THE NV-INTEGRALS

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Abstract. In 1960, Henstock defined the N and NV-integrals by an inductive process beginning by Ward integration [9]. These integrals generalize the Cessáro-Perron and the Jeffery and Miller integrals. In his book, [11], Henstock revisited these integrals but now using intervals of an "additive division space". How much has he changed the former theory? This question and some results obtained in the past are explored for the particular case of real functions on the real line.

1. INTRODUCTION

"The general solution of the equation of vibrating strings is

$$u(x,t) = f(x + vt) + g(x - vt)$$

where f and g are arbitrary functions. The word *arbitrary* is too loose since these functions must be twice continuously differentiable. If we look the solutions in the distributions sense we are led to accept functions f and g of one variable which are not necessarily differentiable but only locally summable, or even distributions of one variable. The solution so obtained has a noteworthy physical interpretation" [18].

On the other hand, A. P. Calderón and A. Zygmund [1] established pointwise estimates for solutions of elliptic partial differential equations, giving inequalities for solutions and their derivatives at isolated individual points. They also obtained results of almost everywhere type in Lebesgue sense (shortly *a.e.*). For this particular purpose, they defined generalized derivatives using certain notions of differentiability developed later by L. Gordon in [3].

The derivatives used by Gordon allow the definition of Perron style integrals, which are related to the Cessàro-Perron and also to the NV-integrals. Using increments instead of derivatives, R. Henstock defined the N-integrals in [9], where he generalized the Ward integral by using convergence factors. He further defined the N-variational integral in [10] giving an easy handling descriptive definition.

We used the definition of the N-variational integral to try to express any distribution of compact support as NV-integrals. We proved that "*Each distribution in the Schwartz sense with compact support and of order one can be expressed as an N-variational integral*". We also extended this result to all distributions of punctual support [5, 6, 7].

Using these last results it is possible to express the distributions of support of null measure, in a general case, as N-variational integrals. These results are extended to distributions with some kind of "generalized density $f^{\#}$".

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If successive derivatives of f exist in some generalized way, then, with a sequential induction and some hard work, the same result as in [5] follows. A lack of interest of possible users and some unavoidable activities of the author left this project unfinished.

The information that the NV-integrals could give to the distributions with "generalized density $f^{\#}$" would then be *a.e.* and punctual information for elements of a set of positive measure.

In his definitions of integrals, Henstock, in [9, 10], used "right and left complete families of intervals" given by a gauge. He observed, however, that one could replace these interval neighborhoods by smaller ones: the so-called L, R complete families of intervals. "For example, products of them with sets having density 1 at x, but this would be effectively the same as altering the functions N and using the neighborhoods as they stand." (cf. [9], p. 110). This was done by ourselves in cases where the families of intervals corresponding at certain single points were reduced to a sequence.

When Henstock revisited the N and NV-integrals in his book [11], he redefined the same N and NV-integrals on a division space however. Thus some of the techniques we used were not necessary any more. Nevertheless, there is some meaning in the use of convergence factors which brings step by step the difficulties that lead to the very complicated families of intervals that must be considered simultaneously if the division space is used. On the other hand, some integrals as that of L. Gordon, although contained in the Cessàro-Perron scale of integrals (being less general than that of Burkill which is only approximately continuous), cannot be defined neither by system of intervals, nor as N or NV-integrals in a simple way. But there exist N and NV-integrals (the Cessàro-Perron integral of order one, for instance) which contain it.

By a generalization of the families of intervals (which could be families of generalized sets subject to certain rules), R. Henstock, in his book [11], defined a very wide set of integrals using families of intervals called "*division systems*". These integrals include the Kurzweil-Henstock, the Denjoy-Khintchine and almost all integrals usually used. He extended, therefore, the definition of N and N-variational integrals considering these families of generalized intervals.

The sequence in this survey is as follows.

(1) An overview on how generalized integrals can give some information beyond the information of the distribution in the sense of punctual behaviors or *a.e.*. The Gordon integral and its use.
(2) The N and NV-integrals of Henstock defined in 1960 versus other integrals, in particular, the Gordon integral; the comparison between these and Henstock's new definitions (1991).
(3) Some considerations on Henstock integrals with "division system".

