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SEIBERG-WITTEN EQUATIONS ON $\mathbb{R}^6$

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Abstract. It is known that Seiberg-Witten equations are defined on smooth four dimensional manifolds. In the present work we write down a six dimensional analogue of these equations on $\mathbb{R}^6$. To express the first equation, the Dirac equation, we use a unitary representation of complex Clifford algebra $\mathbb{C}\mathrm{l}_{2n}$. For the second equation, a kind of self-duality concept of a two-form is needed, we make use of the decomposition $\Lambda^2(\mathbb{R}^6) = \Lambda^2_1(\mathbb{R}^6)\oplus\Lambda^2_6(\mathbb{R}^6)\oplus\Lambda^2_8(\mathbb{R}^6)$. We consider the eight-dimensional part $\Lambda^2_8(\mathbb{R}^6)$ as the space of self-dual two-forms.

1. Introduction

The Seiberg-Witten equations defined on four-dimensional manifolds yield some invariants for the underlying manifold. There are some generalizations of these equation to higher dimensionsinal manifolds. In [2, 7] some eight-dimensional analogies were given and a seven-dimensional analog was presented in [5]. In this work we write down similar equations to Seiberg-Witten equations on $\mathbb{R}^6$.

2. spin^c–structure and Dirac Operator on $\mathbb{R}^{2n}$

Definition 1. *A* spin^c*-structure on the Euclidian space* $\mathbb{R}^{2n}$ *is a pair* (S,Γ) *where* S *is a* 2^n*−dimensional complex Hermitian vector space and* $\Gamma : \mathbb{R}^{2n} \to \mathrm{End}(S)$ *is a linear map which satisfies*

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^*\Gamma(v) = |v|^2 1$$

for every $v \in \mathbb{R}^{2n}$.

The 2^n-dimensional complex vector space S is called spinor space over $\mathbb{R}^{2n}$.

From the universal property of the complex Clifford algebra $\mathbb{C}\mathrm{l}_{2n}$ the map Γ can be extended to an algebra isomorphism Γ: $\mathbb{C}\mathrm{l}_{2n} \to \mathrm{End}(S)$ which satisfies $\Gamma(\tilde{x}) =$

$\Gamma(x)^*$, where $\tilde{x}$ is conjugate of x in $\mathbb{C}l_{2n}$ and $\Gamma(x)^*$ denotes the Hermitian conjugate of $\Gamma(x)$.

If (S, Γ) is a spin^c-structure on $\mathbb{R}^{2n}$, then there is a natural splitting of the spinor space S. Let $e_1, e_2, ..., e_{2n}$ be the standard basis of $\mathbb{R}^{2n}$, define a special element ε of $\mathbb{C}l_{2n}$ by

$$\varepsilon = e_{2n}...e_2e_1.$$

Note that $\varepsilon^2 = (-1)^n$, so we can decompose S as follows

$$S = S^+ \oplus S^-$$

where $S^{\pm}$ are the eigenspaces of $\Gamma(\varepsilon)$ by

$$S^{\pm} = \{\phi \in S;\, \Gamma(\varepsilon)\phi = \pm\, \mathrm{i}^n\phi\}.$$

The space S^+ is called **positive spinor space** and the space S^- is called **negative spinor space**. The map $\Gamma(v)$ interchanges these subspaces that is, $\Gamma(v)S^+ \subset S^-$ and $\Gamma(v)S^- \subset S^+$ for each $v \in \mathbb{R}^{2n}$. The restrictions of $\Gamma(v)$ to S^+ for $v \in \mathbb{R}^{2n}$ determine a linear map $\gamma : \mathbb{R}^{2n} \to \mathrm{Hom}(S^+, S^-)$ which satisfies

$$\gamma(v)^*\gamma(v) = |v|^2 1$$

for every $v \in \mathbb{R}^{2n}$. On the other hand the map Γ: $\mathbb{R}^{2n} \to \mathrm{End}(S)$ can be recovered from γ via $S = S^+ \oplus S^-$ and

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix}.$$

If (S, Γ) is a spin^c structure on $\mathbb{R}^{2n}$, we can define an action of the space of two-forms $\Lambda^2(\mathbb{R}^{2n})$ on S as follows

