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MODULAR FORMS ON BALL QUOTIENTS OF NON-POSITIVE KODAIRA DIMENSION*

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Abstract. The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of an arithmetic ball quotient admits projective embeddings by Γ-modular forms of sufficiently large weight. We are interested in the target and the rank of the projective map Φ, determined by Γ-modular forms of weight one. This paper concentrates on the finite H-Galois quotients $\mathbb{B}/\Gamma_H$ of a specific $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, birational to an abelian surface A_{-1}. Any compactification of $\mathbb{B}/\Gamma_H$ has non-positive Kodaira dimension. The rational maps Φ^H of $\widehat{\mathbb{B}/\Gamma_H}$ are studied by means of the H-invariant abelian functions on A_{-1}.

CONTENTS

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1. Introduction

The modular forms of sufficiently large weight are known to provide projective embeddings of the arithmetic quotients of the **two-ball**

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 ; |z_1|^2 + |z_2|^2 < 1\} \simeq \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}_2 \times \mathrm{U}_1).$$

The present work studies the projective maps, given by the modular forms of weight one on certain Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_H}$ of Kodaira dimension $\kappa(\widehat{\mathbb{B}/\Gamma_H}) \leq 0$. More precisely, we start with a fixed smooth Picard modular surface $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ with abelian minimal model $A_{-1} = E_{-1} \times E_{-1}$, $E_{-1} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\mathrm{i}$. Any automorphism group of A'_{-1}, preserving the toroidal compactifying divisor $T' = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)' \setminus \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)$ acts on A_{-1} and lifts to a ball lattice Γ_H, normalizing $\Gamma_{-1}^{(6,8)}$. The ball quotient compactification $A'_{-1}/H = \overline{\mathbb{B}/\Gamma_H}$ is birational to A_{-1}/H. We study the Γ_H-modular forms $[\Gamma_H, 1]$ of weight one by realizing them as H-invariants of $[\Gamma_{-1}^{(6,8)}, 1]$. That allows to transfer $[\Gamma_H, 1]$ to the H-invariant abelian functions, in order to determine $\dim_{\mathbb{C}}[\Gamma_H, 1]$ and the transcendence dimension of the graded $\mathbb{C}$-algebra, generated by $[\Gamma_H, 1]$. The last one is exactly the rank of the projective map $\Phi : \widetilde{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$.

2. The Transfer of Modular Forms to Meromorphic Functions is Inherited by the Finite Galois Quotients

Definition 1. *Let* $\Gamma < \mathrm{SU}(2,1)$ *be a lattice, i.e., a discrete subgroup, whose quotient* $\mathrm{SU}(2,1)/\Gamma$ *has finite invariant measure. A* Γ*-modular form of weight* n *is a holomorphic function* $\delta : \mathbb{B} \to \mathbb{C}$ *with transformation law*

$$\gamma(\delta)(z) = \delta(\gamma(z)) = [\det \mathrm{Jac}(\gamma)]^{-n}\delta(z), \qquad \gamma \in \Gamma, \quad z \in \mathbb{B}.$$

Bearing in mind that a biholomorphism $\gamma \in \mathrm{Aut}(\mathbb{B})$ acts on a differential form $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ of top degree as a multiplication by the Jacobian determinant $\det \mathrm{Jac}(\gamma)$, one constructs the linear isomorphism

$$j_n : [\Gamma, n] \longrightarrow H^0(\mathbb{B}, (\Omega^2_{\mathbb{B}})^{\otimes n})^{\Gamma}$$

with the Γ-invariant holomorphic sections of the canonical bundle $\Omega^2_{\mathbb{B}}$ of $\mathbb{B}$. Thus, the graded $\mathbb{C}$-algebra of the Γ-modular forms can be viewed as the tensor algebra of the Γ-invariant volume forms on $\mathbb{B}$. For any $\delta_1, \delta_2 \in [\Gamma, n]$ the quotient $\frac{\delta_1}{\delta_2}$ is a correctly defined holomorphic function on $\mathbb{B}/\Gamma$. In such a way, $[\Gamma, n]$ and $j_n[\Gamma, n]$ determine a projective map

$$\Phi_n : \mathbb{B}/\Gamma \longrightarrow \mathbb{P}([\Gamma, n]) = \mathbb{P}(j_n[\Gamma, n]).$$

The Γ-cusps $\partial_\Gamma \mathbb{B}/\Gamma$ are of complex co-dimension two, so that Φ_n extends to the Baily-Borel compactification

$$\Phi_n : \widehat{\mathbb{B}/\Gamma} \longrightarrow \mathbb{P}([\Gamma, n]).$$

