Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

HYPERCOMPLEX STRUCTURES ON SPECIAL CLASSES OF NILPOTENT AND SOLVABLE LIE GROUPS

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1. Introduction

A hypercomplex structure on a manifold M is a family $\{J_\alpha\}_{\alpha=1,2,3}$ of complex structures on M satisfying the following relations:

$$J_\alpha^2 = -I, \quad \alpha = 1, 2, 3, \qquad J_3 = J_1 J_2 = -J_2 J_1 \tag{1.1}$$

where I is the identity on the tangent space T_pM of M at p for all p in M. A riemannian metric g on a hypercomplex manifold $(M, \{J_\alpha\}_{\alpha=1,2,3})$ is called hyper-Hermitian when $g(J_\alpha X, J_\alpha Y) = g(X, Y)$ for all vectors fields X, Y on M, $\alpha = 1, 2, 3$.

Given a manifold M with a hypercomplex structure $\{J_\alpha\}_{\alpha=1,2,3}$ and a hyper-Hermitian metric g consider the 2-forms $\omega_\alpha, \alpha = 1, 2, 3$, defined by

$$\omega_\alpha(X, Y) = g(X, J_\alpha Y).$$

The metric g is said to be hyper-Kähler when $d\omega_\alpha = 0$ for $\alpha = 1, 2, 3$.

It is well known (cf. [5]) that a hyper-Hermitian metric g is conformal to a hyper-Kähler metric $\tilde{g}$ if and only if there exists an exact 1-form $\theta \in \Lambda^1 M$ such that

$$d\omega_\alpha = \theta \wedge \omega_\alpha, \quad \alpha = 1, 2, 3 \tag{1.2}$$

where, if $g = e^f \tilde{g}$ for some $f \in C^\infty(M)$, then $\theta = df$.

A hypercomplex structure on a real Lie group G is said to be invariant if left translations by elements of G are holomorphic with respect to J_1, J_2 and J_3.

Given $\mathfrak{g}$ a real Lie algebra, a hypercomplex structure on $\mathfrak{g}$ is a family $\{J_\alpha\}_{\alpha=1,2,3}$ of endomorphisms of $\mathfrak{g}$ satisfying the relations (1.1) and the following conditions:

$$N_\alpha = 0, \quad \alpha = 1, 2, 3 \tag{1.3}$$

where I is the identity on $\mathfrak{g}$ and N_α is the Nijenhuis tensor corresponding to J_α:

$$N_\alpha(x, y) = [J_\alpha x, J_\alpha y] - J_\alpha([x, J_\alpha y] + [J_\alpha x, y]) - [x, y] \tag{1.4}$$

for all $x, y \in \mathfrak{g}$. Clearly, if G is a Lie group with Lie algebra $\mathfrak{g}$, a hypercomplex structure on $\mathfrak{g}$ induces by left translations an invariant hypercomplex structure on G.

Two hypercomplex structures $\{J_\alpha\}_{\alpha=1,2,3}$ and $\{J'_\alpha\}_{\alpha=1,2,3}$ on $\mathfrak{g}$ are said to be equivalent if there exists an automorphism ϕ of $\mathfrak{g}$ such that $\phi J_\alpha = J'_\alpha \phi$ for $\alpha = 1, 2, 3$. The classification of the four-dimensional real Lie algebras carrying hypercomplex structures was done in [2], where the equivalence classes of hypercomplex structures were determined and the corresponding left invariant hyper-Hermitian metrics were studied. It turns out that all such metrics are conformal to hyper-Kähler metrics (cf. [4]).

In the present work we study some remarkable properties of a special hyper-Hermitian metric which corresponds to a four-dimensional solvable Lie group. We also sketch a procedure for constructing hypercomplex structures on certain nilpotent and solvable Lie groups, following the lines of [3].

Acknowledgement The author would like to thank the organizers of the meeting *Quaternionic Structures in Mathematics and Physics* for their kind invitation to take part in this event.

