NON-ABSOLUTE INTEGRALS IN THE TWENTIETH CENTURY

1 Introduction

It is a little surprising to be giving a talk with this title at the end of the twentieth century, since at the beginning of the century two blows were dealt to the theory of non-absolute integrals. Either of these blows should have been mortal.

Since the time of Cauchy integration theory has in the main been an attempt to regain the Eden of Newton. In that idyllic time, a time celebrated each fall in every first year calculus course, derivatives and integrals were, as a result of *Newton's fundamental theorem of calculus*, but different aspects of, the same thing.

EXAMPLES (i)

$$(x^3)' = 3x^2 \qquad \int 3x^2 \, \mathrm{d}x = x^3.$$

When later the integral was given an independent definition it became necessary to reunite the now distinct concepts, integral and derivative. As the concept of function became wider this task became more difficult, and led to the need for more and more general integrals.

Integrals of positive functions always give the area under the graph–and can fail to exist either because the function is too badly behaved or because the area is not finite.

EXAMPLES (ii) On $[0,1]$ the indicator function $1_{\mathbb{Q}}$ is too badly behaved to be integrable;

$$1_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(iii) The function f defined on $[0,1]$ by

$$f(x) = \begin{cases} x^{-1}, & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0. \end{cases}$$

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has an infinite area under its graph, and so is not integrable.

If a function has variable sign the integral measures the difference between the area under the graph and above the axis, and the area below the axis and above the graph.

Examples (iv)

$$\int_0^{2\pi} \sin x \,\mathrm{d}x = \int_0^{\pi} \sin x \,\mathrm{d}x - \int_{\pi}^{2\pi} (-\sin x) \,\mathrm{d}x.$$

For elementary integrals both of these areas are finite, and so the integral of the absolute value also exists; the integral for this reason is called an *absolute integral*.

Examples (v)

$$\int_0^{2\pi} |\sin x| \,\mathrm{d}x = \int_0^{\pi} \sin x \,\mathrm{d}x + \int_{\pi}^{2\pi} (-\sin x) \,\mathrm{d}x.$$

However, we can integrate, in a slightly less elementary way by using what is called the *Cauchy extension* of the elementary integral—the oddly named improper, or infinite integral of first year calculus. If then the function to be integrated is unbounded with an infinite number of oscillations we are in effect evaluating the sum of an alternating series that may be conditionally convergent.

Examples (vi) Consider $f = F'$, where

$$F(x) = \begin{cases} x^2 \sin(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then,

$$\int_0^x f = \lim_{a\to 0} \int_a^x f = \lim_{a\to 0} \left(x^2 \sin(x^{-2}) - a^2 \sin(a^{-2})\right) = x^2 \sin(x^{-2}) = F(x), \qquad (1)$$

but

$$\int_0^x |f| = \lim_{a\to 0} \int_a^x |f| = \infty.$$

The example shows that to handle derivatives an integral needs to be a *non-absolute integral*.

2 The Work of Lebesgue and Denjoy

After Cauchy and until the end of the nineteenth century the integral was extended to handle more and more complicated derivatives. The integral of Riemann could handle bounded derivatives that were continuous almost everywhere, and the Cauchy extension, as in 1(1), could be elaborated to handle derivatives, continuous almost everywhere, with a closed set S of points of unboundedness provided S was nowhere dense and countable; a so-called *reducible set*.

If the set S is uncountable then it has a non-empty perfect kernel, and on each contiguous interval of this kernel S is reducible. So the Cauchy extension can be used to evaluate the integral on each such contiguous interval. If then S is of measure zero and if the sum of these integrals converges absolutely, we could again define an integral that would integrate the derivative. This process is called the *Harnack extension*; and we need the series to converge absolutely as the set of contiguous intervals is unordered.

The process, as you see, is getting more and more complex. Further the set S, as well as the set of points of discontinuity can have positive measure, and the series in the Harnack extension need not converge absolutely; for details see [*Hawkins, Pesin*].

In 1901 Lebesgue announced a new simple integral, and delivered the first blow to non-absolute integration theory.[*Hawkins, Pesin*], [*Lebesgue 1901, 1902*].

If he did not take us back to the lost Eden, he certainly led us into the promised land. No field of analysis was not enriched by the insights the Lebesgue integral brought. It did not integrate all derivatives — but given its many other gifts, who cared. In any case it integrated all bounded derivatives, or even all Lebesgue integrable derivatives. Then using the Cauchy and Harnack extensions defined above this class of integrable derivatives can be extended even further. All this was known by 1904 when Lebesgue published his famous book with its significant title *"Leçons sur l'Intégration et la Recherche des Fonctions Primitives"*; [*Lebesgue 1904*].

A second blow to the life of non-absolute integral theory was delivered eight years later when Denjoy published two notes — a total of eight pages — in the Comptes Rendus; [*Pesin*], [*Denjoy 1912*]

In these he defined an integral that handled all derivatives, he recovered the lost Eden, a place of decidedly less attractive aspect than when it had been left.