If the lattice $\Gamma < \mathrm{SU}_{2,1}$ is torsion-free then the toroidal compactification $X' = (\mathbb{B}/\Gamma)'$ is a smooth surface. Denote by $\rho : X' = (\mathbb{B}/\Gamma)' \rightarrow \widehat{X} = \widehat{\mathbb{B}/\Gamma}$ the contraction of the irreducible components T'_i of the toroidal compactifying divisor T' to the Γ-cusps $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$. The tensor product $\Omega^2_{X'}(T')$ of the canonical bundle $\Omega^2_{X'}$ of X' with the holomorphic line bundle $\mathcal{O}(T')$, associated with the toroidal compactifying divisor T' is the logarithmic canonical bundle of X'. In [2] Hemperly has observes that

$$H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_n[\Gamma, n] \simeq [\Gamma, n].$$

Let $K_{X'}$ be the canonical divisor of X'

$$\mathcal{L}_{X'}(nK_{X'} + nT') = \{f \in \quad (X'); \ (f) + nK_{X'} + nT' \geq 0\}$$

be the linear system of the divisor $n(K_{X'} + T')$ and s be a global meromorphic section of $\Omega^2_{X'}(T')$. Then

$$s^{\otimes n} : \mathcal{L}_{X'}(nK_{X'} + nT') \longrightarrow H^0(X', \Omega^2_{X'}(T')^{\otimes n})$$

is a $\mathbb{C}$-linear isomorphism. Let $\xi : X' \rightarrow X$ be the blow-down of the (-1)-curves on $X' = (\mathbb{B}/\Gamma)'$ to its minimal model X. The Kobayashi hyperbolicity of $\mathbb{B}$ requires X' to be the blow-up of X at the singular locus T^{sing} of $T = \xi(T')$. The canonical divisor $K_{X'} = \xi^* K_X + L$ is the sum of the pull-back of K_X with the exceptional divisor L of ξ. The birational map ξ induces an isomorphism $\xi^* : \quad (X) \rightarrow \quad (X')$ of the meromorphic function fields. In order to translate the condition $\xi^*(f) + nK_{X'} + nT' \geq 0$ in terms of $f \in \quad (X)$, let us recall the notion of a multiplicity of a divisor $D \subset X$ at a point $p \in X$. If $D = \sum\limits_i n_i D_i$ is the decomposition of D into irreducible components then $m_p(D) = \sum\limits_i n_i m_p(D_i)$, where

$$m_p(D_i) = \begin{cases} 1 & \text{for } p \in D_i \\ 0 & \text{for } p \notin D_i. \end{cases}$$

Let $L = \sum\limits_{p \in T^{\mathrm{sing}}} L(p)$ for $L(p) = \xi^{-1}(p)$ and $f \in \quad (X)$. The condition $\xi^*(f) + nL \geq 0$ is equivalent to $m_p(f) + n \geq 0$ for all $p \in T^{\mathrm{sing}}$. Thus, $\mathcal{L}_{X'}(nK_{X'} + nT')$ turns to be the pull-back of the subspace

$$\mathcal{L}_X(nK_X + nT, nT^{\mathrm{sing}})$$

$$= \{f \in \quad (X); \ (f) + nK_X + nT \geq 0, \ \ m_p(f) + n \geq 0, \ \ p \in T^{\mathrm{sing}}\}$$

of the linear system $\mathcal{L}_X(nK_X + nT)$. The $\mathbb{C}$-linear isomorphism

$$\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_X(nK_X + nT, nT^{\mathrm{sing}})$$

introduced by Holzapfel in [3], is called **transfer of modular forms**.

Bearing in mind Hemperly's result $H^0(X', \Omega^2_{X'}(T')^{\otimes n}) = \rho^* j_1[\Gamma, n]$ for a fixed point free Γ, we refer to

$$\Phi_n^H : \widehat{\mathbb{B}/\Gamma_H} \longrightarrow \mathbb{P}([\Gamma_H, n]) = \mathbb{P}(j_n[\Gamma_H, n])$$

as the n-th logarithmic-canonical map of $\widehat{\mathbb{B}/\Gamma_H}$, regardless of the ramifications of $\mathbb{B} \to \mathbb{B}/\Gamma_H$.

The next lemma explains the transfer of modular forms on finite Galois quotients $\mathbb{B}/\Gamma_H$ of $\mathbb{B}/\Gamma$ to meromorphic functions on X/H. In general, the toroidal compactification $X'_H = (\mathbb{B}/\Gamma_H)'$ is a normal surface. The logarithmic-canonical bundle is not defined on a singular X'_H, but there is always a logarithmic-canonical Weil divisor on X'_H.

