Geometry, Integrability and Quantization
September 1–10, 1999, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
**Coral Press**, Sofia 2000, pp 159-162

# TWISTOR INTEGRAL REPRESENTATIONS OF SOLUTIONS OF THE SUB-LAPLACIAN

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## Introduction

The twistor integral representations of solutions of the complex Laplacian on the complex space $\mathbb{C}^{2n}$ of even dimension $2n$ are well-known. We also showed them on $\mathbb{C}^{2n-1}$ of odd dimension $2n-1$ before. The purpose of this article is to generalize the results above to that of the complex sub-Laplacian on $\mathbb{C}^{2n-1}$ with the standard contact structure. The details and further discussion will appear elswhere.

Let $(x_i, y_i, z)$ $i = 1, \dots, n-1$ be the standard coordinate system of $\mathbb{M} = \mathbb{C}^{2n-1}$. We give $\mathbb{M}$ a contact structure defind by

$$\theta = \mathrm{d}z - \sum_{i=1}^{n-1} (y_i \, \mathrm{d}x_i - x_i \, \mathrm{d}y_i)$$

called a contact form. The contact distribution $D$ on $\mathbb{M}$ is defined by $\theta = 0$. The vector fields

$$X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}\,, \quad Y_i = \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z}\,, \quad i = 1, \dots, n-1$$

furnish a basis of $D$. Let us join $Z = \dfrac{\partial}{\partial z}$ to them. By $[Y_i, X_i] = 2Z;\ i = 1, \dots, n-1$ they form a basis of the Heisenberg algebra.

Let $g$ be a complex sub-Riemannian metric on $D$ such that

$$g(X_i, Y_j) = \delta_{ij},$$
$$g(X_i, X_j) = 0, \quad g(Y_i, Y_j) = 0\,.$$

Let $\mathbb{P}$ be the set of all totally null affine $(n-1)$-planes in $\mathbb{M}$ in the sense of the Heisenberg group. The space $\mathbb{P}$ is called the twistor space of $\mathbb{M}$. Either of the folloing equations represents a generic element belonging to $\mathbb{P}$:

$$\mathbb{P}_1 : \begin{cases} y_i = \sum_{j=1}^{n-1} a_{ij} x_j + b_i, \qquad a_{ij} = -a_{ji} \quad i = 1, \ldots, n-1 \\ z = \sum_{j=1}^{n-1} b_j x_j + c \\ \quad = \sum_{j=1}^{n-1} x_j y_j + c \end{cases}$$

$$\mathbb{P}_2 : \begin{cases} y_i = \sum_{j=1}^{n-1} a_{ij} x_j + b_i, \qquad a_{ij} = -a_{ji} \quad i = 1, \ldots, n-1 \\ z = -\sum_{j=1}^{n-1} b_j x_j + c \\ \quad = -\sum_{j=1}^{n-1} x_j y_j + c \end{cases}$$

Remark that each totally null affine $(n-1)$-plane is not tangent to $D$, but the projection to the $(x_i, y_i)$-space is totally null affine $(n-1)$-plane in the usual sense. We can take $(a_{ij}, b_i, c)$ as generic parameters of $\mathbb{P}$. Therefore the dimension of $\mathbb{P}$ is $\dfrac{n^2 - n + 2}{2}$. By the natural projection $(a_{ij}, b_i, c) \longmapsto (a_{ij})$, the $(a_{ij})$-space is of $\dfrac{(n-1)(n-2)}{2}$ dimension.

Let $\Box_R$, $\Box_L$ and $\Box$ be complex sub-Laplacians associated with $g$ as follows:

$$\Box_R \phi = \left( \sum_{i=1}^{n-1} Y_i X_i \right) \phi$$
$$\Box_L \phi = \left( \sum_{i=1}^{n-1} X_i Y_i \right) \phi$$
$$\Box \phi = (\Box_L + \Box_R)\phi = \sum_{i=1}^{n-1} (X_i Y_i + Y_i X_i)\phi$$