The full theory was published in 1915-1917, in four long papers, totaling about 400 pages, [*Denjoy 1954*], [*Denjoy 1915a,1915b, 1916, 1917*].

So the search was over and on the way the beautiful and useful Lebesgue integral had been found. Why did not the theory of non-absolute integral remain a historical curiosity—rather like the theory of Jordan content? What has kept it alive?

3 Developments to the Denjoy Theory

First let us consider just what Denjoy had done.

His process of integration, that he called *totalization*, consisted of essentially three operations, the evaluation of sequences of Lebesgue integrals, Cauchy extensions of these integrals, and summing of subsequences of these integrals, the Harnack extension.

For the method to work these series in the Harnack extensions have to converge absolutely. While this requirement led to an integral that would integrate derivatives, the integral itself was not necessarily differentiable almost everywhere — this is the *general Denjoy integral*, the $\mathcal{D}$-integral. In order to remove this flaw Denjoy added a condition restricting the size of the terms in the series. This led to another integral that would integrate derivatives, the *restricted Denjoy integral*, the $\mathcal{D}^*$-integral, and all $\mathcal{D}^*$-primitives were differentiable almost everywhere; further two $\mathcal{D}^*$-primitives with derivatives equal almost everywhere differed by constant.

In 1912, in the same volume as Denjoy, page 1688 to his page 1075, just three months later, Luzin introduced two classes of functions: the ACG* and ACG functions, and as their names suggest both are generalizations, hence the G, of the class of absolutely continuous functions, hence the AC; [*Luzin*]. These classes do for the Denjoy integrals just what the class of AC functions does for the Lebesgue integral— they give descriptions of the primitives.

The class ACG is perhaps the most general class of wide interest in analysis and much research was done on its properties, by Saks in particular; [*Saks 1937 pp.221, 228*], [*Saks 1923, 1930, 1931*]

Denjoy had recognized that his more general $\mathcal{D}$-primitives were not necessarily differentiable almost everywhere, and introduced the concept of approximate limits, and so approximate derivatives. $\mathcal{D}$-primitives were approximately differentiable almost everywhere and approximate derivatives of continuous functions were $\mathcal{D}$-integrable.

In 1916 Hinčin, [*Hinčin*], added a condition to the Denjoy $\mathcal{D}$-totalization process, weaker than the one giving the $\mathcal{D}^*$-integral. This gives an integral, the $\mathcal{DH}$-integral[1], that has continuous ACG primitive with a derivative almost everywhere. This integral integrates a function that is the almost everywhere existing derivative of a continuous ACG function, and will be referred to below as it is a borderline integral in a certain sense.

Denjoy's work was both monumental and detailed as he was not one to leave any stone unturned. However, by hindsight, he left an obvious gap.

The missing point in Denjoy's work was brought about by his insistence that primitives had to be continuous. Surely if we are handling approximate derivatives we only need approximately continuous functions for primitives. This was taken up by Verblunsky and Ridder in 1933–34. In different ways they defined a totalization process that gives an approximately continuous ACG function;[*Ridder 1933,Verblunsky*].

In 1915 Perron introduced a completely different approach to integration–based on certain ideas of de la Vallée Poussin. His main application was to the theory of differential equations and in particular potential theory; [*Perron*]. In the hands of Bauer, Perron's method of major and minor functions led to an integral, equivalent to the $\mathcal{D}^*-$integral; [*Bauer*]. This equivalence is usually called the *Hake-Looman-Aleksandrov*, or just *HLA-,theorem.* The papers by these mathematicians giving this equivalence appeared between 1921 and 1925; [*Aleksandrov, Hake, Looman 1925*].

[1] Because of different transliterations this is also called the $\mathcal{DK}$-integral.

It is one of the deep basic results of the theory.
It was natural then to continue studying the Perron approach for other integrals–for instance the $\mathcal{D}$- and $\mathcal{DH}$-integrals. The most interesting use of this method was by Burkill in 1932; his approximate Perron, or $\mathcal{AP}$-, integral does for approximate derivatives what the Perron integral did for the classical derivative. The $\mathcal{AP}$-primitive is approximately continuous, ACG and has an approximate derivative almost everywhere, and the integral will integrate all approximate derivatives; [*Gordon pp.259–270*], [*Burkill 1931*].
A thorough review of all these ideas was given its final and very detailed form in various papers of Ridder in the years 1931-1936, [*Ridder 1931, 1932, 1933, 1934, 1936*]. See also various papers of Kubota; [*Kubota 1960, 1964a,b, 1966, 1967, 1968, 1970, 1971, 1972, 1976, 1978, 1986, 1989, 1994*].
Two interesting points can be made, one rather trivial, the other deep.
While the Perron approach is simple, until the eighties, it appeared that the only way to prove the elementary integration by parts theorem was to use the deep HLA-theorem, together with the fact that integration by parts is fairly easy to prove for the $\mathcal{D}^*$-integral, in either the Denjoy or Luzin form; [*Hobson p.711, McShane 1944, pp.332–335, Saks 1937, p.246*], [*Bullen 1985, 1986, Gordon & Lasher, Jeffrey 1942, McShane 1942*].
A deeper result is that of Marcinkiewicz and Tolstov, 1937–1939; [*Gordon 1994, pp.131–132, Saks 1937, p. 253*], [*Bullen & Výborný, Sarkhel, Tolstov*]. A function is Perron integrable if and only if for all $\epsilon > 0$ there is a measurable major function M, and a measurable minor function m such that $0 < M - m < \epsilon$. The deep and remarkable result of Marcinkiewicz and Tolstov says that : a function is Perron integrable if and only if there exist one continuous major function and one continuous minor function. Let us refer to this as the *Marcinkiewicz property*[2].
So by the late thirties the theory was essentially complete but no properties made these integrals very useful. The monotone converge theorem holds for all of the new integrals, but as non-negative functions integrable in any of these senses is Lebesgue integrable, see for instance [*Gordon 1994, pp.109, 127, Saks 1937, p.242*], any application of this result really meant we were in a Lebesgue integral situation. In the 1950's Džvaršeĭšvili gave a convergence theorem that was really in the Denjoy theory; [*Čelidze & Džvaršeĭšvili, transl.p. 40*]. An interesting result in the Lebesgue theory is the following, see [*Gordon 1994, p.203, Natanson p.152*].