2. PART 1

As expressed in the introduction, when it is possible to associate some ordinary function to a Schwartz distribution by the use of an integral in such a way that the function gives information punctually or *a.e.*, some information beyond the one obtained with distributions is reached. This happens when there exists a certain "density $f^{\#}$" which allows us to define generalized integrals of the type

$$< T, \varphi >= (I) \int_{-\infty}^{\infty} \varphi f^{\#} dt,$$

with the function $f^{\#}$ giving such information as it was done in [1] by using the Gordon integral [3] and also for more general integrals

$$< T, \varphi >= (I) \int_{-\infty}^{\infty} \varphi \, df.$$

The pieces of information given in the author's papers [5, 6, 7] are more related to the theory of integration rather than to practical applications. But because the N-integrals contain the Gordon integral, our results are, in some way, related to applications. Since we also gave the integral form for distribution of support of null measure for which the results *a.e.* are obvious [6, 7], the use of such expressions may only be used in application to justify the fact that such distribution are expressed symbolically in such a form. From the point of view of the theory of integration, the integral representations of distributions of compact support may have some utility. After all, distributions satisfy theorems of Fubini-type!

On one occasion, Prof. Laurent Schwartz asked us about the reason of that work and whether it would not be the same if we used the distributions as they were. We became embarrassed since our intentions were only to use the wonderful results of Prof. L. Schwartz in the theory of integration and not the converse. After reading A. P. Calderón and Zygmund paper [1] many years later, we realized that we could have said something clever on that opportunity. But it was too late!

In the sixties, A. P. Calderón showed us Gordon's results and we told him that such an integral was contained in Henstock's integral. We also told him that the study of that particular case in such a detail was very beautiful. At that time, we did not know about A. P. Calderón's applications, which perhaps were known by everyone else in the field. So we continued our research on Schwartz distributions in a popular art form ignoring the clever applications already done by other mathematicians (maybe because our job was in other fields of a rather domestic type). But some surprises arose from that research which encouraged us to insist on the theme. The kindness and patience of Prof. R. Henstock made possible such attempts.

An integral of Perron type is defined in [3], where properties are studied and applications are given. For this purpose, some generalized derivatives are defined because "every generalization of the derivative can serve as a basis of a generalization of the Perron integral". The notion of a derivative in L^r was introduced by Calderón and Zygmund and, "unlike the idea of the approximate derivative, it has proved to be quite effective in applications (partial differential equations, area of surfaces, etc)" [3]. See also [10] for Roussel derivatives.

We add here some comments on S. Lojasievicz paper [13]. In that paper Lojasievics defined what he called the "value of a distribution in one point." This value $T(x_0) = C$ is defined as the limit

$$\lim T(x + \lambda x) \quad \text{as} \quad \lambda \to 0$$

when it exists, where $< T, \varphi >$ is a distribution in Schwartz sense. If a distribution with values in each point of its support satisfies $T(x) = f(x)$ *a.e.* and f is locally summable, then the function $f(x)$ is equal to T in the sense of distributions. Furthermore, if $T(\xi) = 0$ *a.e.*, then we have $T = 0$. If $T(\xi) = S(\xi)$ *a.e.*, then $T = S$ (i.e., a distribution is determined completely by its values, when they exist, in every point). It is possible to associate biunivocaly one distribution T with a function $f(x)$, even if the function is non-summable as, for instance,

$$f(x) = (1/|x^\alpha|)\sin(1/|x^\beta|).$$

Non-summable functions are not integrable in Denjoy's sense, but they have an integral with the Lojasiewicz definition which relates the definition of derivatives of distributions to "values in each point of its support with the correspondent measurable function of its values". See also [19]. The integral defined in this manner is contained in the NV-integral. In order to prove this assertion, one can use the more extensive definition of value of a distribution given in [13], p. 9.

2.1. **SOME DEFINITIONS OF L. GORDON PAPER.** The right-hand upper Dini derivative of f at x (denoted by f_r^{u+}, $1 \le r < \infty$) in the metric of L^r is the lower bound of α such that

$$\int_0^h [f(x+t) - f(x) - \alpha t]_+^r dt = o(h^{r+1}) \quad \text{as} \quad h \to +0.$$

If no such α exists, then it is set $f_r^{u+} = \infty$.

Similar notation and definitions are given for the lower Dini derivative. The letters u and l stand for "upper" and "lower" respectively.

When all the limits are equal, the derivative exists and it satisfies

$$\int_{-h}^h |f(x+t) - f(x) - \alpha t|^r \, dt = o(h^{r+1}) \quad \text{as} \quad h \to +0.$$

L. Gordon proved that

$$f_l^+ \le f_{ls}^+ \le f_{lr}^+ \le f_{l,app}^+ \le f_{app}^{u+} \le f_r^{u+} \le f_s^{u+}(x) \le f^{u+}$$

for all $1 \le r < s < \infty$, where $l =$ lower and $u =$ upper. Such generalized derivatives were defined in [3]. And for each one of these derivatives, a definition of majorants and minorants and the generalized Perron integral was determined. These integrals are contained in certain N-integrals (but they probably do not integrate the same functions in all cases). The Cessàro-Perron type integrals [9] of order $0 < \lambda \le 1$ are related to the Gordon integral.