First let us identify $\Lambda^2(\mathbb{R}^{2n})$ with the spaces of second order elements of Clifford algebra $C_2(\mathbb{R}^{2n})$ via the map

$$\Lambda^2(\mathbb{R}^{2n}) \to C_2(\mathbb{R}^{2n}), \qquad \eta = \sum_{i<j} \eta_{ij} e_i^* \wedge e_j^* \mapsto \sum_{i<j} \eta_{ij} e_i e_j.$$

Then we compose this map with Γ to obtain a map $\rho : \Lambda^2(\mathbb{R}^{2n}) \to \mathrm{End}(S)$

$$\rho(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j).$$

The map $\rho(\eta)$ respects the decomposition $S^+ \oplus S^-$ for each $\eta \in \Lambda^2(\mathbb{R}^{2n})$ so we can define new maps by restriction

$$\rho^{\pm}(\eta) = \rho(\eta)|_{S^{\pm}}.$$

The map ρ extends to a map

$$\rho : \Lambda^2(\mathbb{R}^{2n}) \otimes \mathbb{C} \to \mathrm{End}(S)$$

on the space of complex valued two-forms.

By using an $\mathrm{i}\mathbb{R}-$valued one-form $A \in \Omega^1(\mathbb{R}^{2n}, \mathrm{i}\mathbb{R})$ and the Levi-Civita connection ∇ on $\mathbb{R}^{2n}$ we can obtain a connection ∇^A on S which is called spinor covariant derivative operator that satisfies the relation

$$\nabla_V^A(\Gamma(W)\Psi) = \Gamma(W)\nabla_V^A\Psi + \Gamma(\nabla_V W)\Psi$$

in which Ψ is spinor, (a section of S) and V, W are vector fields on $\mathbb{R}^{2n}$. The spinor covariant derivative ∇^A respects the decomposition $S = S^+ \oplus S^-$. At this point we can define Dirac operator $D_A\colon C^\infty(\mathbb{R}^{2n}, S^+) \to C^\infty(\mathbb{R}^{2n}, S^-)$ by the formula

$$D_A(\Psi) = \sum_{i=1}^{2n} \Gamma(e_i)\nabla_{e_i}^A(\Psi).$$

3. Seiberg-Witten Equations on $\mathbb{R}^4$

The Seiberg-Witten equations constitute of two equations. The first equation is the harmonicity of the spinor with respect to the Dirac operator, that is

$$D_A(\Psi) = 0. \tag{1}$$

The second equation couples the self-dual part of the curvature two-form F_A^+ of the connection one-form A with the traceless endomorphism $(\Psi\Psi^*)_0$ associated to the spinor field Ψ. And it is expressed as

$$\rho^+(F_A^+) = (\Psi\Psi^*)_0. \tag{2}$$

Let us write these equations on $\mathbb{R}^4$. The following form of these equations can be found in many books and papers [8, 10]. The spin^c connection $\nabla = \nabla^A$ on $\mathbb{R}^4$ is given by

$$\nabla_j\Psi = \frac{\partial\Psi}{\partial x_j} + A_j\Psi$$

where $A_j : \mathbb{R}^4 \longrightarrow \mathrm{i}\mathbb{R}$ and $\Psi : \mathbb{R}^4 \longrightarrow \mathbb{C}^2$. Then the associated connection on the line bundle $L_\Gamma = \mathbb{R}^4 \times \mathbb{C}$ is the connection one-form

$$A = \sum_{i=1}^{4} A_i \,\mathrm{d}x_i \in \Omega^1\big(\mathbb{R}^4, \mathrm{i}\mathbb{R}\big)$$

and its curvature two-form is given by

$$F_A = \mathrm{d}A = \sum_{i<j} F_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j \in \Omega^2\big(\mathbb{R}^4, \mathrm{i}\mathbb{R}\big)$$

where $F_{ij} = \dfrac{\partial A_j}{\partial x_i} - \dfrac{\partial A_i}{\partial x_j}$ for $i, j = 1, ..., 4$.