Lemma 1. *Let $A' = (\mathbb{B}/\Gamma)'$ be a neat toroidal compactification with an abelian minimal model A and H be a subgroup of $G = \mathrm{Aut}(A, T) = \mathrm{Aut}(A', T')$. Then*

i) *the transfer* $\mathrm{Trans}_n := (\xi^*)^{-1} s^{\otimes(-n)} j_n : [\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})$ *of Γ-modular forms to abelian functions induces a linear isomorphism*

$$\mathrm{Trans}_n^H : [\Gamma_H, n] \longrightarrow \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H$$

of Γ_H-modular forms with rational functions on A/H, called also a transfer

ii) *the projective maps*

$$\Phi_n^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, n]), \quad \Psi_n^H : A/H \dashrightarrow \mathbb{P}(\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H)$$

coincide on an open Zariski dense subset.

Proof: i) Note that $j_n[\Gamma_H, n] = j_n[\Gamma, n]^H$. The inclusion $j_n[\Gamma_H, n] \subseteq j_n[\Gamma, n]$ follows from $\Gamma \leq \Gamma_H$. If $\Gamma_H = \cup_{j=1}^n \gamma_j \Gamma$ is the coset decomposition of Γ_H modulo Γ, then $H = \{h_i = \gamma_i \Gamma;\ 1 \leq i \leq n\}$. A Γ-modular form $\omega \in j_n[\Gamma, n]$ is Γ_H-modular exactly when it is invariant under all γ_i, which amounts to the invariance under all h_i.

One needs a global meromorphic G-invariant section s of $\Omega^2_{A'}(T')$, in order to obtain a linear isomorphism

$$(\xi^*)^{-1} s^{\otimes(-n)} = \mathrm{Trans}_n^H j_n^{-1} : j_n[\Gamma_H, n] = j_n[\Gamma, n]^H \to \mathcal{L}_A(nT, nT^{\mathrm{sing}})^H.$$

The global meromorphic sections of the logarithmic-canonical line bundle $\Omega^2_{A'}(T')$ are in a bijective correspondence with the families $(f_\alpha, U_\alpha)_{\alpha \in S}$ of local meromorphic defining equations $f_\alpha : U_\alpha \to \mathbb{C}$ of the logarithmic-canonical divisor $L + T'$. We construct local meromorphic G-invariant equations $g_\alpha : V_\alpha \to \mathbb{C}$ of T and pull-back to $(f_\alpha = \xi^* g_\alpha, U_\alpha = \xi^{-1}(V_\alpha))_{\alpha \in S}$. Let $F_A : \widetilde{A} = \mathbb{C}^2 \to A$ be the universal covering map of A. Then for any point $p \in A$ choose a lifting $\widetilde{p} \in F_A^{-1}(p)$ and a sufficiently small neighborhood $\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$, which is contained in the interior of a $\pi_1(A)$-fundamental domain on $\widetilde{A}$, centered at $\widetilde{p}$. The G-invariant open neighborhood $W = \cap_{g \in G} g\widetilde{W}$ of $\widetilde{p}$ on $\widetilde{A}$ intersects $F_A^{-1}(T)$ in lines with local equations $l_j(u,v) = a_j(\widetilde{p})u + b_j(\widetilde{p})v + c_j(\widetilde{p}) = 0$. The holomorphic function $g(u,v) = \prod_{g \in G} \prod_j (l_j(u,v))$ on W is G-invariant and can be viewed as a G-invariant local defining equation of T on $V = F_A(W)$. Note that F_A is locally biholomorphic, so that $V \subset A$ is an open subset, after an eventual shrinking of $\widetilde{W}$. The family $(g, V)_{p \in A}$ of local G-invariant defining equations of T pullbacks to a family $(f = \xi^* g, U = \xi^{-1}(V))_{p \in A}$ of local G-invariant sections of $\Omega^2_A(T')$.

ii) Towards the coincidence $\Psi_n^H|_{[(A \setminus T)/H]} \equiv \Phi_n^H|_{[(\mathbb{B}/\Gamma_H) \setminus (L/H)]}$, let us fix a basis $\{\omega_i;\ 1 \leq i \leq d\}$ of $j_n[\Gamma_H, n]$ and apply i), in order to conclude that the set $\{f_i = \mathrm{Trans}_n^H j_n^{-1}(\omega_i);\ 1 \leq i \leq d\}$ is a basis of $\mathcal{L}_A(nT, nT^{\mathrm{sing}})^H$. Tensoring by $s^{\otimes(-n)}$ does not alter the ratios $\frac{\omega_i}{\omega_j}$. The isomorphism $\xi :\quad (A) \to \quad (A')$ is identical on $(A \setminus T)/H$. $\square$

3. Preliminaries

In order to specify $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$ let us note that the blow-down $\xi : A'_{-1} \to A_{-1}$ of the (-1)-curves maps T' to a divisor $T = \xi(T')$ with smooth elliptic irreducible components T_i. Such T are called multi-elliptic divisors. Any irreducible component T_i of T lifts to a $\pi_1(A_{-1})$-orbit of complex lines on the universal cover $\widetilde{A'_{-1}} = \mathbb{C}^2$. That allows to represent

$$T_j = \{(u(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}));\ a_j u + b_j v + c_j = 0\}.$$

If T_j is defined over the field $\mathbb{Q}(\mathrm{i})$ of Gauss numbers, there is no loss of generality in assuming $a_j, b_j \in \mathbb{Z}[\mathrm{i}]$ to be Gaussian integers.