3. PART 2

3.1. **THE N AND NV-INTEGRALS.** In [9], R. Henstock presented a simplified theory of the use of convergence factors in integration. Convergence factors have been already used in integration theory for many years. The theory presented in that paper was new and it included old results which were reconsidered by the new theory.

The N-integral is a generalization of the Ward integral and, as a consequence, of the Perron integral. Henstock did not use upper and lower derivatives, but increments. He defined *major* and *minor functions* (in Ward sense). He also defined the N-integral using left and right complete families of intervals and a process of Stieltjes integration.

A family L of intervals is *left complete* in $[a, b]$ if there is an $h_1(L, x) = h_1(x) > 0$, $a < x \le b$, such that every interval $(x - h, x)$ in $[a, b]$, with $0 < h \le h_1(x)$, lies in L.

A family R of intervals is *right complete* in [a,b] if there is an $h_2(R,x) = h_2(x) > 0$, $a \le x < b$, such that every interval $(x, x+h)$ in $[a,b]$, with $0 < h \le h_2$, lies in R.

Suppose a Stieltjes integration process $(N_0), \ldots, (N_{r-1}) = (S)$ is defined beginning with the Ward integration (W). Let

$$N(x,h;t) \qquad (0 \le t \le h,\ a \le x < x+h \le b)$$

and

$$N(x,-h;t) \qquad (-h \le t \le 0,\ a \le x-h < x \le b)$$

satisfy the conditions

(i) $N(x,h;h) = 1$ and $N(x,h;0) = N(x,h;0+) = 0 \quad (a \le x < x+h \le b)$,
(ii) $N(x,-h;0) = N(x,-h;0-) = 1$ and
(iii) $N(x,-h;-h) = 0 \quad (a \le x-h < x \le b)$,

where $N(x,h;t)$ and $N(x,-h;t)$ are monotone increasing in t. It is also required that

(iv) if F is an N-major function of $0, \varphi$ in $[a,b]$, then F is monotone increasing there.

In [10], Henstock defined the NV-integral and other integrals of Perron and Denjoy types and proved that they are equivalent. In order to define the NV-integral, which we deal with in our papers, Henstock introduced the concept of N-variation.

A pair $h = (h_l, h_r)$ of interval functions is of bounded *N-variation* over $[a,b]$, if there exist a left complete family L and a right complete family R of intervals in $[a,b]$ and there is a monotone increasing function χ such that

(v) $\left|(S)\int_0^h h_r(x,x+t)d_tN(x,h;t)\right| \le (S)\int_0^h\{\chi(x+t) - \chi(x)\}d_tN(x,h;t)$, $((x,x+h) \in R)$,
(vi) $\left|(S)\int_{-h}^0 h_l(x+t,x)d_tN(x,-h;t)\right| \le (S)\int_{-h}^0\{\chi(x) - \chi(x+t)\}d_tN(x,-h;t)$, $((x-h,x) \in L)$.

Thus, the N-variation of h over [a,b] is defined to be

$$V(N;h;[a,b]) = \inf\{\chi(b) - \chi(a)\}$$

for all such χ. If X is a set of real numbers we write

$$V(N;h;[a,b];X) = V(N;(h_l.ch(X), h_r.ch(X));[a,b])$$

where $ch(X)$ is the characteristic function of X. Furthermore, h is *N-variational* equivalent to 0 in $[a,b]$, if $V(N;h;[a,b]) = 0$. Then, given $\varepsilon > 0$, we can choose the χ of conditions (v) and (vi) fulfilling

(vii) $0 \le \chi(b) - \chi(a) < \varepsilon$.

Given three functions H, f, φ in that order, we say that H is *N-variational* equivalent to f, φ in $[a,b]$, if h is N-variational equivalent to 0, where

$$h_l(u,v) = H(v) - H(u) - f(v)\{\varphi(v) - \varphi(u)\},$$

$$h_r(u,v) = H(v) - H(u) - f(u)\{\varphi(v) - \varphi(u)\}.$$

The difference $H(b) - H(a)$ is called the *N-variational* integral of f, φ in $[a,b]$.

