers with mathematics tasks that are rich in context and encouraging them to develop drawings and representations to convey the meaning of their solution methods to other students appeared to have great potential to help them ground their mathematical reasoning. The results of this study also point to the need to discuss linear and area concepts earlier in the mathematics courses for prospective elementary school teachers, rather than in the second course as typically done in the U.S.. The hope of the mathematics education community is to place reasoning and justification at the central focus of mathematics instruction (Ball & Bass, 2003.) To support this goal, more studies are needed to investigate how various mathematics topics can be reasoned and justified by prospective elementary school teachers, and also to determine how mathematics courses can be specifically designed to enhance the reasoning and justifying abilities of future teachers.

References

Ball, D. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21,* 132-144.

Ball, D. & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston: VA: National Council of Teachers of Mathematics.

Diezmann, C. M. & English, L. D. (2001). Promoting the use of diagrams as tools for thinking (pp. 77-89). In A. A. Cuoco (Ed), *The Roles of representation in school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Kaput, J. & West, M. (1994). Missing-value proportional reasoning problems: factors affecting informal reasoning patterns. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. (pp. 235-287). Albany, NY: State University of New York Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics; Teachers' understanding of fundamental mathematics in China and United States*. Hillsdale, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2000). P*rinciples and standards for school mathematics*. Reston, VA: Author.

National Research Council. (2002). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swaford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education, Washington, DC: National Academy Press.

Post, T. R., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), Integrating research on teaching and learning mathematics. (pp. 177-198). Ithaca, NY: SUNY Press.

Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*, 233-253.

Simon, M. & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, *25*, 472-494.

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In Kilpatrick, J., Martin, G. W., and Schifter D. (Eds.) *A research companion to Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W. & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education*, *25*, 279-303.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *27*, 2-24.