Let $\Gamma : \mathbb{R}^4 \longrightarrow \mathrm{End}(\mathbb{C}^4)$ be the classical spin^c structure which is given by the map

$$\Gamma(w) = \begin{bmatrix} 0 & \gamma(w) \\ -\gamma(w)^* & 0 \end{bmatrix}$$

where $\gamma\colon \mathbb{R}^4 \longrightarrow \mathrm{End}(\mathbb{C}^2)$ is defined on the generators e_1, e_2, e_3, e_4 by the following rule

$$\gamma(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma(e_2) = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \quad \gamma(e_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma(e_4) = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

Note that in the definition of γ, the 2×2 identity matrix and i multiplies the well-known Pauli matrices $\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}$ and $\sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The classical spin^c-structure has been used, in many works (see for instance [8–10]).

Note that $\Gamma(e_4e_3e_2e_1) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and the eigenspaces of $\Gamma(e_4e_3e_2e_1)$ are

$$\begin{aligned} S^+ &= \{(\psi_1, \psi_2, 0, 0)\ ;\ \psi_1, \psi_2 \in \mathbb{C}\} \\ S^- &= \{(0, 0, \psi_3, \psi_4)\ ;\ \psi_3, \psi_4 \in \mathbb{C}\}. \end{aligned}$$

The corresponding vector bundles which are called half spinor bundles on the manifold $\mathbb{R}^4$ are $\mathbb{S}^+ = \mathbb{R}^4 \times S^+$ and $\mathbb{S}^- = \mathbb{R}^4 \times S^-$. The sections of these bundles are called spinor fields on $\mathbb{R}^4$ and we will denote them by

$$\begin{aligned} \Gamma(\mathbb{S}^+) &= \left\{(\psi_1, \psi_2, 0, 0)\ ;\ \psi_1, \psi_2 \in C^\infty\left(\mathbb{R}^4, \mathbb{C}\right)\right\} \\ \Gamma(\mathbb{S}^-) &= \left\{(0, 0, \psi_3, \psi_4)\ ;\ \psi_3, \psi_4 \in C^\infty\left(\mathbb{R}^4, \mathbb{C}\right)\right\}. \end{aligned}$$

According to the above data Seiberg-Witten equations on $\mathbb{R}^4$, i.e., equations (1) and (2), are as follows (see [9, 10])

The first of these equations, $D_A\Psi = 0$, can be expressed as

$$-\nabla_1\Psi + \mathrm{i}\sigma_1\nabla_2\Psi + \mathrm{i}\sigma_2\nabla_3\Psi + \mathrm{i}\sigma_3\nabla_4\Psi = 0$$

or more explicitly

$$\begin{aligned} \frac{\partial\psi_1}{\partial x_1} + A_1\psi_1 &= \mathrm{i}\left(\frac{\partial\psi_1}{\partial x_2} + A_2\psi_1\right) + \left(\frac{\partial\psi_2}{\partial x_3} + A_3\psi_2\right) + \mathrm{i}\left(\frac{\partial\psi_2}{\partial x_4} + A_4\psi_2\right) \\ \frac{\partial\psi_2}{\partial x_1} + A_1\psi_2 &= -\mathrm{i}\left(\frac{\partial\psi_2}{\partial x_2} + A_2\psi_2\right) - \left(\frac{\partial\psi_1}{\partial x_3} + A_3\psi_1\right) + \mathrm{i}\left(\frac{\partial\psi_1}{\partial x_4} + A_4\psi_1\right) \end{aligned} \tag{3}$$

where $\Psi = (\psi_1, \psi_2, 0, 0)$. The second one is

$$\rho^+\left(F_A^+\right) = (\Psi\Psi^*)_0$$

which can be expressed explicitly as

$$\begin{aligned} F_{12}+F_{34} &= -\tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_1-\psi_2\overline{\psi}_2\right) \\ F_{13}-F_{24} &= \tfrac{1}{2}\left(\psi_1\overline{\psi}_2-\psi_2\overline{\psi}_1\right) \\ F_{14}+F_{23} &= -\tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_2+\psi_2\overline{\psi}_1\right) \end{aligned} \tag{4}$$

where $F_A=\mathrm{d}A$.

4. Seiberg-Witten Equations on $\mathbb{R}^6$

The first one in Seiberg-Witten equations can be written on any $2n-$ dimensional spin^c manifold. But the second one is meaningful in four-dimensional cases. Because the self duality of a two-form in Hodge sense is meaningful in four-dimension. On the other hand there are some various generalizations of self-duality concept of a two-form to higher dimensions (see [3,4]).