In [10], Henstock proved that the N-variational integral is equivalent to the N-integral defined in [9]. The latter generalizes the Ward integral by using convergence families $N(x,h;t)$ and $N(x,-h;t)$ as defined above.

In order to prove the uniqueness of the integrals already defined, Henstock stated Theorem 4 in [9]. But if the integrals and, in particular, the convergence factors do not satisfy the conditions of [9], Theorem 4, then it is necessary to make a new proof for the particular case. This job is rather difficult. It was done by Henstock and the author for some cases (see [5, 6, 7]). The proof of uniqueness of the integrals was improved by Henstock [11] after redefining N and NV-integrals.

3.2. **RELATIONS BETWEEN THE GORDON INTEGRAL AND THE NV- INTEGRAL.** The Gordon integral for the case $r = 1$ coincides with the NV-integral when the convergence factor corresponds to the Cessàro-Perron integration of order $r = 1$ ([10], p. 298). Therefore every other Gordon integral is contained in such NV-integral. It is interesting to ask if "there exist convergence factors which, for a given r, enable the Gordon integral of order r to be equivalent to some NV-integral". In other words, "do they integrate the same functions?"

Remark 1. *Let $F(x)$ be as in* [3]*, p. 305, that is,*

$$F(x) = (P_r)\int_a^x f(t)dt, \quad 1 \leq r < \infty,$$

where $f : [a, b] \to R$. Then the integral is contained in the integral of order 1*, P_1 which coincides with the Cessàro-Perron integral of order one and, therefore, with the NV-integral. But for each r, it looks like the answer is negative or at least difficult in general.*

Let us suppose that the P_r integral of f exists for some fixed r, $1 \leq r < \infty$. Then the primitive F belongs to L^r and $F' = f$ *a.e.* which is defined *a.e.* (see [3], p. 303). From [3], p. 296-297, we have

$$\int_0^h |F(x+t) - F(x) - f(x)t|^r \, dt =$$

$$= \int_0^h |F(x+t) - F(x) - f(x)t| \, . \, |F(x+t) - F(x) - f(t)t|^{r-1} \, dt$$

which equals $0(h^{r+1})$ as $h \to +0$ and is equivalent to

$$\int_0^h |F(x+t) - F(x) - f(x)t| \, dN(x, h, t),$$

with

$$N(x, h, t) = \frac{\int_0^t |F(x+t) - F(x) - f(t)t|^{r-1} \, dt}{\int_0^h |F(x+t) - F(x) - f(t)t|^{r-1} \, dt}, \tag{3.1}$$

and analogously to the left (see [9]).

The problem is that equation (3.1) is good for f, but the convergence factors contain f. Hence, if we apply these convergence factors to another function $g = G'$, then we obtain a primitive G in L^r that may not be a P_r primitive. It would be necessary to obtain a family $\{N\}$ not so involved with the function f. The use of functions obtained from functions in the dual of L^r leads to the same kind of problem.

3.3. **NV-INTEGRALS AND SCHWARTZ DISTRIBUTIONS.** The theory of distributions used here can be found in chapters II and III of reference [17]. But the elementary theory in the last chapter of [12] is enough, since $(\mathcal{D})$ is the vector space of complex functions $\varphi(x)$ of a real variable x with derivatives of any order and compact support and $(\mathcal{D}^m)$ is the vector space of complex functions $\varphi(x)$ with continuous derivatives up to (including) m-th order and compact support.

The basic properties of distributions used here are the equality

$$< T\,', \varphi >= - < T\ , \varphi' >$$

and the fact that *"every distribution of order $\leq m$ with compact support is a finite sum of derivatives of order $\leq m$ of measures whose supports can be taken in an arbitrary neighborhood of the support and conversely"*. Everyone who once read our papers [5, 6, 7] asked us about their purpose and possible applications. Perhaps what compelled us to that direction was the last chapter of [12] and the idea of proving properties like the Riesz representation theorem. Everything started one night. My son Daniel (2) was ill and I had to stay with him in the dark. At one moment my mind went blank and the NV-integrals appeared together with the theorem of integration by parts with the derivatives of distributions. It was then that I wrote in my mind the following theorem:

Theorem 1. *Let $< T, \varphi >$ be an absolutely continuous measure in $[a, b]$, i.e.*

$$< T, \varphi >= (L) \int_{-\infty}^{\infty} \varphi(x)\ dF(x) = (L) \int_{-\infty}^{\infty} \varphi(x)\ f(x)\ dx$$

where $\varphi \in \mathcal{D}^1$, $f(x) = F'(x)$ whenever $F'(x)$ exists, and $f(x) = 0$ otherwise. Then $V(N; \varphi(v) - \varphi(u))(f(v) - f(u); [a, b]) = 0$ (all $[a, b]$), and the distribution $< T', \varphi >$ can be expressed as an NV-integral by taking $N(x, h; t) = t/h$, for $a \leq x < b$, and $N(x, -h; t) = 1 + t/h$, for $a < x \leq b$. In other words, the derivatives in Schwartz' sense of an absolutely continuous measure of compact support can be expressed as an NV-integral ([5], Theorem 1).

