TOWARDS A DEFINITION OF FUNCTION

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*This paper points up, in the case of a particular class discussion, the crucial role that the **Trace** tool could play as potential semiotic mediator for the notion of function. In particular, the episode we are presenting here want to show how the idea of trajectory developed through a specific sequence of activities, carried out in Cabri and centered on the use of this tool can substantially contribute to building the meaning of function as a point by point correspondence. It also shows the conceptual difficulties attached to a complete construction of this meaning and how the role of the teacher is based on and complements the pragmatic experience of the students in Cabri.*

INTRODUCTION

Similarly to what happened for other basic mathematical notions, a formal definition of function, as correspondence between two sets, dates back to the beginning of the nineteenth century. Actually, within set theory, the definition of function as a particular triplet (**E**, **F**, **A**) in which **A** is a subset of **EXF**, is due to Bourbaki and it has been given in 1939 (Bourbaki 1939).

As shown by Malik (1980): "a deep gap separates early notions of function, based on an implicit sense of motion, and the modern definition of function, that is "algebraic" in spirit, appeals to discrete approach and lacks a feel for variable".

Nevertheless, it's interesting to remark that traces of the fertile nexus of this concept with the sense of motion can be still identified in the work of famous mathematicians that contributed to the elaboration of this modern definition. In fact, in 1837 Dirichlet writes: *"Soient a et b, deux nombres fixes et soit x, une grandeur variable, qui prend successivement toutes les valeurs comprises entre a et b. Si à chaque x correspond un y fini unique de façon que, quand x parcourt continûment l'intervalle entre a et b, y=(x) varie aussi progressivement, alors y est dite fonction continue de x sur cette intervalle. Pour celà, il n'est pas du tout obligatoire que y, sur tout l'intervalle, dépend de x par une seule et même loi, ni qu'elle soit représentable par une relation exprimée à l'aide d'opérations mathématique.[1]"* (Youschkevitch 1976). As Youschkevitch underlies, the general characteristic of this definition of continuous function and its possibility to be directly generalized to the discontinuous one is evident; nevertheless a dynamical point of view is still present and Dirichlet found

[1] "Let *a* and *b* be two fixed numbers et let *x* be a variable quantity that takes successively all the values between *a* and *b*. If for each given *x*, a unique finite *y* corresponds to it in the way that, when *x* moves continuously along the interval between *a* and *b*, *y*=(*x*) varies progressively too, then *y* is said to be the continuous function of *x* over this interval. For this, it is not obligatory at all either that *y*, on all over the interval, would depend on *x* according to the same unique law, or that it would be represented by a relation expressed with the help of some mathematical operations" (translated by the authors)

the necessity to add a geometrical explanation to this definition.

On the contrary, the modern definition of function refers to a static notion that has lost every relation with the primitive dynamic intuition tightly tied to time and movement.

This research report presents a meaningful episode of a larger teaching and learning project, the first part of which has already been presented in PME27 (Mariotti et al. 2003). There were four classes involved in the project, two in France and two in Italy; the students were 15-16 year old with a major in scientific studies. In the previous research report (Mariotti et al. 2003) we have shown a particular aspect of the role of semiotic mediator played by some Cabri tools, in constructing a net of interrelated and indispensable meanings for the notion of function. In particular, both the asymmetrical nature of the independent versus the dependent variable and the twofold conception of trajectory (as both a "sequence of position of a moving point" and "a globally perceived geometrical object") were identifiable in the analysis of pupils' productions**.** Such meanings are crucial components for grounding the notion of function as a co-variation (between two variables, one depending on the other, and between two sets, the domain and the image) and they clearly emerged in relation to the internalization of the ***Dragging*** and ***Trace tools.***

Without solution of continuity to what we have already presented, the episode we are going to analyze here, enables to highlight an other aspect of the potentialities of the ***Trace tool*** as semiotic mediator. In particular, it shows the contribution of this tool in the emergence of the idea of function as point by point correspondence, by its simply evocation, in the case of a classroom discussion. This episode shows, also, how the achievement of this mathematical definition is difficult for the students and how the teacher manages to exploit such tools potentialities in order to attain this objective. Finally, this episode points up the role played by the problem of "defining two equal functions" in the construction of the meaning and of the definition of the function itself.