[VITALI] *If $f_n \to f$ almost everywhere, f_n $\mathcal{L}$-integrable on $[a, b]$, and if the primitives $F_n = \int f_n$ are uniformly AC then f is $\mathcal{L}$ integrable and $F_n \to \int f$.*

It is easy to state a generalization for either the $\mathcal{D}$- or $\mathcal{D}^*$-integral by replacing

[2] An important result of Saks says that there is no difference between requiring the major and minor functions to measurable and requiring them to be continuous; see [*Gordon 1994, pp.127–134, Saks 1937, pp.247–252*].

"uniformly AC" by uniformly continuous and uniformly ACG or ACG*. This very nice theorem was rediscovered in the seventies by Lee Peng Yee and called *controlled convergence* — much work on this has been done by Lee and his colleagues and students in South East Asia; [*Gordon 1994, pp. 205–207, Lee 1989 pp.39–41, Lee & Výborný pp.190–197*], [*Lee & Chew 1985, 1986, 1987*].

The original integral definitions of Denjoy and Luzin are wedded to basic properties of the real line and extensions to higher dimensions, and abstract spaces are very unsatisfactory, particularly when compared to situation for the Lebesgue integral.

Some work on a two dimensional Denjoy totalization was done in the twenties by Looman, producing an integral that would invert the mixed second derivative; [*Looman 1923*]. Later Čelidze gave a definition of a two dimensional version of the general $\mathcal{D}$-integral and obtained a sufficient condition for this integral to have a Fubini theorem; [*Čelidze & Džvaršeĭšvili, Chap.2*]. In this case there is not going to be a Fubini theorem as Tolstov showed by an example; [*Tolstov 1949,1950*].

In the thirties Kempisty defined a two dimensional Denjoy integral that was used in the fundamental work of Romanovskiĭ in 1941. This led to the idea of *Romanovskiĭ spaces* defined in 1969 by Solomon; [*Solomon*], [*Romanovskiĭ*]. Still this abstract theory does not have a life of its own — as is the case in Lebesgue theory. A series of works by Trijitzinsky on abstract generalizations of the totalization concept have left almost no mark on the theory.[*Trijitzinsky 1963*], [*Trijitzinsky 1969*].

The Perron approach is easier to extend but even here there are problems. An example can be given of a function of two variables that is Perron integrable but such that if the axes were rotated by $\pi/4$ the function was no longer integrable. In other words the definition is very much tied to the axes; [*Kurzweil 1980, p.99 11.4*]. The Fubini theorem can be given but is in any case of very limited value. There is no easy way of telling when a function will be integrable. In the case of absolute integrals we have Tonelli's theorem that allows us to carry out the repeated integration, and if this works then the function is integrable — such a theorem does not exist for non-absolute integrals; see [*McLeod pp.35–36, 156*]. Also, as above, we get a theory that is tied to the axes. Further such an extension seems to have a fatal flaw. An example can be given as follows: if a multi-dimensional integral satisfies Fubini's theorem, and if the integral in one dimension reduces to the $\mathcal{D}^*$-integral then there is a differentiable function whose partial derivatives are not integrable; [*Pfeffer 1993, p.207*], [*Pfeffer 1986, Ex.5.7*]. So if we have Fubini's theorem we lose the raison d'être of the non-absolute integral.

There are also definitions of of Denjoy extensions of the Bochner and Pettis integrals; [*Kurzweil 1980 pp.116–179*], [*Gordon 1981*]. Finally mention should be made of the deep and extensive work of Nakanishi on totalization, and other approaches to non-absolute integrals, on ranked spaces;[*Nakanishi 1954a,b,c, 1955, 1956, 1957, 1958, 1968, 1969, 1970, 1988, 1991a,b, 1992a,b,c, 1993a,b, 1994, 1995a,b,c, 1997, 1998*].