Let $f = f(a_{ij}, b_i, c)$ be a suitable analytic function on $\mathbb{P}$. Then we can define a function

$$\phi(x_i, y_i, z) = \int_{\Delta} f(a_{ij}, y_i - \sum_{j=1}^{n-1} a_{ij}x_j, z \mp \sum_{j=1}^{n-1} x_j y_j) \wedge \mathrm{d}a_{ij}$$

where $b_i = y_i - \sum_{j=1}^{n-1} a_{ij}x_j$, $c = z \mp \sum_{j=1}^{n-1} x_j y_j$, and $\wedge \, \mathrm{d}a_{ij}$ is an exterior $k$-form by any of $\mathrm{d}a_{ij}$ while $\Delta$ is a $k$-chain. The function $\phi$ on $\mathbb{M}$ is not necessarily a solution of $\Box_R$, $\Box_L$, $\Box$ for any $f$.

First, we have the following.

**Proposition 1.** *Take a form* $f = f(a_{ij}, b_i) = f(a_{ij}, b_i, \gamma)$*, where* $\gamma$ *is a constant. We have* $\phi(x_i, y_i, z) = \varphi(x_i, y_i)$*. Then we have*

$$\Box_R \phi = 0, \quad \Box_L \phi = 0.$$

These are nothing but the twistor integral representations of solutions of the complex Laplacian on $\mathbb{C}^{2n-2}$. We call them type 1 and write them as $f_1$ and $\phi_1$.

Next, we have the following.

**Proposition 2.** *Take a form* $f = f(c) = f(\alpha_{ij}, \beta_i, c)$*, where* $\alpha_{ij}$ *and* $\beta_i$ *are constants. We have* $\phi(x_i, y_i, z) = \varphi(z \mp \sum_{j=1}^{n-1} x_j y_j)$*. Then we have*

i) *for* $\phi = \varphi\left(z - \sum_{j=1}^{n-1} x_j y_j\right)$

$$X_i \phi = 0 \ (i = 1, \dots, n-1), \quad i\,e. \ \Box_R \phi = 0\,,$$

ii) *for* $\phi = \varphi\left(z + \sum_{j=1}^{n-1} x_j y_j\right)$

$$Y_i \phi = 0 \ (i = 1, \dots, n-1), \quad i\,e. \ \Box_L \phi = 0\,.$$

We call them type 2 and write them as $f_2$ and $\phi_2$.

Combining the above two propositions, we have the following.

**Theorem 1.** *Take a form*

$$f = f(a_{ij}, b_i, c) = f_1(a_{ij}, b_i) + f_2(c) = f_1 + f_2$$

*on* $\mathbb{P}_1$. *We have*

$$\phi(x_i, y_i, z) = \phi_1(x_i, y_i) + \phi_2\left(z - \sum_{j=1}^{n-1} x_j y_j\right) = \phi_1 + \phi_2$$

*on* $\mathbb{M}$. *Then we have*

$$\Box_R \phi = 0\,.$$

*Conversely, a solution* $\phi$ *of* $\Box_R \phi = 0$ *is represented by* $\phi = \phi_1 + \phi_2$ *by some* $f = f_1 + f_2$. *Similarly, from* $f = f_1 + f_2$ *on* $\mathbb{P}_2$, $\phi = \phi_1 + \phi_2$ *satisfies* $\Box_L \phi = 0$.

We embed $(a_{ij}, b_i, c, c') \in \mathbb{P}_0$ into $\mathbb{P}_1 \times \mathbb{P}_2$ as $(a_{ij}, b_i, c) \times (a_{ij}, b_i, c')$. Taking a function

$$F = F(a_{ij}, b_i, c, c') = F(c, c') = (cc')^{-\frac{n-1}{2}}$$

on $\mathbb{P}_1 \times \mathbb{P}_2$, we have

$$\Phi(x_i, y_i, z) = \mathrm{const}\left(\left(\sum_{i=1}^{n-1} x_i y_i\right)^2 - z^2\right)^{-\frac{n-1}{2}}.$$

This is the (complex) fundamental solution of $\Box$.

## References


[1] Aomoto K. and Machida Y., *Twistor Integral Representations of Fundamental Solutions of Massless Field Equations*, J. Geom. Phys. **32** (1999) 189–210.