4.1. The First Equation: Dirac Equation

The main objective of the present work is to write down Seiberg-Witten like equations on $\mathbb{R}^6$. In order to achieve this we consider the following spin^c-structure Γ on $\mathbb{R}^6$ which is coming from the representation of the complex Clifford algebra $\mathbb{C}\mathrm{l}_6$. Let $\Gamma:\mathbb{R}^6\longrightarrow\mathrm{End}(\mathbb{C}^8)$ be the spin^c structure which is given by

$$\Gamma(w)=\begin{bmatrix} 0 & \gamma(w) \\ -\gamma(w)^* & 0 \end{bmatrix}$$

where $\gamma:\mathbb{R}^6\longrightarrow\mathrm{End}(\mathbb{C}^4)$ is defined on generators e_1,e_2,e_3,e_4,e_5,e_6 as follow

$$\gamma(e_1)=\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix},\quad \gamma(e_2)=\begin{bmatrix} \mathrm{i}&0&0&0\\ 0&-\mathrm{i}&0&0\\ 0&0&\mathrm{i}&0\\ 0&0&0&-\mathrm{i} \end{bmatrix},\quad \gamma(e_3)=\begin{bmatrix} 0&\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&\mathrm{i}\\ 0&0&\mathrm{i}&0 \end{bmatrix}$$

$$\gamma(e_4)=\begin{bmatrix} 0&1&0&0\\ -1&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{bmatrix},\ \gamma(e_5)=\begin{bmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&1&0&0\\ -1&0&0&0 \end{bmatrix},\ \gamma(e_6)=\begin{bmatrix} 0&0&0&\mathrm{i}\\ 0&0&-\mathrm{i}&0\\ 0&-\mathrm{i}&0&0\\ \mathrm{i}&0&0&0 \end{bmatrix}.$$

Then the special element $\varepsilon = e_6 \ldots e_2 e_1$ in $\mathbb{C}l_6$ satisfies $\varepsilon^2 = -1$ and its image under Γ is

$$\Gamma(\varepsilon) = \begin{bmatrix} -\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i} \end{bmatrix}.$$

The decomposition $S = \mathbb{C}^8 = S^+ \oplus S^-$ with respect to $\Gamma(\varepsilon)$ is given by

$$S^+ = \{(\psi_1, \psi_2, \psi_3, \psi_4, 0, 0, 0, 0)\,;\ \psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{C}\}$$

and

$$S^- = \{(0, 0, 0, 0, \psi_5, \psi_6, \psi_7, \psi_8)\,;\ \psi_5, \psi_6, \psi_7, \psi_8 \in \mathbb{C}\}.$$

These spaces give the following vector bundles on the manifold $\mathbb{R}^6$

$$\mathbb{S}^+ = \mathbb{R}^6 \times S^+ \quad \text{and} \quad \mathbb{S}^- = \mathbb{R}^6 \times S^-.$$

The sections of these bundles can be interpreted as follows

$$\Gamma(\mathbb{S}^+) = \{(\psi_1, \psi_2, \psi_3, \psi_4, 0, 0, 0, 0) | \psi_1, \psi_2, \psi_3, \psi_4 \in C^\infty\left(\mathbb{R}^6, \mathbb{C}\right)\}$$

and

$$\Gamma(\mathbb{S}^-) = \{(0, 0, 0, 0, \psi_5, \psi_6, \psi_7, \psi_8) | \psi_5, \psi_6, \psi_7, \psi_8 \in C^\infty\left(\mathbb{R}^6, \mathbb{C}\right)\}.$$

The spin^c connection ∇^A on $\mathbb{R}^6$ is given by

$$\nabla_j^A \Psi = \frac{\partial \Psi}{\partial x_j} + A_j \Psi$$

where $A_j : \mathbb{R}^6 \longrightarrow \mathrm{i}\mathbb{R}$ and $\Psi : \mathbb{R}^6 \longrightarrow \mathbb{C}^4$ are smooth maps. Then the associated connection on the line bundle $L_\Gamma = \mathbb{R}^6 \times \mathbb{C}$ is the connection one-form

$$A = \sum_{i=1}^{4} A_i \, \mathrm{d}x_i \in \Omega^1\left(\mathbb{R}^6, \mathrm{i}\mathbb{R}\right)$$