Then we have

$$< T', \varphi >= - < T, \varphi' >= -(LS) \int_{-\infty}^{\infty} \varphi'(x) f(x) dx = (NV) \int_{-\infty}^{\infty} \varphi(x) df(x) \tag{3.2}$$

A long time passed until we wrote a paper containing Theorem 1 above. Actually, the first time we tried to relate distributions to NV-integrals was after we had read [10]. We expressed $< \delta', \varphi >$ as an NV-integral. But our proof of Theorem 1 was written in a rather "broken English" and Prof. Henstock could not understand it. This happened only a few months before he left Bristol. After that, we had been corresponding with one another for almost two years, and we were then able to communicate better. Finally, in 1962 he gave our research the green light. However, after the third year in Bristol, new problems started at home and things progressed very slowly because of lack of time.

At last, part of our ideas were published, after a very kind referee rewrote the "woolly" original version in proper English. We also published two more papers on distributions with compact support of null measure. Because the time was so tight, we only finished to write our work in 1973, 13 years after everything appeared in our mind.

The completed results are the material of [5, 6, 7]. The case of a general distribution was far from being solved, although we explored some problems related to this subject.

Let us connect the results in [5] with other results.

Remark 2. *Any Gordon integral P_r, $1 \leq r < s < \infty$, is contained in the Cessàro-Perron integral $P^{(1)} = P_1$ (by $P^{(1)}$, we mean the Cessàro-Perron integral of order 1, see* [9]*). In* [5]*, we used $P^{(1)}$ which is an NV-integral. Any primitive P_r, say F, has a P_r derivative a.e. which is P_r integrable. F' is also NV-integrable (P_1 integrable), but not Lebesgue integrable in general. In these cases, the P_1 integral defines a distribution which has a sort of "generalized density $F' = f^{\#}$" (*[5]*, p. 339, (8)). By taking $F(x) = P_r \int f^{\#} dx$, $F \in L^r$, it follows that*

$$P_1 \int_{-\infty}^{\infty} \varphi(x) dF(x) = NV \int_{-\infty}^{\infty} \varphi(x) f^{\#}(x) dx, \tag{3.3}$$

where $F(x)$ is P_r-differentiable with derivative $F'(x) = f^{\#}$. We see that equation (3.3) is a particular case of equation (3.2) above, provided $F(x) \in L^r$ is the function $f(x)$ in equation (3.2) and $f^{\#}$ in equation (3.3) is the generalized derivative of $f(x)$. In this particular case, the same process used in [5] *can be repeated at least one step more.*

Remark 3. *When $f(x)$ in equation (3.2) has a generalized derivative, as for instance in the case of the Cessàro-Perron class of integrals or some other NV-integral, we shall be able to continue the process.*

Remark 4. *One says that a distribution T is of order m, if T belongs to $(\mathcal{D}^m)'$, the dual of $(\mathcal{D}^m)$. The measures are distributions of order* 0*.*

Remark 5. *In the definitions given above, the Perron integral corresponds to the integral of order zero in the process of Cessàro-Perron integrals. A Gordon integral of order r, $1 \leq r \leq s < \infty$, which is not a measure, represents a distribution of class 1. The Perron integral represents a distribution of order 1, and it has an ordinary derivative a.e. which is a generalized density.*

¿From equation (3.2) above we have that, in the case of a derivative of a measure with density f, the distribution T' (the *"derivative"* of f) can be written as a generalized Stieltjes integral

$$< T', \varphi >= NV_1 \int \varphi df(x). \tag{3.4}$$

This distribution T' will be, in general, the sum of certain distributions with supports of null measure plus another distribution. In particular, if the support of T is $[a, b]$, then either $f(a) = f(b) = 0$, or there is one jump of f at each endpoint of the interval $[a, b]$. Notice that, in this case, when the distribution is the derivative of a measure but it is not a measure, the value of the distribution $- < T, \varphi'(x) >=< T', \varphi >$ is zero, whenever $\varphi(t) = 1$ on the support of T'. Hence, without loss of generality, we can study the case when the distribution T' has a support of positive measure and $f(a) = f(b) = 0$. At this stage, the subtraction of all the deltas and other distributions with support of null measure has been disregarded. The remaining summable function has discontinuities of the second class only, and it is precisely this part that will be considered in the sequential induction. For this purpose, we need a convenient type of derivative (as suggested by Ceder paper, [2]) to define $f^{\#}$ as a generalized derivative of f. Notice that f