THE EXPERIMENTAL CONTEXT

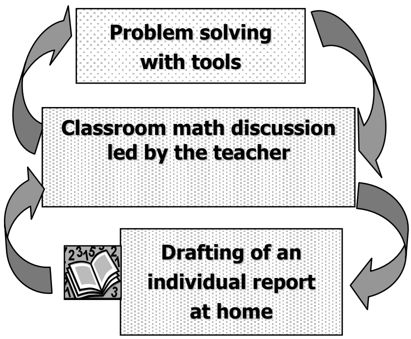
The sequence of activities, the episode presented here is part of, is based on four fundamental hypotheses:

1. One crucial aspect of the notion of function is the idea of variation or more precisely of co-variation, that is to say a relation between two variations one depending on the other one.
2. The primitive metaphor of co-variation is motion, that is to say the change of space according to the change of time.
3. A DGS environment, such as Cabri-géomètre, can provide a semantic domain of space and time within which variation can be experienced as motion.
4. According to the Vygotskian theoretical perspective of the semiotic mediation (Vygotsky, 1978, Mariotti, 2002), the computational tools and objects students

interact with, can be thought as signs referring to this notion of function as co-variation and, as such, they may become tools of "semiotic mediation", specifically implied by the teacher in class activities.

As a consequence of these theoretical assumptions, the general structure of the experimental sequence consists of four stages:

- At the beginning, students are faced with tasks to be carried out with Cabri tools;
- secondly, the various solutions are discussed collectively under the guidance of the teacher. These collective discussions play an essential part in the teaching and learning process. They are real "mathematical discussions" in the sense that their main characteristic is the cognitive dialectics between different personal senses and the general meaning which is introduced and promoted by the teacher (Bartolini Bussi 1998).



- Thirdly, it's required to the students to write at home an individual report specifying, on the one hand, what one has experienced and understood, and, on the other hand, doubts and questions arisen. This third stage is important because it constitutes a first externalization of internalized meanings and enables a reflective feedback on the solving task activity with tools.
- In a fourth time, students are asked to discuss about their productions. This phase is aimed at pursuing the processes of internalization and social inter-subjective construction of meanings.

THE EPISODE

A definition of "equal functions"

In the first part of the sequence, the activities with the Cabri-tools made the students perceive the difference between points that can be dragged directly, by taking them with the mouse, and points that can be moved only indirectly, by dragging those that these latter points depend on. This has become a reference situation, where a system of signs has been established, on the basis of which the meaning of variable has been introduced by the teacher. The points that can be moved directly correspond to the independent variables, whilst the points that can be moved only indirectly correspond to the dependent variables. Similarly and accordingly to our main fourth hypothesis, the use of the ***Trace tool*** contributed to the emergence of the twofold meaning of trajectory. In fact, both the conception of trajectory as a "globally perceived object" and as "an ordered sequence of position of a moving point" can be found in pupils' formulations and individual reports at home. (Mariotti, et al. 2003; Falcade 2003).

After the first phase of activities, a collective discussion was carried out, with the explicit aim of elaborating a definition of function. The discussion was articulated into two parts and took place during three lessons (lasting approximately 5 hours), with a twofold aim (it is possible to recognize a cognitive and a meta-cognitive level) corresponding to:

- clarifying and systematizing the ideas emerged during the previous activities.
- expressing these ideas into a 'mathematical statement', i. e. the definition of function.

At the very beginning pupils were asked to characterize a function. Different elements in play are highlighted by the students: the (independent and dependent) variables, the range domain, the image. Both the pupils and the teacher refer to Cabri tools and Cabri phenomena, as experienced during the first activities.

The difficulties arising in entering the mathematical world make the role of the teacher become relevant; the teacher has the difficult task of mediating between culture and pupils, between mathematics, as a product of human activities, and pupils' learning. Thus when the crucial point arises and the pupils realize that characterizing a function implies determining when two functions can be said to be "equal", the teacher shifted the focus of the discussion and asked the student to try and give a "definition of equal functions". A first attempt of definition simply stated: « Two functions are equal if they have the same range domain and the same image ».