As is well known the Lebesgue theory the class of integrable functions defines some very useful Banach spaces, the theory of the space $\mathcal{D}^*(a,b)$ of all $\mathcal{D}^*$-integrable functions on $[a,b]$ is less satisfactory. Alexiewicz, [*Alexiewicz*], introduced the norm:

$$\|f\|_{\mathcal{D}^*(a,b)} = \max_{a\le x\le b} \int_a^x f.$$

In this way $\mathcal{D}^*(a,b)$ is an incomplete normed space, a subspace of the Banach space of continuous functions on $[a,b]$ given the sup norm. It is however a barreled space and so there are tools that can be used to study the linear functionals on this space; [*Henstock 1967, pp.340–342*], [*Sargent 1950, 1953*]. A different approach to this topic has recently been given by Thomson; [*Thomson 1999-2000*].

Another way of extending the theory was to define various Stieltjes integrals. The first person to do this was Lebesgue in the second edition of his book where he extends Denjoy totalization to functions that are the derivatives of continuous functions with respect to a function of bounded variation, [*Lebesgue 1928, p.296*], [*Jeffery 1932, Ridder 1939*]. The most important work on an integral that generalizes both the Lebesgue-Stieltjes and Riemann-Stieltjes integral was done by Ward; he used the Perron method and allowed the integration to be with respect to to any function; [*Jacobs pp.515–533, Saks 1937, pp.207–212*], [*Ward*]. This work was extended to the general Denjoy integral by Ridder;[*Ridder 1935, 1938*]. As the Lebesgue-Stieltjes integral is not additive on intervals the values of this integral and the Perron-Stieltjes integral do not necessarily agree, see [*Saks 1937, p.210*]. Finally their are definitions of such Stieltjes integrals in higher dimension by Kempisty and Ridder; [*Kempisty, Ridder 1943*].

A more detailed account of the topics in this section can be found in [*Bullen 1979, 1980*] although these papers are a bit dated.

4 Trigonometric Integrals

What really kept non-absolute integrals alive during the second quarter of the 20th century was not the above work but another monumental achievement of Denjoy.

It has been known — since basic work in the nineteenth century — that if for all $x, -\pi \le x \le \pi$, the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \tag{2}$$

converges, to $f(x)$ say, then : (i) its coefficients converge to zero, and (ii) these coefficients are uniquely determined by the sum function; [*Zygmund I, pp. 316, 326*]. Over the years more and more results calculated these coefficients from the sum function by the Fourier formulæ;

$$\begin{aligned} a_k =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx \,\mathrm{d}x,\ k = 0,1,\dots,\\ b_k =& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \,\mathrm{d}x,\ k = 1,\dots. \end{aligned}$$

If the sum, f, is Riemann integrable, then du Bois Reymond proved that the Riemann integral can be used, [*Hawkins, pp.53, 213*], and if it is the Lebesgue integrable then de la Vallée-Poussin showed that the Lebesgue integral can be used, [*Zygmund, I, pp.326,382, II p.349*]. In 1917 Nalli showed if f is $\mathcal{D}^*$-integrable then the formulæ hold with the $\mathcal{D}^*$-integral; [*Zygmund II, pp.84–86*], [*Nalli*]. Then progress stopped and later Sklyarenko, [*Sklyarenko 1973*], gave an example of a series with a $\mathcal{DH}$-integrable sum, f say, for which the coefficients are not given by the Fourier formulæ using the $\mathcal{DH}$-integral; in particular,

$$a_0 \neq \frac{1}{\pi}\mathcal{DH}\text{–}\int_{-\pi}^{\pi} f.$$

It had been known for some time that the sum of an everywhere convergent trigonometric series need not be $\mathcal{D}^*$-integrable, or indeed integrable in any of the senses mentioned above.

Examples (i) The everywhere convergent series

$$\sum_{n\geq 2}\frac{\sin nx}{\log n}$$

has a sum that is not integrable in any sense mentioned so far. It is Riemann integrable on any interval $[a, b]$, $0 < a < b < 2\pi$, as the series converges uniformly on such an interval. However, the limit of the integral obtained in this way, as $a \to 0$ does not exist, as an indefinite integral on $[a, b]$ is

$$\sum_{n\geq 2}\frac{\cos nx}{n\log n},$$

a series which diverges at the origin. This example is due to Fatou who presented it as a convergent trigonometric series with a sum function that is not $\mathcal{L}$-integrable; [*Denjoy 1949, p.42, Thomson 1994, p.336*].

The obvious integral for the sum of a trigonometric series is the sum of the once integrated series. However, the once integrated series can diverge, see Examples(i), and can do so on a set of measure zero. So Denjoy, with his insistence that primitives be continuous, looked at the second integrated series which converges uniformly, and its sum has the original series as a derivative — a *second order symmetric derivative.* If f is the sum of (2), and if F is the sum of the twice integrated series then

$$f(x) = \lim_{h\to 0}\frac{F(x+h) + F(x-h) - 2F(x)}{h^2},\ -\pi \leq x \leq \pi;$$

[*Zygmund I, pp.328–329*].