in equation (3.4) has been defined only *a.e.* We also remind the reader that the family of intervals are a left and right-complete family of intervals (too much!). The next steps are to reduce, in a convenient way, the family of intervals, and to study certain properties of distributions with regular compact supports. Then we will be able to know when an inductive process can be defined.

A few words about distributions of compact support of null measure follow.

Remark 6. *Distributions of punctual support are treated in* [5]. *The general case is treated in* [6] *and* [7]. *In these three papers, the construction of NV-integrals of Stieltjes-type is given. The constructions are straightforward. The difficulty appears in the proofs of uniqueness of integrals defined for given convergence factors, or equivalently, in the proof of the fact that the integral of the zero function is zero. In these cases, the information a.e., which gives the integral, is obvious. Although the usual right and left families of intervals are used together with convergence factors, the novelty is that the convergence factors act as if there were less intervals corresponding to the points in the support of such distributions. This is a consequence of the very special kind of convergence factors applied.*

3.3.1. *SOME PROPERTIES OF DISTRIBUTIONS WITH REGULAR COMPACT SUPPORT.* ¿From [17], Theorem 34, we have

Theorem 2. *(i) Every distribution with regular closed support F_0 (in our case compact and convex) can be decomposed, in infinite ways, in a finite sum of derivatives of measures*

$$T = \sum_j D^p \mu_j$$

with support contained in F_0.
(ii) If $\{F_\nu\}$ is a finite or countable covering of closed sets, then every distribution with regular closed support F_0 can be decomposed, in infinite ways, in a convergent infinite sum

$$T = \sum_\nu T_\nu$$

where the support of T_ν lies in F_ν.

The idea is to relate "intervals" to closed sets F_ν. Notice that, for each interval $[u, v] \subset [a, b]$, we may introduce one jump at each end-point of the interval. Also, we need to define convenient systems of intervals.

Remark 7. *Let us take a finite partition of $[a, b]$ and the corresponding decomposition $T = \sum_1^n T_\nu$. We suppose that the NV-integral, $NV \int_a^b \varphi(t)\ df(t)$, exists. Then,*

$$T = \sum_{\nu=1}^{n} T_\nu = \sum_{\nu=1}^{n} \left(\varphi(u_\nu) f(u_\nu) - \varphi(v_\nu) f(v_\nu) + NV \int_{u_\nu}^{v_\nu} \varphi(t) df(t) \right). \tag{3.5}$$

Some terms in the sum $\sum_{\nu=1}^n$ can be canceled remaining only the first and the last ones, that is, $\varphi(a)f(a) - \varphi(b)f(b) + \sum_{\nu=1}^n NV \int_{u_\nu}^{v_\nu} \varphi(t) df(t)$.

Here, the function f is not summable in general, but it is NV-integrable. We assume that both the distribution and decomposition exist. If, in equation (3.5), the terms with integrals satisfy the condition $NV_\nu \int_u^v \varphi(t)\ df(t) = NV_\nu \int_u^v \varphi(t)\ f^\#(t)\ dt$, then the function f is the NV_ν*-integral* of $f^\#$, or else, there exists a function which is the NV-primitive of $f^\#$. In this case, we can continue the process.

The following theorem ([16], p. 297) can help us to see what kind of functions are such generalized derivatives $f^{\#}$.

Theorem 3. *If a finite function F is measurable on a set E then, at almost every point x of E, we have*
(i) either the function F is approximately differentiable,
(ii) or there exists a measurable set $R(x)$ whose right and left-hand upper densities are both equal to one at x and with respect to which the two upper unilateral derivatives of F at x are $+\infty$ and the two lower derivatives are $-\infty$.

In case (i) of Theorem 3, the process can be continued. However, in the second case (ii), some complications arise. A sketch of a possible proof follows below.