Theorem 1 (Holzapfel [4]). *Let $A_{-1} = E_{-1} \times E_{-1}$ be the Cartesian square of the elliptic curve $E_{-1} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\mathrm{i}$, $\omega_1 = \frac{1}{2}$, $\omega_2 = i\omega_1$, $\omega_3 = \omega_1 + \omega_2$ be half-periods,*

$$Q_0 = 0(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}),\ \ Q_1 = \omega_1(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}),\ \ Q_2 = \mathrm{i}Q_1,\ \ Q_3 = Q_1 + Q_2$$

be the two-torsion points on E_{-1}, $Q_{ij} = (Q_i, Q_j) \in A_{-1}^{2-\mathrm{tor}}$ and

$$T_k = \{(u(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i});\ u - i^k v = 0\} \quad \textit{with} \quad 1 \leq k \leq 4,$$

$$T_{4+m} = \{u(\mathrm{mod}\,\mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\,\mathbb{Z} + \mathbb{Z}\mathrm{i});\ u - \omega_m = 0\}\ \textit{for} \ \ 1 \leq m \leq 2 \ \ \textit{and}$$

$$T_{6+m} = \{u(\mathrm{mod}\,\mathbb{Z} + \mathbb{Z}\mathrm{i}), v(\mathrm{mod}\,\mathbb{Z} + \mathbb{Z}\mathrm{i});\ v - \omega_m = 0\}\ \textit{for} \ \ 1 \leq m \leq 2.$$

Then the blow-up of A_{-1} *at the singular locus* $\left(T_{-1}^{(6,8)}\right)^{\text{sing}} = Q_{00} + Q_{33} + \sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij}$ *of the multi-elliptic divisor* $T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ *is a neat toroidal ball quotient compactification* $A'_{-1} = \left(\mathbb{B}/\Gamma_{-1}^{(6,8)}\right)'$.

Theorem 2 (Kasparian and Kotzev [6]). *The group* $G_{-1} = \operatorname{Aut}(A_{-1}, T_{-1}^{(6,8)}) = \operatorname{Aut}(A'_{-1}, T')$ *of order* 64 *is generated by the translation* τ_{33} *with* Q_{33}*, the multiplications*

$$I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}, \quad \textit{respectively} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

with $\mathrm{i} \in \mathbb{Z}[\mathrm{i}]$ *on the first, respectively, the second factor* E_{-1} *of* A_{-1} *and the transposition*

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of these factors.

Throughout, we use the notations from Theorem 1 and Theorem 2, without mentioning this explicitly. With a slight abuse of notation, we speak of Kodaira-Enriques classification type, irregularity and geometric genus of A_{-1}/H, $H \leq G_{-1}$, referring actually to a smooth minimal model Y of A_{-1}/H.

Theorem 3 (Kasparian and Nikolova [7]). *Let*

$$\mathcal{L} : G_{-1} \to \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}]) = \{g \in \mathbb{Z}[\mathrm{i}]_{2\times 2}\,;\ \det(g) \in \mathbb{Z}[\mathrm{i}]^* = \langle \mathrm{i} \rangle\}$$

be the homomorphism, associating to $g \in G_{-1}$ *its linear part* $\mathcal{L}$ *and*

$$L_1(G_{-1}) = \{g \in G_{-1}\,;\ \mathrm{rk}(\mathcal{L}(g) - I_2) = 1\}$$

$$= \{\tau_{33}^n I^k, \tau_{33}^n J^k, \tau_{33}^n I^l J^{-l}\theta\,;\ 0 \leq n \leq 1,\ \ 1 \leq k \leq 3,\ \ 0 \leq l \leq 3\}.$$