So if we could integrate such derivatives we obtain F, the sum of the twice integrated series, and so the coefficients of the original series (2). This Denjoy did in a series of 5

notes in 1921 — a total of 12 pages; [*Denjoy 1921*]. There he described an incredibly complex generalization of his original totalization process. The full details were not published until 1941–1949 in four volumes totaling over 600 pages; [*Denjoy 1949*].
Meantime his original papers caused much interest and during the next twenty years Verblunsky, Burkill, and Marcinkiewicz & Zygmund gave various integrals that did the same as Denjoy did with his process; [*Burkill 1951, Marcinkiewicz & Zygmund, Verblunsky*]. The most important of these are Burkill's $\mathcal{SCP}-$integral and the Marcinkiewicz & Zygmund $\mathcal{T}-$integral. Both used a generalized first order derivative, and lived with a primitive that failed to be defined in a set of measure zero.
In addition James gave a Perron integral that used the same derivative as Denjoy used, the $\mathcal{P}^2$-integral; [*Zygmund II pp.86–9*], [*James & Gage, James*].
All these integrals existed and did the job they were designed for — but little else was known. It is clear from Sklyarenko's example that these integrals were generally inconsistent with the classical integrals. This example gives a function that is $\mathcal{DH}$-integrable, and also integrable in each of these senses; but the value of the $\mathcal{DH}$-integrals can differ from the value of these other integrals.

Examples (ii) In particular if f is Sklyarenko's example:

$$a_0 = \frac{1}{\pi}\mathcal{SCP}-\int_{-\pi}^{\pi} f, \quad \text{but} \quad a_0 \neq \frac{1}{\pi}\mathcal{DH}-\int_{-\pi}^{\pi} f$$

The relationships between these trigonometric integrals was not clear — no Luzin came along, and there is no HLA. Rather it appears that small changes in definitions can affect these relationships — without of course spoiling their main character of integrating sums of everywhere convergent trigonometric series; see for instance [*Mukhpadhyay*].
Finally Skvorcov, and his students Sklyarenko and Vinogradova, found various properties of the James $\mathcal{P}^2$-integral and the $\mathcal{SCP}$-integral that enabled them to determine the exact connection between these integrals:
A function f is $\mathcal{SCP}$-integrable on $[a,b]$ if and only if it is $\mathcal{P}^2$-integrable and the $\mathcal{P}^2$-primitive F is ACG^*, and $F'(a)$, $F'(b)$ exist.[3]
The relationship of either of these integrals to Denjoy's totalization is still not known; [*Sklyarenko 1972, Skvorcov 1966, 1967, Vinogradova & Skvorcov*].
Sklyarenko, and Skvorcov & Thomson showed that these integrals do not possess the Marcinkiewicz property— that is, for example there are functions with an $\mathcal{SCP}$-major and an $\mathcal{SCP}$-minor function that are not $\mathcal{SCP}$-integrable; [*Sklyarenko 1973, Skvorcov & Thomson*]
Another historical oddity was that again integration by parts caused trouble— the original proof given by Burkill for his $\mathcal{SCP}$-integral was wrong, as was noted by

[3] In general a $\mathcal{P}^2$-primitive need only be ACG.

Skvortsov, a student at that time visiting Burkill in Cambridge; [*Burkill 1983*]. Much later a proof was found by Sklyarenko, and it was not at all easy to give; [*Sklyarenko 1981*].

5 Riemann Redux

By the mid-fifties the non-absolute world was in the doldrums. It cannot be claimed that the publication of two compendious volumes by Denjoy putting into one source work he had done 30 or 40 years earlier was any sign of vigour. There were a few stirrings in Canada, and very interesting work in Moscow; but all in all mere flickers of a dying fire that had raged earlier in the century. Then a new and extremely simple idea reinvigorated the field, re-ignited the fire, so that now it is burning merrily. The whole field — the work of Denjoy, Luzin, the classical primitive problem, trigonometric integrals — was overhauled, and looked at from a new and exciting perspective. The point of view had been there all along and missed — just why is not clear. Maybe Riemann dismissed it as an unnecessary complication, that Denjoy who did work in the Riemann integral did not seize on this idea is probably due to his constructive bias.

In the late fifties two mathematicians— Kurzweil in Prague, and Henstock in Bristol— had the same brilliant idea. It would appear that Kurzweil had this idea first in 1957 in a paper with the unlikely title of *"Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter"*[4]; [*Kurzweil 1957*]. Very shortly after, in 1960, Henstock started a long series of papers using the same idea culminating in his basic book on the subject; [*Henstock 1963*].

While Kurzweil defined almost in passing an integral, that I will call the $\mathcal{HK}$-integral[5], Henstock came to it in a long series of papers on integrals. He developed all the basic tools that have been used ever since; so H before K, also it is the alphabetical order. He grasped early on just how important this new integral was; his 1968 paper is called, *"A Riemann-type integral with Lebesgue power"*, [*Henstock 1968*]; and I will always remember at the 1962 ICM in Stockholm, his cry, "Lebesgue is dead". A little histrionic, but he knew the power of his ideas.