At this point it is impossible to say whether, when the pupils speak about range and image, they are thinking globally or punctually.

The teacher asked the students to go back to Cabri and to look for different examples that can corroborate or invalidate their first conjectured definition. After this working moment in pairs, pupils were asked to express the new ideas arisen from their activity in Cabri. The following definitions about "two equal functions" were proposed.

Andrea – Alessandro: "Two functions are equal if they have the same range domain and the same image for all the domains subsets of the original domain which define the functions"
Gioia – Federica: "Two functions are equal if they have the same number of variables, the same range domain, and the same procedure (in the construction of the macro)".
Marco- Gabriele: "Two functions are equal when they have the same image and (when) the same range domain is fixed (for both).
Tiziano – Sebastiano: "In our opinion two functions are equal if having the same range domain and the same definition procedure they have the same image. If either the domain, or the definition procedure, or the image are not equal, neither the functions are equal."

Apart from that of Gabriele & Marco, who do not take into account the procedure, all the other definitions do consider the main elements in play: the range domain, the procedure and the image. For the majority of the students, the attention is focused on the difference between the procedures, but, surprisingly, the definition of Andrea &

Alessandro presents a characterization in which the domain is thought in terms of subsets. It's a static definition that shows no traces of variations and uses a quantifier ("for all"). This way of thinking may appear quite strange, if one does not take into account the very peculiar experience that pupils had in the previous activities and the relation built between the idea of trajectory and that of image: in the previous work, in order to compare two different functions pupils have compared two procedures on the same range domain observing that each procedure produced a different 'trace – trajectory' (image). Nevertheless, the link between what it has been done in Cabri and Andrea's and Alessandro's formulation is not immediate at all, we can suppose that process of internalization of this tool, which has transformed the Dragging in Cabri into such a static formulation ("for all the domain subsets") has been quite important.

Let see how the way of thinking was shared in the class and evolved.

A "reliable" definition hard to be accepted

The discussion starts when pupils are asked to compare the different definitions they have produced.

1. Ins: we must find an agreement on a definition, which can be one of these, or an improvement of one of these, or the fusion of these … We must decide.
2. Andrea : According to me, Gabriele's and Marco's definition is wrong.
3. Ins: So, Andrea, according to you, Gabriele's and Marco's definition is wrong. Let's read it again (she reads again) "two functions are equal when they have the same image and (when) the same range domain is fixed for both".
4. Andrea : Because to get to the same image, someone could pass through… we could have several journeys; in fact, if there were a subset of the domain… we can't say that the functions are…
5. Ins :… Tiziano, could you try to explain it?
6. Tiziano: Yesterday, we saw that we can, by doing the same domain, we can create the same image and this, with different functions (procedures).

The teacher redirects the discussion on the comparison between the definition of Andrea and Alessandro and those referring to the procedures.

44. I Ins: Let's read the text. You say that if they have the same domain and the same image for each subset of the domain…
45. Tiziano: But, here it's like to have the same procedure.
46. Ins: Hum, and why it's like to have the same procedure?
47. Several voices: …Because…
48. Gabriele: …As we go further, the subsets of the domain and vice versa…
49. Ins: Do you agree, Andrea?
50. Gioia: The domain is the plane, then you have the straight line, then a segment…
51. Ins: What are these?
52. Andrea: The domain can be whatever.

53. Gioia: They are subsets.
54. Ins: And then, the procedure, what does it do? That is to say, I…. Where does it start from?
55. Andrea The domain can be one point too… if we want!
56. Ins: The subset of the domain can be one point too. Oh!
57. Andrea: For whatever point, we get the same point of the image.
58. Ins: And this give the idea to say that…
59. Gioia: I'm doing the same procedure.
60. Andrea (together with Gioia) I'm doing the same procedure
61. Ins: I'm doing the same procedure. Therefore, for whatever point of what?
62. Andrea: For each point of the domain we have the same… as the result of the function, the same point of the image.
63. Ins: Do you agree? (referring to Tiziano)
64. Perplexed silences
65. The teacher writes at the blackboard and reads: "For each point of the domain, we have as the result of the function, the same point as the image".