Then

i) A_{-1}/H *is an abelian surface for* $H = \langle \tau_{33} \rangle$

ii) A_{-1}/H *is a hyperelliptic surface for* $H = \langle \tau_{33} I^2 \rangle$ *or* $H = \langle \tau_{33} J^2 \rangle$

iii) A_{-1}/H *is a ruled surface with an elliptic base for*

$$H = \langle h \rangle,\ \ h \in L_1(G_{-1}) \setminus \{\tau_{33} I^2, \tau_{33} J^2\} \quad \textit{or} \quad H = \langle \tau_{33}, h_o \rangle,\ \ h_o \in \mathcal{L}(L_1(G_{-1}))$$

iv) A_{-1}/H *is a* K3 *surface for* $\langle \tau_{33}^n \rangle \neq H \leq K = \ker \det \mathcal{L}$*, where*

$$K = \{\tau_{33}^n I^k J^{-k}, \tau_{33}^n I^k J^{2-k}\theta\,;\ 0 \leq n \leq 1,\ \ 0 \leq k \leq 3\}$$

v) A_{-1}/H *is an Enriques surface for* $H = \langle I^2 J^2, \tau_{33} I^2 \rangle$

vi) *A_{-1}/H is a rational surface for*

$$\langle h\rangle \leq H, \;\; h \in \{\tau_{33}^n IJ, \tau_{33}^n I^2 J, \tau_{33}^n IJ^2;\, 0 \leq n \leq 1\} \;\; or \;\; \langle \tau_{33}^n I^2 J^2, h_1\rangle \leq H$$

$$h_1 \in \{I^{2m}J^{2-2m}, \tau_{33}^m I, \tau_{33}^m J, \tau_{33}^m I^l J^{-l}\theta;\, 0 \leq m \leq 1,\;\; 0 \leq l \leq 3\},\;\; 0 \leq n \leq 1.$$

The following lemma specifies some known properties of Weierstrass σ-function over Gaussian integers $\mathbb{Z}[\mathrm{i}]$.

Lemma 2. *Let* $\sigma(z) = z \prod_{\lambda\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}} (1-\frac{z}{\lambda})^{\frac{z}{\lambda}+\frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$ *be the* ***Weierstrass σ-function****, associated with the lattice* $\mathbb{Z}[\mathrm{i}]$ *of* $\mathbb{C}$. *Then*

i) $\sigma(\mathrm{i}^k z) = \mathrm{i}^k\sigma(z), \quad z \in \mathbb{C}, \quad 0 \leq k \leq 3$

ii) $\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda)e^{-\pi\overline{\lambda}z-\frac{\pi}{2}|\lambda|^2}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{Z}[\mathrm{i}]$, *where*

$$\varepsilon(\lambda) = \begin{cases} -1 & \textit{if } \lambda \in \mathbb{Z}[\mathrm{i}] \setminus 2\mathbb{Z}[\mathrm{i}] \\ 1 & \textit{if } \lambda \in 2\mathbb{Z}[\mathrm{i}]. \end{cases}$$

Proof: i) follows from

$$\prod_{\lambda\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}} \left(1-\frac{\mathrm{i}^k z}{\lambda}\right)^{\frac{\mathrm{i}^k z}{\lambda}+\frac{1}{2}\left(\frac{\mathrm{i}^k z}{\lambda}\right)^2} = \prod_{\mu=\frac{\lambda}{\mathrm{i}^k}\in\mathbb{Z}[\mathrm{i}]\setminus\{0\}} \left(1-\frac{z}{\mu}\right)^{\frac{z}{\mu}+\frac{1}{2}\left(\frac{z}{\mu}\right)^2}.$$

ii) According to Lang's book [8]

$$\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda)\mathrm{e}^{\eta(\lambda)\left(z+\frac{\lambda}{2}\right)}, \qquad z \in \mathbb{C}, \qquad \lambda \in \mathbb{Z}[\mathrm{i}]$$

where $\eta : \mathbb{Z}[\mathrm{i}] \to \mathbb{C}$ is the homomorphism of $\mathbb{Z}$-modules, related to **Weierstrass ζ-function** $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}$ by the identity $\zeta(z+\lambda) = \zeta(z) + \eta(\lambda)$. It suffices to establish that $\eta(\lambda) = -\pi\overline{\lambda}$, $\lambda \in \mathbb{Z}[\mathrm{i}]$. Recall from [8] Legendre's equality $\eta(\mathrm{i}) - \mathrm{i}\eta(1) = 2\pi\mathrm{i}$, in order to derive

$$\eta(\lambda) = \frac{\lambda+\overline{\lambda}}{2}\eta(1) + \frac{\lambda-\overline{\lambda}}{2\mathrm{i}}\eta(\mathrm{i}) = (\eta(1)+\pi)\lambda - \pi\overline{\lambda}, \quad \lambda \in \mathbb{Z}[\mathrm{i}].$$