The basic theory is so simple some foolhardy souls have dared to displace Newton from the first year calculus rites. This is extreme but the ideas are natural for a first analysis course — see for instance the books [*Depree & Swartz, Mawhin*].

[4] Important results in papers with unlikely titles might make an interesting article. Another example is the Riesz convexity theorem that appears in the paper "Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires". This certainly did not catch the attention of Hardy & Littlewood, although Paley did notice it; compare the original Hardy & Littlewood proof of [*Zygmund (3.19)*] with Paley's proof of the much more general [*Zygmund (5.2)*].

[5] The name of this integral varies from reference to reference; $\mathcal{KH}$-integral, the Riemann complete integral, the Henstock integral, the Kurzweil integral. Perhaps the best generic name for an integral defined this way, using partitions and Riemann sums, is a *generalized Riemann integral.*

To see how easy it is let us introduce some notations: a *(tagged) partition of* $[a, b]$ is $\varpi = \{a_0, \ldots, a_n; x_1, \ldots .x_n\}$ where $a = a_0 < \cdots < a_n = b$, and the *tags* satisfy $a_{i-1} \le x_i \le a_i$, $1 \le i \le n$. If $\delta(x) > 0, a \le x \le b$, a positive function called a *gauge*[6], then ϖ is said to be a *δ-fine partition* if :

$$\begin{gathered} a = a_0 < \cdots < a_n = b \\ x_i - \delta(x_i) < a_{i-1} \le x_i \le a_i < x_i + \delta(x_i),\ 1 \le i \le n \end{gathered} \tag{3}$$

The the definition of the $\mathcal{HK}$-integral follows that of the Riemann integral in any elementary analysis course but using these δ-fine partitions instead of the standard ones that use a constant gauge.

Replacing a classical uniform partition by one that allows for various conditions across the interval is a simple but, as it turns out, a very powerful change that gives an integral equivalent to the $\mathcal{D}^*$-integral. All the basic properties are as easy to prove as they are for the elementary Riemann integral. A little harder to prove is the monotone convergence theorem but it can be done —and the proof is probably no harder than the one in Lebesgue theory; [*Gordon pp.161–162, Henstock 1988,pp.90–91, Kurzweil 1980, pp.41–42, Lee p.12–15, Lee & Výborný p.224–226, Lukeš & Malý pp.75–80, McLeod pp.86–88, 96–101, Pfeffer 1993, pp.18–20*].

With this fantastic start one might be forgiven for thinking that there was nowhere to go — the starting line being also the finishing line. However, the flood of basic concepts and tools introduced, mainly by Henstock, has lead to what is virtually a new field of research that for the purpose of this talk I will call *partition theory*[7]. This theory has solved or clarified every problem in non-absolute integration. Some questions are immediate and were solved, or at least stated, by Henstock. Can the method be used to obtain other non-absolute integrals? Can we move as easily to double integrals as we can for the Riemann integral? Others arose as the field was developed. The fact that partition theory gave another way of defining the $\mathcal{D}^*$-integral was nowhere nearly as important as the opening up of these new ideas and methods.

6 Partition Theory

It is a little surprising that this simple approach to integration has not replaced Lebesgue theory, or at least the usual consideration of the Riemann integral, in first analysis courses. Why waste time on this now obsolete tool — as obsolete as Jordan content — when the same amount of work will give you the powerful tool,

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[6] Care must be exercised as the usage *gage* also occurs in the literature. This is usually occurs with generalized Riemann integrals in higher dimensions, see 7.1, and indicates a non-negative gauge; a gauge that can be zero on a countable set.

[7] This is not a standard terminology, rather a shorthand for quick reference in this talk. Perhaps the *theory of partitions of intervals* would be better, but it is cumbersome.

the $\mathcal{HK}$-integral? All I can say is that we are all lazy, and too busy to learn new tricks.
However, putting it in first year calculus courses is going too far. The concept of a δ-fine partition is subtle, and right at the beginning is the problem — does one always exist? Well the answer is yes, but the existence for all positive functions δ of a δ-fine partition is equivalent to the completeness of $\mathbb{R}$ — a fact noticed in the nineteenth century by a Belgian mathematician, Cousin, [*Cousin*][8].

[Cousin's Lemma] *If δ is a gauge on $[a, b]$ then every closed sub-interval of $[a, b]$ has a δ-fine partition.*

The existence of δ-fine partitions gives a very simple form of completeness and it can be used to prove most basic results in analysis in a very succinct and uniform way;[*Lee & Výborný pp.26–29, 71, McLeod pp.16–17, 38–39, Thomson 1994, pp.64–65*], [*Botsko 1987, 1989, Bullen 1983, Thomson 1980-1981*].
In particular every derivative has a natural set of δ-fine partitions associated with it. Consider the derivative of f and a positive ϵ then for each x define $\delta(x)$ by:

$$\left|\frac{f(v)-f(u)}{v-u} - f'(x)\right| < \epsilon;\ \forall u, v\ x-\delta(x) < u \leq x \leq v < x+\delta(x), u \neq v. \tag{4}$$