2. A special hyper-Hermitian metric

Consider the four-dimensional real Lie algebra $\mathfrak{s} = \operatorname{span}\{e_j\}_{j=1,\dots,4}$ with the following Lie bracket:

$$[e_3, e_4] = \frac{1}{2}\, e_2, \qquad [e_1, e_2] = e_2, \qquad [e_1, e_j] = \frac{1}{2}\, e_j, \quad j = 3, 4$$

and let g be the inner product with respect to which $\{e_j\}_{j=1,\dots,4}$ is an orthonormal basis. It follows from [2] that g is hyper-Hermitian. Let $\{e^j\}_{j=1,\dots,4} \subset \mathfrak{s}^*$ be the dual basis of $\{e_j\}_{j=1,\dots,4}$ and write $e^{ij\cdots}$ to denote $e^i \wedge e^j \wedge \dots$. In this case we have the following equations for ω_α:

$$\omega_1 = -e^{12} + e^{34}, \qquad \omega_2 = -e^{13} - e^{24}, \qquad \omega_3 = e^{14} - e^{23}.$$

To calculate $d\omega_\alpha$, $\alpha = 1, 2, 3$, we compute first:

$$de^1 = 0, \quad de^2 = -e^{12} - \frac{1}{2}\, e^{34}, \quad de^j = -\frac{1}{2}\, e^{1j}, \quad j = 3, 4$$

to obtain:

$$d\omega_1 = -\frac{3}{2}\, e^{134}, \qquad d\omega_2 = \frac{3}{2}\, e^{124}, \qquad d\omega_3 = \frac{3}{2}\, e^{123}$$

so that (1.2) is satisfied for $\theta = -\frac{3}{2}\, e^1$. We can therefore conclude that the left invariant hyper-Hermitian metric induced by g on the corresponding simply connected solvable Lie group S is conformally hyper-Kähler. We recall from [2] that g is neither symmetric nor conformally flat. The Levi-Civita connection ∇^g is given as follows:

$\nabla^g_{e_1} \equiv 0,$

$$\nabla^g_{e_2} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \frac14 \\ & & -\frac14 & 0 \end{pmatrix}, \quad \nabla^g_{e_3} = \begin{pmatrix} & & 1 & 0 \\ & & 0 & \frac12 \\ -\frac12 & 0 & & \\ 0 & -\frac14 & & \end{pmatrix},$$

$$\nabla^g_{e_4} = \begin{pmatrix} & & 0 & \frac12 \\ & & -\frac14 & 0 \\ 0 & \frac14 & & \\ -\frac12 & 0 & & \end{pmatrix}.$$

Using these formulas we calculate the curvature tensor R^g : $R^g(e_1, v) = -\nabla^g_{[e_1,v]}$ for all v in $\mathfrak{s}$ and

$$R^g(e_2,e_3) = \begin{pmatrix} & & 0 & \frac18 \\ & & -\frac{7}{16} & 0 \\ 0 & \frac{7}{16} & & \\ -\frac18 & 0 & & \end{pmatrix}, \qquad R^g(e_2,e_4) = \begin{pmatrix} & & -\frac18 & 0 \\ & & 0 & -\frac{7}{16} \\ \frac18 & 0 & & \\ 0 & \frac{7}{16} & & \end{pmatrix},$$

$$R^g(e_3,e_4) = \begin{pmatrix} 0 & -\frac14 & & \\ \frac14 & 0 & & \\ & & 0 & -\frac{7}{16} \\ & & \frac{7}{16} & 0 \end{pmatrix}$$

and after some tedious calculations one can verify that the sectional curvature K satisfies $K \leq -\frac14$.

It is possible to show that the connected component of the identity $I_0(S,g)$ of the isometry group of (S,g) is a semidirect product $S^1 \times S$. To see this, one shows that every isometry f of S fixing the identity e of S satisfies that $(df)_e$ is an automorphism of $\mathfrak{s}$. It follows from this fact that $I_0(S,g)$ has no discrete co-compact subgroups and therefore, since S is solvable, S itself does not admit such a discrete subgroup.

3. Hypercomplex structures on certain nilpotent and solvable Lie groups

An *abelian* complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ satisfying

$$J^2 = -I, \qquad [Jx, Jy] = [x,y], \quad \forall x, y \in \mathfrak{g}. \tag{3.1}$$

The above conditions automatically imply the vanishing of the Nijenhuis tensor. By an abelian hypercomplex structure we mean a pair of anticommuting abelian complex structures. Our main motivation for studying abelian hypercomplex structures comes from the fact that such structures provide examples of homogeneous HKT-geometries (where HKT stands for hyper-Kähler with torsion, cf. [8]).