Combining with homogeneity $\eta(\mathrm{i}\lambda) = \frac{1}{\mathrm{i}}\eta(\lambda)$, $\lambda \in \mathbb{Z}[\mathrm{i}]$ (cf.[8]), one obtains

$$(\eta(1)+\pi)\mathrm{i}\lambda + \pi\mathrm{i}\overline{\lambda} = \eta(\mathrm{i}\lambda) = -\mathrm{i}\eta(\lambda) = -(\eta(1)+\pi)\mathrm{i}\lambda + \pi\mathrm{i}\overline{\lambda}, \quad \lambda \in \mathbb{Z}[\mathrm{i}].$$

Therefore $\eta(1) = -\pi$ and $\eta(\lambda) = -\pi\overline{\lambda}$, $\lambda \in \mathbb{Z}[\mathrm{i}]$. □

Corollary 1.

$$\frac{\sigma(z+\omega_m)}{\sigma(z-\omega_m)} = -\mathrm{e}^{2(-1)^m\omega_m\pi z}$$

$$\frac{\sigma(z+\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)} = (-1)^{m+1}\varepsilon \mathrm{i} \mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z+2(-1)^m\omega_m\pi z}$$

$$\frac{\sigma(z-\omega_m+2\varepsilon\omega_{3-m})}{\sigma(z-\omega_m)} = (-1)^{m+1}\varepsilon \mathrm{i} \mathrm{e}^{-\frac{\pi}{2}+2(-1)^{m+1}\varepsilon\omega_{3-m}\pi z}.$$

for the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \mathrm{i}\omega_1$ *and* $\varepsilon = \pm 1$.

Corollary 2.

$$\frac{\sigma(z+2\varepsilon\omega_m)}{\sigma(z-1)} = \mathrm{e}^{-\pi z+(-1)^m 2\varepsilon\pi\omega_m z}$$

$$\frac{\sigma(z+(-1)^m\omega_m+\varepsilon(-1)^m\omega_{3-m})}{\sigma(z-(-1)^m\omega_m+(-1)^m\omega_{3-m})} = -\mathrm{i}^{(-1)^m\frac{(1+\varepsilon)}{2}}\mathrm{e}^{2\omega_m\pi z+(1-\varepsilon)\omega_{3-m}\pi z}.$$

for the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \mathrm{i}\omega_1$ *and* $\varepsilon = \pm 1$.

Corollary 1 and Corollary 2 follow from Lemma 2 ii) and $\bar{\omega}_m = (-1)^{m+1}\omega_m$, $\omega_m^2 = \frac{(-1)^{m+1}}{4}$.

In [5] the map $\Phi : \widetilde{\mathbb{B}/\Gamma_{-1}^{(6,8)}} \to \mathbb{P}([\Gamma_{-1}^{(6,8)}, 1])$ is shown to be a regular embedding. This is done by constructing a $\mathbb{C}$-basis of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$, consisting of binary parallel or triangular σ-quotients. An abelian function $f_{\alpha,\beta} \in \mathcal{L}$ is binary parallel if the pole divisor $(f_{\alpha,\beta})_\infty = T_\alpha + T_\beta$ consists of two disjoint smooth elliptic curves T_α and T_β. A σ-quotient $f_{i,\alpha,\beta} \in \mathcal{L}$ is triangular if $T_i \cap T_\alpha \cap T_\beta = \emptyset$ and any two of T_i, T_α and T_β intersect in a single point.

Theorem 4 (Kasparian and Kotzev [5]). *Let*

$$\Sigma_{12}(z) = \frac{\sigma(z-1)\sigma(z+\omega_1-\omega_2)}{\sigma(z-\omega_1)\sigma(z-\omega_2)}, \qquad \Sigma_1 = \frac{\sigma(u-\mathrm{i}v+\omega_3)}{\sigma(u-\mathrm{i}v)}$$

$$\Sigma_2 = \frac{\sigma(u+v+\omega_3)}{\sigma(u+v)}, \quad \Sigma_3 = \frac{\sigma(u+\mathrm{i}v+\omega_3)}{\sigma(u+\mathrm{i}v)}, \quad \Sigma_4 = \frac{\sigma(u-v+\omega_3)}{\sigma(u-v)}$$

$$\Sigma_5 = \frac{\sigma(u-\omega_2)}{\sigma(u-\omega_1)}, \ \Sigma_6 = \frac{\sigma(u-\omega_1)}{\sigma(u-\omega_2)}, \ \Sigma_7 = \frac{\sigma(v-\omega_2)}{\sigma(v-\omega_1)}, \ \Sigma_8 = \frac{\sigma(v-\omega_1)}{\sigma(v-\omega_2)}.$$