It is possible to observe the emergence of the idea of coincidence point by point, as it is originated by the coincidence trajectory by trajectory, passing to the limit situation when the subset is a point. This is the case for Andrea, in which the process of internalization of the ***Trace*** tool turns out to be quite substantial. The development of this definition is achieved by thinking aloud and other pupils seem to participate to its elaboration (see in particular the interventions 50 – 59, when Gioia completes the sentence of Andrea or answers instead of him), but we can't be sure about them.

At the beginning, pupils seem clearly to accept that "to have the same domain and the same image for each subset of the domain it's like to have the same domain and the same procedure (lines 45, 48, 50). This is probably due to the fact they have already experienced in Cabri different phenomena according to which, Andrea's and Alessandro's definition appears sensible. Their agreement with Andrea is based on their actions with the tools; it is not theoretical at all.

During the second part of the dialogue, involving mainly the teacher and Andrea, but also Goia, the definition emerges in a quite "logical" and reliable way. Nevertheless, at the end, when the conclusion is written on the blackboard and read by the teacher, it becomes difficult, for the other students to accept it. Actually, this corresponds to deny the key role of the procedure in the definition of a function. It corresponds also to overcome the conceptual move from an experience based definition, tightly tied with Cabri activities, to a purely mathematical definition. And, indeed, a further discussion was, needed to reach the acceptance of confronting point by point a function. The role of the teacher is crucial in helping students to face this move. Indeed, her role is determinant all aver the discussion. At the beginning (line 1) she states the didactical contract within which the discussions should be developed. In different occasions (lines 44, 46, 51, 54, 58, 61), she intervenes or poses very specific questions, in order to redirect the discussion and focuses on the main mathematical points. In particular, at line 56, she repeats, with emphasis, Andrea's statement. She is aware about the important mathematical implications of Andrea's observation and pushes further the discussion in this direction. In other moments (lines 5, 49, 61) she tries to involve into the discussion students that seems not to participate to it. In general, what she tries to do is to orchestrate all the interventions in order to obtain that certain mathematical meanings emerge from particular students and then are discussed (and possibly shared) by all the other ones.

CONCLUSIONS

The excerpt, we have just presented, shows, in the case of a class discussion, a particular way the ***Trace*** tool can function as potential semiotic mediator. In fact, the idea of trajectory, as it emerges from the activities carried out in Cabri and centered on the use of this tool, substantially contributes, at least for Andrea, to building the meaning of function as a point by point correspondence. The same idea leads the other students to conceive that "to have the same domain and the same image for each subset of the domain it's like to have the same domain and the same procedure". The definition of function as correspondence is not far from there but it isn't immediate at all. Indeed, the procedural aspect seems to remain dominant for the other students and the correspondence between the two points, far from being arbitrary, must be related to a well stated procedure. The ultimate perplexity to accept Andrea's definition of "equal functions" shows also the difficulty to shift to a formulation that is theoretical and completely detached from the sensible experience in which it has been originated

Maybe the activities developed in the Cabri environment could have even reinforced a natural procedural tendency. But, on the other hand, within Cabri, the available tools (***Dragging***, ***Trace, Macro***,…) and the particular signs (segments, rays, Cabri figures representing the range domain, on which the independent variable varies, or the image, on which the dependent variable varies) offer a common semiotic system that the pupils and the teacher can elaborate. The Cabri tools and the related signs allow a discourse on them and on their behavior which gives a fundamental contribution to the construction of the net of interconnected meanings concerning the notion of function.

This excerpt shows also the importance of the teacher's role. On the one hand she has to organize a sequence of tasks involving tools to activate and support the process of internalization. On the other hand, she has to orchestrate the discussions in order to guide this process towards the construction, necessarily inter-subjective, of a certain specific mathematical meaning which may be, sometimes, quite different from the students' personal meaning. For this reason, in some cases, and this is the case, she has even to induce certain conceptual moves in order to help student to completely accomplish this processes.

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