It was proved in [1] that if $\dim[\mathfrak{g},\mathfrak{g}] \leq 2$ then every hypercomplex structure on $\mathfrak{g}$ must be abelian. To complete the classification of the Lie algebras $\mathfrak{g}$ with $\dim[\mathfrak{g},\mathfrak{g}] \leq 2$ carrying hypercomplex structures (cf. [1]) it remained to give a characterization in the case when $\mathfrak{g}$ is 2-step nilpotent and $\dim[\mathfrak{g},\mathfrak{g}] = 2$: this is obtained by taking $m = 2$ in Theorem 3.1 below.

It is a result of [7] that the only 8-dimensional non-abelian nilpotent Lie algebras carrying abelian hypercomplex structures are trivial central extensions of H-type Lie algebras. We show in [3] that this does not hold for higher dimensions: there exist 2-step nilpotent Lie algebras which are not of type H carrying such structures.

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent Lie algebra endowed with an inner product $\langle\ ,\ \rangle$ and consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v},\mathfrak{v}] \subseteq \mathfrak{z}$. Define a linear map $j : \mathfrak{z} \to \text{End}\,(\mathfrak{v})$, $z \mapsto j_z$, where j_z is determined as follows:

$$\langle j_z v, w\rangle = \langle z, [v,w]\,\rangle, \quad \forall v, w \in \mathfrak{v}. \tag{3.2}$$

Observe that $j_z,\ z \in \mathfrak{z}$, are skew-symmetric so that $z \to j_z$ defines a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$. Note that $\text{Ker}(j)$ is the orthogonal complement of $[\mathfrak{n},\mathfrak{n}]$ in $\mathfrak{z}$. In particular, $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$ if and only if j is injective. Conversely, any linear map $j : \mathbb{R}^m \to \mathfrak{so}(k)$ gives rise to a 2-step nilpotent Lie algebra $\mathfrak{n}$ by means of (3.2). It follows that the center of $\mathfrak{n}$ is $\mathbb{R}^m \oplus (\cap_{z\in\mathbb{R}^m} \text{Ker}\, j_z)$ and $[\mathfrak{n},\mathfrak{n}] \subseteq \mathbb{R}^m$ where equality holds precisely when j is injective. We say that $(\mathfrak{n}, \langle\ ,\ \rangle)$ is irreducible when $\mathfrak{v}$ has no proper subspaces invariant by all $j_z,\ z \in \mathfrak{z}$.

It follows that a two-step nilpotent Lie algebra carrying an abelian complex structure amounts to a linear map $j : \mathfrak{z} \to \mathfrak{u}(k)$ (where $\dim \mathfrak{v} = 2k$ and $\mathfrak{u}(k)$ denotes the Lie algebra of the unitary group $U(k)$). As a consequence of this we obtain the following result, where we denote by $\mathfrak{sp}(k)$ the Lie algebra of the symplectic group $Sp(k)$:

Theorem 3.1 ([3]). Every injective linear map $j : \mathbb{R}^m \to \mathfrak{sp}(k)$ $(m \leq k(2k+1))$ gives rise to a two-step nilpotent Lie algebra $\mathfrak{n}$ with $\dim[\mathfrak{n},\mathfrak{n}] = m$ carrying an abelian hypercomplex structure. Conversely, any two step nilpotent Lie algebra carrying an abelian hypercomplex structure arises in this manner.

Using the same idea as in the above theorem it is possible to construct hypercomplex structures on certain solvable Lie algebras. In fact, given a two step nilpotent Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ set $\mathfrak{s} = \mathbb{R}a \oplus \mathfrak{n}$ with $[a,z] = z,\ \forall\, z \in \mathfrak{z},\ [a,v] = \frac{1}{2}v,\ \forall\, v \in \mathfrak{v}$, where the inner product on $\mathfrak{v}$ is extended to $\mathfrak{s}$ by decreeing $a \perp \mathfrak{v}$ and $\langle a, a\rangle = 1$. This solvable extension of $\mathfrak{n}$ has been studied by various authors ([6]). In the special case when $\dim \mathfrak{z} \equiv 3 \pmod 4$, $\dim \mathfrak{v} = 4k$ and the the endomorphisms $j_z,\ z \in \mathfrak{z}$, defined as in (3.2), belong to $\mathfrak{sp}(k)$, it can be shown that $\mathfrak{s}$ carries a hypercomplex (hyper-Hermitian) structure. The procedure is analogous to that in the preceding theorem. It should be noted that these structures cannot be abelian and the corresponding metrics are not hyper-Kähler (since they are not flat).

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