and its curvature two-form is given by

$$F_A = \mathrm{d}A = \sum_{i<j} F_{ij} \, \mathrm{d}x_i \wedge \mathrm{d}x_j \in \Omega^2\left(\mathbb{R}^6, \mathrm{i}\mathbb{R}\right)$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ for $i, j = 1, \ldots, 6$. Now we can write the Dirac operator $D_A : \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ on $\mathbb{R}^6$ with respect to given spin^c-structure Γ and spin^c-connection ∇^A as follows

$$D_A\Psi = \sum_{i=1}^{6} e_i \cdot \nabla^A_{e_i}\Psi = \sum_{i=1}^{6} \Gamma(e_i)\left(\nabla^A_{e_i}\Psi\right) = \sum_{i=1}^{6} \Gamma(e_i) \begin{pmatrix} \frac{\partial \psi_1}{\partial x_i} + A_i\psi_1 \\ \frac{\partial \psi_2}{\partial x_i} + A_i\psi_2 \\ \vdots \\ \frac{\partial \psi_4}{\partial x_i} + A_i\psi_4 \end{pmatrix}.$$

Introducing the notation $\nabla_i = \partial_i + A_i,\ i = 1, \ldots, 6$ the equation $D_A\Psi = 0$ can be expressed as

$$\begin{aligned}
\nabla_1\psi_1 &= \mathrm{i}\nabla_2\psi_1 + \mathrm{i}\nabla_3\psi_2 + \nabla_4\psi_2 + \nabla_5\psi_4 + \mathrm{i}\nabla_6\psi_4 \\
\nabla_1\psi_2 &= \mathrm{i}\nabla_3\psi_1 - \mathrm{i}\nabla_2\psi_2 - \nabla_4\psi_1 - \nabla_5\psi_3 - \mathrm{i}\nabla_6\psi_3 \\
\nabla_1\psi_3 &= \mathrm{i}\nabla_2\psi_3 + \mathrm{i}\nabla_3\psi_4 - \nabla_4\psi_4 + \nabla_5\psi_2 - \mathrm{i}\nabla_6\psi_2 \\
\nabla_1\psi_4 &= \mathrm{i}\nabla_3\psi_3 - \mathrm{i}\nabla_2\psi_4 + \nabla_4\psi_3 - \nabla_5\psi_1 + \mathrm{i}\nabla_6\psi_1.
\end{aligned}$$

4.2. The Second Equation: Curvature Equation

Now we want to define the second Seiberg-Witten equation on $\mathbb{R}^6$. To achieve this we need a kind of self-duality notion for two-forms on $\mathbb{R}^6$. Let us consider the following decompositions of two-forms on $\mathbb{R}^6$. We denote by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ the standard basis of $\mathbb{R}^6$ and by $\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4, \mathrm{d}x_5, \mathrm{d}x_6\}$ the dual one. Fix the standard symplectic form

$$\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6$$

and the standard complex volume form

$$\varphi_0 = (\mathrm{d}x_1 + \mathrm{i}\mathrm{d}x_2) \wedge (\mathrm{d}x_3 + \mathrm{i}\mathrm{d}x_4) \wedge (\mathrm{d}x_5 + \mathrm{i}\mathrm{d}x_6)$$

the complex structure J_0 give by

$$J_0(e_1) = e_2, \qquad J_0(e_3) = e_4, \qquad J_0(e_5) = e_6$$

on $\mathbb{R}^6$. The space of two-forms $\Lambda^2(\mathbb{R}^6)$ decomposes as follows

$$\Lambda^2(\mathbb{R}^6) = \Lambda^2_1(\mathbb{R}^6) \oplus \Lambda^2_6(\mathbb{R}^6) \oplus \Lambda^2_8(\mathbb{R}^6)$$

where

$$\begin{aligned}
\Lambda^2_1(\mathbb{R}^6) &= \{r\omega_0\ ;\ r \in \mathbb{R}\}, \qquad \Lambda^2_6(\mathbb{R}^6) = \{F \in \Lambda^2(\mathbb{R}^6\ ;\ J_0(F) = -F\} \\
\Lambda^2_8(\mathbb{R}^6) &= \{F \in \Lambda^2(\mathbb{R}^6)\ ;\ J_0(F) = F \text{ and } F \wedge \omega_0 \wedge \omega_0 = 0\}.
\end{aligned}$$