Using these partitions we can readily prove very subtle monotonicity theorems, and also see that all derivatives are $\mathcal{HK}$-integrable; [*Gordon pp.141–142, Lee & Výborný pp.46–48, McLeod pp.40–44, Pfeffer 1993, pp.26–27, 103*], [*Bullen 1983, Thomson 1980-1981*].
It is only natural then to ask if the same idea will apply to other derivatives — approximate, symmetric etc.
Consider approximate derivatives first where we would have to require that the u, v in (4) above lie in a set E_x of density 1 at x. Can we get a partition using these intervals? Henstock said yes, and Thomson proved this was so, [*Henstock 1967, p.223*], [*Thomson 1980*]. This gives what we might call an approximate δ-fine partitions that can be used to give monotonicity theorems for approximate derivatives, and to define a Riemann type integral equivalent to the $\mathcal{AP}$-integral of Burkill;[*Gordon pp.245–258*], [*Bullen 1983*].
Now consider the case of symmetric derivatives: then the u, v above would have to satisfy $(u + v)/2 = x$. Again, can we get a partition using these intervals? Preiss & Thomson showed we could, defining δ-fine symmetric partitions, that he used to get a monotonicity theorem for the symmetric derivative, and to define an integral that would integrate such derivatives;[*Thomson 1994, pp.65–77, 329–330, 354–362*], [*Preiss & Thomson 1988–1989, 1989, Thomson 1989-1990*]. A totalization method

[8] Another piece of interesting "pre-history" is that the concept of a δ-fine partition is at the heart of Goursat's proof of the Cauchy integral theorem; see[*Goursat*].

for this derivative had been given earlier by Denjoy, [*Denjoy 1955*]. The relationship between these two integrals is not known; see also [*Thomson 1994, pp.336–340*].

The existence of these more general partitions, approximate and symmetric, lies much deeper — for instance the existence of approximate δ-fine partitions needs the use of the Baire Category theorem.

Now we come to a completely new result: what if we combine the two last ideas and consider approximate symmetric derivatives? Preiss & Thomson obtained partitions associated with this very general derivative and used it to give a proof of a deep monotonicity theorem. The approximate symmetric derivative had lacked a monotonicity theorem until very recently when Freiling & Rinne finally gave such a result. The work of Preiss & Thomson again showed the power of the use of partitions and gave a different proof of the Freiling & Rinne result; [*Thomson 1994, p.83*], [*Freiling & Rinne*]. The existence of these approximate symmetric partitions was then used to define a new trigonometric integral, an integral that could integrate all approximate symmetric derivatives; [*Thomson 1994. pp.362–388*], [*Preiss & Thomson 1989*].

It had been known for a long time that if f is the sum of a trigonometric series then $F'_{ap,sy} = f$, where F is the sum of the once integrated series; see [*Zygmund I, p.324*]. This approximate symmetric integral was then another trigonometric integral, the sum of any everywhere convergent trigonometric series was integrable in this sense, and the coefficients were obtained by the Fourier formulæ using this integral.

The whole field of trigonometric integrals has been clarified and unified by the use of the techniques of partition theory in the book by Thomson; [*Thomson 1994*].

7 Higher Dimensions, Stieltjes Integrals etc.

7.1 Gauss-Green and Stokes Theorems The generalized Riemann integral extends very readily to higher dimensions, and a Fubini theorem can be given; [*Henstock 1963 pp.100–112, Kurzweil 1980, Lee & Výborný pp.202–251, McLeod pp.31–3, 164–168*]. This extension suffers from the same problems as were mentioned earlier.

However, the right approach to higher dimensions was found— generalize the problem not the integral. How can we define an integral to handle the gradients etc. of differentiable fields? A very good theory exists if we assume the gradients etc. to be Lebesgue integrable, but what about the general situation? This is the natural extension of the classical primitive problem.

With this new approach, very interesting research has been done in providing an integral, using partitions — a generalized Riemann integral — that is coordinate free and which will give a very general Gauss-Green, and Stokes theorems for differentiable fields in domains with very general boundaries. The first work was done by Mawhin, [*Mawhin 1980, 1981a, 1981b*], who defined a generalized Riemann integral in which the partitions were restricted by the *regularity* of their intervals—an idea

that has its source in work of Lebesgue, see [*Saks 1937, p.106*]. Unfortunately this integral does not have some of the simple properties expected of an integral, [*Jarník, Kurzweil & Schwabik*]. Various modifications were made to Mawhin's definition to remove these drawbacks; in particular see [*Jarník & Kurzweil 1984, 1986*]. An independent approach to the whole problem was made in [*Pfeffer 1986, 1987a*]. A summary of this work can be found in [*Mawhin 1986/87, Pfeffer 1987b*], and a masterly exposition of all but the most recent work can be found in the book [*Pfeffer 1993*].
This whole exciting multi-dimensional theory is wedded to the Riemann approach, and is one of the jewels in the crown of that theory.
It turns out, rather surprisingly, that when the above theory is applied to the one-dimensional case that an integral strictly between the Lebesgue and the $\mathcal{D}^*$-integral is obtained; in fact different definitions of the same integral in higher dimensions can give different integrals in one dimension,[*Bongiorno, Giertz & Pfeffer*].