Then

i) *the space* $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{\sqrt{-1}}^{(6,8)}, \left(T_{\sqrt{-1}}^{(6,8)}\right)^{\mathrm{sing}}\right)$ *contains the binary parallel* σ*-quotients* $f_{56}(u,v) = \Sigma_{12}(u)$, $f_{78}(u,v) = \Sigma_{12}(v)$ *and the triangular* *σ-quotients*

$$f_{157} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u}\Sigma_1\Sigma_5\Sigma_7, \qquad f_{168} = -\,\mathrm{e}^{-\pi-\pi \mathrm{i}u-\pi v-\pi \mathrm{i}v}\Sigma_1\Sigma_6\Sigma_8$$

$$f_{357} = -\,\mathrm{e}^{-\pi+\pi u+\pi v+\pi \mathrm{i}v}\Sigma_3\Sigma_5\Sigma_7, \qquad f_{368} = -\,\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi \mathrm{i}u}\Sigma_3\Sigma_6\Sigma_8$$

$$f_{258} = \mathrm{e}^{-\pi+\pi u-\pi \mathrm{i}v}\Sigma_2\Sigma_5\Sigma_8, \qquad f_{267} = \mathrm{e}^{-\pi-\pi \mathrm{i}u+\pi v}\Sigma_2\Sigma_6\Sigma_7$$

$$f_{458} = -\,\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}+\pi u-\pi v}\Sigma_4\Sigma_5\Sigma_8, \qquad f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}-\pi \mathrm{i}u+\pi \mathrm{i}v}\Sigma_4\Sigma_6\Sigma_7$$

ii) *a $\mathbb{C}$-basis of $\mathcal{L}$ is*

$$f_o := 1,\ f_1 := f_{157},\ f_2 := f_{258},\ f_3 := f_{368},\ f_4 := f_{467},\ f_5 := f_{56},\ f_6 := f_{78}.$$

4. Technical Preparation

The group $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ permutes the eight irreducible components of $T_{-1}^{(6,8)}$ and the $\Gamma_{-1}^{(6,8)}$-cusps. For any subgroup H of G_{-1}, the Γ_H-cusps are the H-orbits of $\partial_{\Gamma_{-1}^{(6,8)}}\mathbb{B}/\Gamma_{-1}^{(6,8)} = \{\kappa_i\,;\ 1 \le i \le 8\}$.

Lemma 3. *If $\varphi : G_{-1} \to S_8(\kappa_1, \ldots, \kappa_8)$ is the natural representation of $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ in the symmetric group of the $\Gamma_{-1}^{(6,8)}$-cusps, then*

$$\varphi(\tau_{33}) = (\kappa_5, \kappa_6)(\kappa_7, \kappa_8), \qquad \varphi(I) = (\kappa_1, \kappa_4, \kappa_3, \kappa_2)(\kappa_5, \kappa_6)$$

$$\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8), \qquad \varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8).$$

Proof: The $\Gamma_{-1}^{(6,8)}$-cusps κ_i are obtained by contraction of the proper transforms T_i' of T_i under the blow-up of A_{-1} at $\left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}$. Therefore the representations of G_{-1} in the permutation groups of $\{T_i\,;\ 1 \le i \le 8\}$, $\{T_i'\,;\ 1 \le i \le 8\}$ and $\{\kappa_i\,;\ 1 \le i \le 8\}$ coincide.

According to $\tau_{33}(u - \mathrm{i}^k v) = u - \mathrm{i}^k v + (1 - \mathrm{i}^k)\omega_3 = u - \mathrm{i}^k v (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i})$, the translation τ_{33} acts identically on T_1, T_2, T_3, T_4. Further, $\tau_{33}(u - \omega_m) = u + \omega_{3-m} \equiv u - \omega_{3-m} (\mathrm{mod}\ \ \mathbb{Z} + \mathbb{Z}\mathrm{i})$ reveals the permutation $(T_5, T_6)(T_7, T_8)$ of the last four components of $T_{-1}^{(6,8)}$.

Due to the identity $I(u - \mathrm{i}^k v) = \mathrm{i}u - \mathrm{i}^k v = \mathrm{i}(u - \mathrm{i}^{k-1} v)$, the automorphism I induces the permutation (T_1, T_4, T_3, T_2) of the first four components of $T_{-1}^{(6,8)}$. Further, $I(u - \omega_m) = \mathrm{i}(u \pm \omega_{3-m})$ reveals that I permutes T_5 with T_6. Note that I acts identically on v and fixes T_7, T_8.

In a similar vein, $J(u - \mathrm{i}^k v) = u - \mathrm{i}^{k+1} v$, $J(v - \omega_m) = \mathrm{i}(v \pm \mathrm{i}\omega_{3-m})$ determine that $\varphi(J) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)(\kappa_7, \kappa_8)$. According to $\theta(u - \mathrm{i}^k v) = v - \mathrm{i}^k u = -\mathrm{i}^k(u - \mathrm{i}^{-k} v)$ and $\theta(u - \omega_m) = v - \omega_m$, one concludes that $\varphi(\theta) = (\kappa_1, \kappa_3)(\kappa_5, \kappa_7)(\kappa_6, \kappa_8)$. □

The following lemma incorporates several arguments, which will be applied repeatedly towards determination of the target $\mathbb{P}([\Gamma_H, 1])$ and the rank of the logarithmic canonical map Φ^H.