For more details see [1]. Then any two-form $F = \sum_{i<j} F_{ij}\, \mathrm{d}x_i \wedge\, \mathrm{d}x_j \in \Lambda^2(\mathbb{R}^6)$ can be decomposed into three parts, we call the one belonging to $\Lambda^2_8(\mathbb{R}^6)$ is the

self-dual part of F and we denote it by F^+. Such a self-duality definition of two-forms in six-dimension is consistent with the widely accepted self-duality notion given in [4]. In their work Corrigan *et al.* consider the eight-dimensional subspace of $\Lambda^2(\mathbb{R}^6)$, given by the following set of equations, as the space of self-dual two-forms on $\mathbb{R}^6$

$$\begin{aligned} &F_{12} + F_{14} + F_{15} = 0, \quad F_{13} - F_{24} = 0, \quad F_{14} + F_{23} = 0 \\ &\quad F_{15} - F_{26} = 0, \quad F_{16} + F_{25}7 = 0, \quad F_{35} - F_{46} = 0, \quad F_{36} + F_{45} = 0. \end{aligned}$$

This eight-dimensional subspace exactly corresponds to $\Lambda^2_8(\mathbb{R}^6)$, because following linearly independent set of vectors belong to both of them

$$\begin{aligned} f_1 &= e_1 \wedge e_3 + e_2 \wedge e_4, & f_5 &= e_3 \wedge e_5 + e_4 \wedge e_6 \\ f_2 &= e_1 \wedge e_4 - e_2 \wedge e_3, & f_6 &= e_3 \wedge e_6 - e_4 \wedge e_5 \\ f_3 &= e_1 \wedge e_5 + e_2 \wedge e_6, & f_7 &= e_1 \wedge e_2 - e_3 \wedge e_4 \\ f_4 &= e_1 \wedge e_6 - e_2 \wedge e_5, & f_8 &= e_3 \wedge e_4 - e_5 \wedge e_6. \end{aligned}$$

Now let us consider the complexified space $\Lambda^2_8(\mathbb{R}^6) \otimes \mathbb{C}$ and F_A be the curvature form of the imaginary valued connection one-form A and F_A^+ be the self-dual part of F_A. Then

$$\begin{aligned} F_A^+ = \frac{1}{2}\sum_{i=1}^{8} < F_A, f_i > f_i = \frac{1}{2}[&(F_{13} + F_{24})f_1 + (F_{14} - F_{23})f_2 + (F_{15} + F_{26})f_3 \\ &+(F_{16} - F_{25})f_4 + (F_{35} + F_{46})f_5 + (F_{36} - F_{45})f_6 \\ &+(F_{12} - F_{34})f_7 + (F_{34} - F_{56})f_8]. \end{aligned}$$

The image of F_A^+ under ρ^+ is

$$\begin{aligned} \rho^+(F_A^+) \;=\; &\frac{1}{2}[(F_{13} + F_{24})\rho^+(f_1) + (F_{14} - F_{23})\rho^+(f_2) + (F_{15} + F_{26})\rho^+(f_3) \\ &+(F_{16} - F_{25})\rho^+(f_4) + (F_{35} + F_{46})\rho^+(f_5) + (F_{36} - F_{45}\rho^+(f_6) \\ &+(F_{12} - F_{34})\rho^+(f_7) + (F_{34} - F_{56})\rho^+(f_8)] \end{aligned}$$

where

$$\rho^+(f_1) = \begin{pmatrix} 0 & 2i & 0 & 0 \\ 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^+(f_2) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^+(f_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^+(f_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\mathrm{i} & 0 \\ 0 & -2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^+(f_5) = \begin{pmatrix} 0 & 0 & -2\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -2\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^+(f_6) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^+(f_7) = \begin{pmatrix} 2\mathrm{i} & 0 & 0 & 0 \\ 0 & -2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \rho^+(f_8) = \begin{pmatrix} -2\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the second equation on $\mathbb{R}^6$ is

$$\rho^+(F_A^+) = (\Psi\Psi^*)_0 \,. \tag{5}$$

The last equation is rather different from the second Seiberg-Witten equation on $\mathbb{R}^4$ and we state it as a theorem:

Theorem 1. *If the pair (A, Ψ) is a solution to (5) then $\Psi = 0$.*

Proof: The left hand side of (5) is

$$\begin{pmatrix} \mathrm{i}(F_{12}-F_{34}) - \mathrm{i}(F_{34}-F_{56}) & \mathrm{i}(F_{13}+F_{24}) + (F_{14}-F_{23}) & -\mathrm{i}(F_{35}+F_{46}) + (F_{36}-F_{45}) & 0 \\ \mathrm{i}(F_{13}+F_{24}) - (F_{14}-F_{23}) & -\mathrm{i}(F_{12}-F_{34}) & -(F_{15}+F_{26}) - \mathrm{i}(F_{16}-F_{25}) & 0 \\ -\mathrm{i}(F_{35}+F_{46}) - (F_{36}-F_{45}) & (F_{15}+F_{26}) - \mathrm{i}(F_{16}-F_{25}) & \mathrm{i}(F_{34}-F_{56}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the endomorphism $\Psi\Psi^*$ of $\mathbb{C}^4$ is given by

$$\Psi\Psi^* = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \begin{pmatrix} \overline{\psi}_1 & \overline{\psi}_2 & \overline{\psi}_3 & \overline{\psi}_4 \end{pmatrix} = \begin{pmatrix} \psi_1\overline{\psi}_1 & \psi_1\overline{\psi}_2 & \psi_1\overline{\psi}_3 & \psi_1\overline{\psi}_4 \\ \psi_2\overline{\psi}_1 & \psi_2\overline{\psi}_2 & \psi_2\overline{\psi}_3 & \psi_2\overline{\psi}_4 \\ \psi_3\overline{\psi}_1 & \psi_3\overline{\psi}_2 & \psi_3\overline{\psi}_3 & \psi_3\overline{\psi}_4 \\ \psi_4\overline{\psi}_1 & \psi_4\overline{\psi}_2 & \psi_4\overline{\psi}_3 & \psi_4\overline{\psi}_4 \end{pmatrix}.$$

The trace free part of $\Psi\Psi^*$ is

$$(\Psi\Psi^*)_0 = \begin{pmatrix} \psi_1\overline{\psi}_1 & \psi_1\overline{\psi}_2 & \psi_1\overline{\psi}_3 & \psi_1\overline{\psi}_4 \\ \psi_2\overline{\psi}_1 & \psi_2\overline{\psi}_2 & \psi_2\overline{\psi}_3 & \psi_2\overline{\psi}_4 \\ \psi_3\overline{\psi}_1 & \psi_3\overline{\psi}_2 & \psi_3\overline{\psi}_3 & \psi_3\overline{\psi}_4 \\ \psi_4\overline{\psi}_1 & \psi_4\overline{\psi}_2 & \psi_4\overline{\psi}_3 & \psi_4\overline{\psi}_4 \end{pmatrix} - \frac{1}{4}|\psi|^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi_1\overline{\psi}_1 - \frac{1}{4}|\psi|^2 & \psi_1\overline{\psi}_2 & \psi_1\overline{\psi}_3 & \psi_1\overline{\psi}_4 \\ \psi_2\overline{\psi}_1 & \psi_2\overline{\psi}_2 - \frac{1}{4}|\psi|^2 & \psi_2\overline{\psi}_3 & \psi_2\overline{\psi}_4 \\ \psi_3\overline{\psi}_1 & \psi_3\overline{\psi}_2 & \psi_3\overline{\psi}_3 - \frac{1}{4}|\psi|^2 & \psi_3\overline{\psi}_4 \\ \psi_4\overline{\psi}_1 & \psi_4\overline{\psi}_2 & \psi_4\overline{\psi}_3 & \psi_4\overline{\psi}_4 - \frac{1}{4}|\psi|^2 \end{pmatrix}.$$