Let us suppose that we have reached some step in the induction. We apply equation (3.5), with f measurable and satisfying condition (i) of Saks theorem above, so that the approximate derivative of f (i.e $f^{\#}$) is assumed to exist *a.e.*. We also assume that the function $f^{\#}(t)$ is NV-integrable. We can take $f^{\#} = 0$ on one set of null measure [4]. The function f is $(ACG)^{\#}$ taken in the more generalized sense (f is only measurable and with discontinuities of the second class). A similar method to that applied in [11], p. 56, Theorem 1.9, can be used with the following modifications: the interval $[a, b]$ can be decomposed in a union of perfect sets P_i plus a set of null measure O. In the points of the union of the perfects sets $\cup P_i$, the approximate derivatives exist, and in O, the complement of the union of these perfect sets (we can take the function $f^{\#}$ without loss of generality) equals zero. Whenever the Baire's density theorem is used in the proof of [11], Theorem 1.9, we must suppose that either some P_i , or O has a dense portion in a subinterval of $[a, b]$. Since the set O is a G_δ, the Baire's density theorem applies (see [16], p. 54). The rest of the proof should follow as in [11], p. 56, Theorem 1.9. The case (ii), in Sacks theorem, is very complicated, and one should treat each situation according to the needs of the application considered. We hope to come back to this matter which requires some extended considerations.

The generalized derivatives are connected to the two following theorems in Ceder paper [2].

Theorem 4 (Bruckner, Ceder and Weiss)**.** *Let f be any real-valued Borel measurable function defined on a perfect set Q of the reals. Then there exists a countable set C such that, for each $x \in Q - C$, there exists a perfect set $P \subset Q$ having x as a bilateral limit point such that the restriction of f to P, $f\mid_P$, is differentiable.*

Theorem 5. *Let f be any real-valued function defined on an uncountable subset A of the reals. Then there exists a countable set C such that, for each $x \in A - C$ (*[2]*, p. 358), there exist a bilaterally dense-in-itself set B containing x such that $f\mid_B$ is monotonic and differentiable. By saying that $f \mid B$ is differentiable, it is meant that $(f\mid_B)'(x)$ exists as an extended real number for each $x \in B$.*

4. PART III

In [11], Prof. Henstock extended the concept of N and NV-integrals using Fully Decomposable Division Spaces instead of the right and left families of intervals. At fist, let us remind the reader of what is meant by Division Spaces (the theory is in [11]). We borrowed the next definitions from [15].

Let $\mathfrak{T}$ be a family of subsets of T (T being some abstract space). If $I \in \mathfrak{T}$, then I is said to be an interval of T. A set $E \subseteq T$ is an elementary set if E is an interval or a finite disjoint union of intervals. A partition of E is a finite collections of mutually disjoint intervals whose union is E. By (I,x), we mean an ordered pair where the first element indicates an interval of $\mathfrak{T}$ and $x \in T$. The dividing collections are families of pairs and we indicate them by

$$\mathcal{S} = \{(I,x)\} \qquad I \in \mathfrak{T}\ , \quad x \in T.$$

If $\mathcal{E}$ is a finite subcollection of $\mathcal{S}$ and the I are mutually disjoint with $\cup I = E$, where $(I,x) \in \mathcal{E}$, then we say that $\mathcal{S}$ divides E and that $\mathcal{E}$ is a division of E from $\mathcal{S}$. The partition $\{I;\ (I,x) \in \mathcal{E}\}$ is called a partition of E from $\mathcal{S}$.

Let $\mathfrak{A}$ be a family of dividing collections $\mathcal{S}$ such that (i) to (iv) below hold:

(i) For each elementary set $E \subseteq T$, there exists $\mathcal{S} \in \mathfrak{A}$ such that $\mathcal{S}$ divides E.

(ii) If $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ belong to $\mathfrak{A}$, both dividing an elementary set E, then there exists $\mathcal{S}^{(3)} \in \mathfrak{A}$ dividing E, with $\mathcal{S}^{(3)} \subseteq \mathcal{S}^{(1)} \cap \mathcal{S}^{(2)}$.

(iii) For each pair of disjoint elementary sets $E^{(1)}, E^{(2)}$ and each $\mathcal{S} \in \mathfrak{A}$ that divides $E^{(1)} \cup E^{(2)}$, the set $\mathcal{S}^{(1)}$ of $(I,x) \in \mathcal{S}$, with $I \subseteq E^{(1)}$, belongs to $\mathfrak{A}$ and $\mathcal{S}^{(1)}$ divides $E^{(1)}$. $\mathcal{S}^{(1)}$ is called the restriction of $\mathcal{S}$ to $E^{(1)}$.