7.2 Stieltjes Integrals It is clear that the Riemann sums can be replaced by Riemann- Stieltjes sums, or even more general sums that involve point-interval functions, when sums of the form $\sum h(x_i,[a_{i-1},a_i])$ replace the more elementary $\sum f(x_i)(a_i-a_{i-1})$. The integrals obtained this way , that include the Ward Perron-Stieltjes integral, are extremely useful in unifying many areas of integration but take us too far from the main topic to be pursued here.
Ii should be noted that these integrals do allow a very simple treatment of the integration by parts theorem.

7.3 The Lebesgue Integral Again Suppose that the definition of δ-fine partition, (3), we make the following change:

$$\begin{gathered} a=a_0<\cdots<a_n=b \\ x_i-\delta(x_i)<a_{i-1}<a_i<x_i+\delta(x_i),\ 1\le i\le n. \end{gathered}$$

That is, we so not require the tag to lie in the interval it tags; in this way we get a larger class of partitions called *McShane partitions.* Using such partitions in the definition of the generalized Riemann integral results in another definition of the Lebesgue integral; see [*Gordon 1994, pp.158–168, Lee & Výborný pp.127–135, McShane 1969, 1983*], [*McShane 1973*].
Such partitions have also been used to define a generalization of the Bochner integral, see [*Gordon 1990*].
Once this idea arose it was natural to manipulate the position of the tag to see if this would lead to other integrals of interest. McShane considered integrals in which the tag is required to lies to the left of the interval it tags, [*McShane 1969*]. This leads to a stochastic integral that includes the Îto integral. Other work on stochastic integrals has been done by Muldowney; [*Henstock 1990, Muldowney 1999*]. Mention

should be made of work by Henstock and Muldowney on integrals related to the Feynman integral; [*Muldowney 1987*], [*Henstock 1991*].

7.4 A Minimal Integral Perhaps one of the more unexpected applications of the idea of McShane is to allow the tag to lie outside the interval it tags but to restrict the distance that it can lie outside. We say that f is *C-integrable* on $[a, b]$ if there exists a constant I such that for each $\epsilon > 0$ there is a gauge δ so that

$$\left|\sum_{i=1}^{p} f(x_i)(a_i - a_{i-1}) - I\right| < \epsilon$$

for each δ-fine McShane partition with $\sum_{i=1}^{p} \operatorname{dist}(x_i, [a_{i-1}, a_i]) < 1/\epsilon$.
The generalized Riemann integral defined this way is the minimal integral that integrates all Lebesgue integrable functions and all derivatives. It is strictly more general than the Lebesgue integral but less general than the $\mathcal{D}^*$-integral; [*Bongiorno 1996, 2000, Bongiorno, Di Piazza & Preiss*]. Such an integral had been first suggested in [*Bruckner, Fleissner & Foran*].

8 Measure and Variation

The one thing that the Riemann approach in a first year analysis course cannot do is allow an immediate entry into measure theory in the subsequent course. However, even here things are more interesting than this simple remark would suggest. Due to an idea of Henstock, developed in particular by Thomson, [*Thomson 1981*], there are associated with each Riemann approach a series of natural metric measures that on analysis are found to contain the basic information about the primitives— more precisely than the classical concepts of ACG^* etc. In particular the concept of variation allows yet another way of defining the various Riemann integrals. These ideas are a little too refined for full mention here, but are part of the excellent reworking of trigonometric integrals by Thomson; [*Thomson 1994*]. A simpler example is given in a recent paper of Pfeffer, [*Pfeffer 1999*]. a paper that builds on a lot of earlier work.

Examples (i) If F is any interval function and if

$$V_* F(E) = \inf_{\delta} \sup_{\varpi} \sum |F(J_i)|$$

where δ is a gauge and ϖ is a partition in I that is anchored in the set E, then V_* is a regular Borel measure.
To show the power and insight given by this idea let me end by quoting a result of Pfeffer, [*Pfeffer 1999*].

With the above notation, if V_* is AC then F is the $\mathcal{HK}$-integral of its derivative, that exists almost everywhere; further, if in addition V_* is finite then F is the $\mathcal{L}$-integral of this derivative.

9 Conclusion

So is this now the end? Well it would be foolhardy to say anything. As I indicated the end should have been a century ago but it did not turn out that way.

There are several areas where work is needed: there are problems at the end of Thomson's book,[*Thomson 1994, p.421*]; in particular Kechris has discussed the logical complexity of the classical Denjoy integrals,[*Dougherty & Kechris, Freiling 1995, 1997, Kechris*], and there remains the same work to be done for the trigonometric integrals; the theory of the Gauss-Stokes theorems is not as complete as the corresponding Lebesgue theory.

Finally I think the talks in this session will show that the next century may have as much to offer as the last. Please attend, find out, and then go home and make your contributions to this lively and surprisingly alive field.

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