Then the equation (5) turns to the following set of equations

$$\begin{aligned}
&(F_{12} - F_{34}) = \mathrm{i}\left(\tfrac{3}{4}|\psi_2|^2 - |\psi_1|^2 - |\psi_3|^2 - |\psi_4|^2\right) \\
&(F_{34} - F_{56}) = -\mathrm{i}\left(\tfrac{3}{4}|\psi_3|^2 - |\psi_1|^2 - |\psi_2|^2 - |\psi_4|^2\right) \\
&(F_{13} + F_{24}) = -\tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_2 + \psi_2\overline{\psi}_1\right) \\
&(F_{14} - F_{23}) = \tfrac{1}{2}\left(\psi_1\overline{\psi}_2 - \psi_2\overline{\psi}_1\right) \\
&(F_{35} + F_{46}) = \tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_3 + \psi_3\overline{\psi}_1\right) \\
&(F_{36} - F_{45}) = \tfrac{1}{2}\left(\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1\right) \\
&(F_{16} - F_{25}) = \tfrac{\mathrm{i}}{2}\left(\psi_2\overline{\psi}_3 + \psi_3\overline{\psi}_2\right) \\
&(F_{15} + F_{26}) = \tfrac{1}{2}\left(-\psi_2\overline{\psi}_3 + \psi_3\overline{\psi}_2\right) \\
&\tfrac{3}{4}|\psi_4|^2 - |\psi_1|^2 - |\psi_2|^2 - |\psi_3|^2 = 0 \\
&\psi_4 = 0.
\end{aligned}$$

From these equations it is clear that $\Psi \equiv 0$. □

Due to the above theorem the equation (5) needs some modification. To do this we follow the method given in [2]. Firstly we consider the space of self-dual complex valued two-forms $\Lambda^2_8(\mathbb{R}^6) \otimes \mathbb{C}$. The image of this space under the map ρ^+ is a subspace of $\mathrm{End}(S^+)$ and denote it by W i.e.,

$$W = \rho^+(\Lambda^2_8(\mathbb{R}^6) \otimes \mathbb{C}).$$

The set of endomorphisms $\{\rho^+(f_1), \rho^+(f_2), \ldots, \rho^+(f_8)\}$ is a basis for the subspace W. Project the endomorphism $\Psi\Psi^*$ onto the subspace W and denote it by $(\Psi\Psi^*)^+$. Then we can explain the second equation

$$\rho^+(F_A^+) = (\Psi\Psi^*)^+.$$

Let us obtain the explicit form of last equation. The projection of the endomorphism $\Psi\Psi^*$ onto the subspace W is given by

$$(\Psi\Psi^*)^+ = \sum_{i=1}^{8} < \rho^+(f_i), \Psi\Psi^* > \frac{\rho^+(f_i)}{|\rho^+(f_i)|^2}.$$

Then the equation $\rho^+(F_A^+) = (\Psi\Psi^*)^+$ turns to the following set of equations

$$\begin{aligned}
F_{12} - F_{34} &= \tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_1 - \psi_2\overline{\psi}_2\right)\\
F_{34} - F_{56} &= -\tfrac{\mathrm{i}}{2}\left(\psi_1\overline{\psi}_1 - \psi_3\overline{\psi}_3\right)\\
F_{13} + F_{24} &= -\tfrac{\mathrm{i}}{2}\left(\psi_2\overline{\psi}_1 + \psi_1\overline{\psi}_2\right)\\
F_{14} - F_{23} &= \tfrac{1}{2}\left(\psi_1\overline{\psi}_2 - \psi_2\overline{\psi}_1\right)\\
F_{35} + F_{46} &= -\tfrac{\mathrm{i}}{2}\left(\psi_3\overline{\psi}_1 + \psi_1\overline{\psi}_3\right)\\
F_{36} - F_{45} &= \tfrac{1}{2}\left(\psi_1\overline{\psi}_3 - \psi_3\overline{\psi}_1\right)\\
F_{16} - F_{25} &= -\tfrac{\mathrm{i}}{2}\left(\psi_3\overline{\psi}_2 + \psi_2\overline{\psi}_3\right)\\
F_{15} + F_{26} &= \tfrac{1}{2}\left(\psi_3\overline{\psi}_2 - \psi_2\overline{\psi}_3\right)
\end{aligned}$$

and they are very similar to the set of equations in (4).

Remark 1. *We have written down Seiberg-Witten like equations on* $\mathbb{R}^6$ *and we observed that these equations are similar to the Seiberg-Witten equations on* $\mathbb{R}^4$*. For the expression of the first equation on* $\mathbb{R}^6$ *we used the* spin^c*-structure on* $\mathbb{R}^6$ *and for the second equation we used the decomposition of the space of two-forms* $\Lambda^2(\mathbb{R}^6)$*. Such equations can be also defined on six-dimensional manifolds with* $\mathrm{SU}(3)$ *structure and it is a subject of a subsequence work* [6].

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