(iv) If $E^{(1)}, E^{(2)}$ are disjoint elementary sets, with $\mathcal{S}^{(j)} \in \mathfrak{A}$ dividing $E^{(j)}$ and $I \subseteq E^{(j)}$ for each $(I,x) \in \mathcal{S}^{(j)}$, $j = 1,2$, then there exists $\mathcal{S} \in \mathfrak{A}$ dividing $E^{(1)} \cup E^{(2)}$, with $\mathcal{S} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)}$.

Any triple $(T, \mathfrak{T}\mathfrak{A})$ satisfying conditions (i) to (iv) is called a division space (see [15], p. 12-15).

At first sight, nothing has changed considering the possibility of expressing distributions as NV-integrals. This is because, as we did before, we can use special convergence factors, which are characteristic functions of sets, to obtain the same generality as when the NV-integral is defined by means of division spaces. But something new appears. When we use right and left families of intervals, there is no relationship between right and left. Now that the convergence factors are defined on generalized intervals, the factors may relate, in some way, the right with the left. This change gives the possibility of providing some relations of symmetry between the sets chosen as generalized intervals. By applying this new method, some integrals may have a different form. However, it does not seem to extend the generality of expressing distributions as integrals.

As an example, let us take

$$< T, \varphi >= Pf.(1/x).\varphi = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} (\varphi(x)/x)dx + \int_{+\varepsilon}^{\infty} (\varphi(x)/x)dx \right] = \tag{4.1}$$

$$= v.p. \int_{-\infty}^{\infty} (\varphi(x)/x)dx.$$

We define the distribution in equation (4.1) by the old method. Then we write

$$< T, \varphi >= Pf(1/x).\varphi = NV \int_{-\infty}^{\infty} \varphi d \log |x| \,,$$

which is a generalized Stieltjes integral. When applying a especial Division System such that the pairs $(I,0)$ appear only with symmetrical intervals with respect to zero, we can write

$$< T, \varphi >= \int_{-\infty}^{\infty} (\varphi(x)/x)dx.$$

Let us prove that such a System exists. Without loss of generality, we can take $T = [-1, 1]$, $\mathfrak{T}$ being the family of all subintervals $[u, v)$ of $[-1, 1]$ which contain $x = 0$ and are symmetrical with respect to zero, plus all the subintervals that neither contains $x = 0$ nor have that point as an end-point of the subintervals. The elementary sets $E \subseteq [-1, 1]$ are the intervals and all the finite disjoint unions of intervals of the family. As a consequence, neither an interval $[r, 0)$, $r < 0$, nor $[0, s)$, $s > 0$, are elementary sets. The dividing collection $\mathcal{S} \in \mathfrak{A}$ contains the pairs (I, x) given by a (δ-fine) gauge and such that $x = 0$ only appears accompanied by a symmetrical interval with center in $x = 0, [-r, r) \subseteq [-\delta(0), \delta(0))$. This system satisfies the next necessary conditions:

(i) $\mathcal{S}$ divides $[a, b]$;

(ii) in order to construct $\mathcal{S}^{(3)}$, we take, for the point $x = 0$, the intervals $[-r, r)$ contained in the intersection of a symmetrical interval $[-u, u) \in \mathcal{S}^{(1)}$ and a the symmetrical interval $[-v, v) \in \mathcal{S}^{(2)}$. For the other points, we act as usual for the systems defined by a gauge.

(iii) If the closure of E^1 does not contain the point $x = 0$, then the result is obvious. If $x = 0$ is in the closure of $E^{(1)}$, then the result follows, since the intervals $[-\alpha, 0), [0, \alpha)$ with $\alpha > 0$, are not in the collection $\mathfrak{T}$ of intervals and the intervals must be only of type $[-r, r)$.

(iv) Let $E^{(1)}, E^{(2)}$ be disjoint elementary sets with $S^{(j)} \in \mathfrak{A}$ dividing $E^{(j)}$ and each $(I, x) \subseteq E^{(j)}$, for each $(I, x) \in S^{(j)}$, $j = 1, 2$. Suppose that one of them contains $x = 0$, and the other does not. The one which contains $x = 0$ also contains an interval of type $[-r, r)$, $r > 0$. We know that the other set cannot contain an interval of type $[0, s)$ and that all its intervals will be of type $[m, n)$. The result then follows.

It can also be proved that the system is decomposable (see [15]). But we do not need this property for the proof above.

The theory about N and NV-integrals has been generalized with the new approach of Henstock in [11]. The question whether considering convergence factors plus the Division Systems would give a more general integral than the one obtained applying only Division Systems was not answered here. The counter-example we are trying to find is still a promise.

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