WEIGHTED INEQUALITIES FOR ONE-SIDED MAXIMAL FUNCTION IN ORLICZ CLASSES

Amiran Gogatishvili

Let Φ be a class of functions $\varphi\colon \mathbb{R}^1 \to \mathbb{R}^1$ which are nonnegative, even, increasing on $[0,\infty)$, and such that $\varphi(0+)=0$, and $\lim_{t\to\infty}\varphi(t)=\infty$.

A function φ is called quasiconvex if there exists a Young function ω and a constant $c>1$ such that the chain of inequalities

$$\omega(t) \le \varphi(t) \le \omega(ct), \quad t \ge 0$$

holds. We can associate every quasiconvex function φ with its complementary function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\ge 0}(st - \varphi(s)).$$

Further, for every quasiconvex function φ we define the number $p(\varphi)$ as follows:

$$\frac{1}{p(\varphi)} = \inf\{\alpha\colon \varphi^\alpha \text{ is quasiconvex}\}.$$

Let g be a positive locally integrable function on the real line. And let the one-sided Hardy-Littlewood maximal functions $M_g^+ f$ and $M_g^- f$ be defined for $f \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$M_g^+ f(x) = \sup_{h>0} \frac{1}{g(x,x+h)} \int_x^{x+h} |f(y)|g(y)\ \mathrm{d}y,$$
$$M_g^- f(x) = \sup_{h>0} \frac{1}{g(x-h,x)} \int_{x-h}^{x} |f(y)|g(y)\ \mathrm{d}y,$$

where $g(x,x+h) = \int_x^{x+h} g(y)\ \mathrm{d}y$.

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Let w be a weight (nonnegative measurable function) in $\mathbb{R}^1$. We say that $w \in A_p^+(g)$, $p \in (1,\infty)$, if

$$\left(\frac{1}{g(a,c)}\int_a^b w(x)g(x)\ \mathrm{d}x\right)\left(\frac{1}{g(a,c)}\int_b^c w(x)^{-1/(p-1)}g(x)\ \mathrm{d}x\right)^{p-1} \le K$$

with some K independent of $a<b<c$.

We say that $w \in A_1^+(g)$ if $M_g^- w(x) \le Kw(x)$ for a.e. x. Similarly, $w \in A_p^-(g)$, $p \in (1,\infty)$, if

$$\left(\frac{1}{g(a,c)}\int_b^c w(x)g(x)\ \mathrm{d}x\right)\left(\frac{1}{g(a,c)}\int_a^b w(x)^{-1/(p-1)}g(x)\ \mathrm{d}x\right)^{p-1} \le K$$

for all $a<b<c$ and $w \in A_1^-(g)$ if $M_g^+ w(x) \le Kw(x)$.

Recently, weighted inequalities for the operator $M_g^+ f$ have been extensively studied [1]–[6].

In particular, the following characterization has been proved.

Theorem ([1], [2], [3], [4], [6])**.** *Let g and w be positive, locally integrable functions on the real line. Let $1<p<\infty$. Then the following conditions are equivalent:*

(i) *There exists a constant $K>0$ such that*

$$\int_{\{x:\ M_g^+ f(x)>\lambda\}} w(x)g(x)\ \mathrm{d}x \le K\lambda^{-p}\int_{-\infty}^{+\infty}|f(x)|^p g(x)\ \mathrm{d}x$$

holds for all $\lambda>0$ and every $f \in L^p(w)$.

(ii) *The weight w satisfies $A_p^+(g)$.*

(iii) *There exists a constant $K>0$ such that for every $f \in L^p(w)$*

$$\int_{-\infty}^{+\infty}\left(M_g^+ f(x)\right)^p w(x)g(x)\ \mathrm{d}x \le K\int_{-\infty}^{+\infty}|f(x)|^p w(x)g(x)\ \mathrm{d}x.$$

If $p=1$ then (i) *is equivalent to saying that w satisfies $A_1^+(g)$.*

The analogous theorem holds for $M_g^- f$ and the corresponding $A_p^-(g)$ classes.

In this paper we present generalization of these results to Orlicz classes. Some results in this direction were obtained in [4], [5]. First we give general weak type inequalities.

Theorem 1. *Let $\varphi \in \Phi$ and w_i $(i = 1, 2, 3, 4)$ be weight functions on the real line. Then the following conditions are equivalent:*

(i) *There exists $c_1 > 0$ such that for all f and λ we have the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x \le c_1 \int_{-\infty}^{+\infty} \varphi\big(c_1 |f(x)| w_3(x)\big) w_4(x) g(x)\ \mathrm{d}x.$$

(ii) *There are constants $c_2 > 0$ and $\varepsilon > 0$ such that*

$$\int_b^c \widetilde{\varphi}\Big(\varepsilon \frac{\int_a^b \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x}{\lambda g(a,c) w_3(x) w_4(x)}\Big) w_4(x) g(x)\ \mathrm{d}x \le c_2 \int_a^b \varphi(\lambda w_1(x)) w_2(x) g(x)\ \mathrm{d}x$$

for every $\lambda > 0$ and all numbers $a < b < c$.

In particular cases various criteria for one-weight inequalities have quite a simple form.

Our results are as follows.

Theorem 2. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

(i) *There is a constant $c > 0$ such that the inequality*

$$\varphi(\lambda) \int_{\{x:\ M_g^+ f(x) > \lambda\}} w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)|) w(x) g(x)\ \mathrm{d}x$$

is fulfilled for every $\lambda > 0$ and f.

(ii) *w is quasiconvex and $w \in A^+_{p(\varphi)}(g)$.*

Theorem 3. *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi\big(\lambda w(x)\big) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)| w(x)) g(x)\ \mathrm{d}x$$

holds for every $\lambda > 0$ and f.

(ii) *φ is quasiconvex, $w^{p(\varphi)} \in A^+_{p(\varphi)}(g)$, and $w^{-p(\widetilde{\varphi})} \in A^-_{p(\widetilde{\varphi})}(g)$.*

Theorem 4. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{\{x:\ M_g^+ f(x) > \lambda\}} \varphi\Big(\frac{\lambda}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi\Big(c\frac{|f(x)|}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x$$

holds for every $\lambda > 0$ and f.
(ii) *φ is quasiconvex and $w \in A^-_{p(\widetilde{\varphi})}(g)$.*

Now we give a complete characterization of weights for which various kinds of strong type inequalities hold.

Theorem 5. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\big(M_g^+ f(x)\big) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)|) w(x) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$ and $w \in A^+_{p(\varphi)}(g)$.*

Theorem 6. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\big(M_g^+ f(x) w(x)\big) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi(c|f(x)|w(x)) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$, $w^{p(\varphi)} \in A^+_{p(\varphi)}(g)$ and $w^{-p(\widetilde{\varphi})} \in A^-_{p(\widetilde{\varphi})}(g)$.*

Theorem 7. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *There is a constant $c > 0$ such that the inequality*

$$\int_{-\infty}^{+\infty} \varphi\Big(\frac{M_g^+ f(x)}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x \le c \int_{-\infty}^{+\infty} \varphi\Big(\frac{|f(x)|}{w(x)}\Big) w(x) g(x)\ \mathrm{d}x$$

holds for every $f \in L_{1,\mathrm{loc}}(\mathbb{R})$.
(ii) *φ^α is quasiconvex for some $\alpha \in (0,1)$ and $w \in A^-_{p(\widetilde{\varphi})}(g)$.*

Analogous theorems hold for $M_g^- f$.

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