Lemma 4. *In the notations from* Theorem 4, *for an arbitrary subgroup H of $G_{-1} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ and any $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$, let $R_H(f) = \sum_{h \in H} h(f)$ be the value of* **Reynolds operator** *R_H of H on f.*

i) *The space $\mathcal{L}^H$ of the H-invariants of $\mathcal{L}$ is spanned by $\{R_H(f_i)\,;\, 0 \leq i \leq 6\}$.*

ii) *Let $T_i \subset (R_H(f_{i,\alpha_1,\beta_1}))_\infty, (R_H(f_{i,\alpha_2,\beta_2}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum_{\alpha=5}^{8} T_\alpha$ for some $1 \leq i \leq 4$, $5 \leq \alpha_j \leq 6$, $7 \leq \beta_j \leq 8$. Then*

$$R_H(f_{i,\alpha_2,\beta_2}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}), R_H(f_{i,\alpha_1,\beta_1})).$$

iii) *Let $\bar{\kappa}_{i_1}, \ldots, \bar{\kappa}_{i_p}$ with $1 \leq i_1 < \ldots < i_p \leq 4$ be different Γ_H-cusps*

$$T_{i_j} \subset (R_H(f_{i_j}))_\infty \subseteq \mathrm{Orb}_H(T_{i_j}) + \sum_{\alpha=5}^{8} T_\alpha \quad \text{for all} \quad 1 \leq j \leq p$$

and B be a $\mathbb{C}$-basis of $\mathcal{L}_2^H = \mathcal{L}_{A_{-1}}\left(\sum_{\alpha=5}^{8} T_\alpha\right)^H$. Then the set

$$\{R_H(f_{i_j,\alpha_j,\beta_j})\,;\, 1 \leq j \leq p\} \cup B$$

consists of linearly independent invariants over $\mathbb{C}$.

iv) *If $R_j = R_H(f_{j,\alpha_j,\beta_j}) \not\equiv \mathrm{const}$, $R_j|_{T_j} = \infty$ and $R_i = R_H(f_{i,\alpha_i,\beta_i})$ has $R_i|_{T_j} \not\equiv \mathrm{const}$ then for any subgroup H_o of H the projective maps*

$$\Psi^{H_o} : X/H_o \dashrightarrow \mathbb{P}(\mathcal{L}^{H_o}), \qquad \Phi^{H_o} : \widehat{\mathbb{B}/\Gamma_{H_o}} \dashrightarrow \mathbb{P}(j_1[\Gamma_{H_o}, 1])$$

are of rank $\mathrm{rk}\Phi^{H_o} = \mathrm{rk}\Psi^{H_o} = 2$.

v) *If the group H' is obtained from the group H by replacing all $\tau_{33}^n I^k J^l \theta^m \in H$ with $\tau_{33}^n I^l J^k \theta^m$, then the spaces of modular forms $j_1[\Gamma_{H'}, 1] \simeq j_1[\Gamma_H, 1]$ are isomorphic and the logarithmic-canonical maps have equal rank $\mathrm{rk}\Phi^H = \mathrm{rk}\Phi^{H'}$.*

Proof: i) By Theorem 4 ii), $\mathcal{L} = \mathrm{Span}_{\mathbb{C}}(f_i\,;\, 0 \leq 6)$. Therefore any $f \in \mathcal{L}$ is a $\mathbb{C}$-linear combination $f = \sum_{i=0}^{6} c_i f_i$. Due to H-invariance of f and the linearity of the representation of H in $\mathrm{Aut}(\mathcal{L})$, Reynolds operator

$$|H| f = R_H(f) = \sum_{i=0}^{6} c_i R_H(f_i).$$

ii) Let $\omega_s \in j_1\left[\Gamma_{-1}^{(6,8)}, 1\right]^H$ are the modular forms, which transfer to $R_H(f_{i,\alpha_s,\beta_s})$, $1 \leq s \leq 2$. Since $\omega_1(\kappa_i) \neq 0$, there exists $c_i \in \mathbb{C}$, such that $\omega_i' = \omega_2 - c_i\omega_1$ vanishes at κ_i. By the assumption $(R_H(f_{i,\alpha_s,\beta_s}))_\infty \subseteq \mathrm{Orb}_H(T_i) + \sum\limits_{\alpha=5}^{8} T_\alpha$, the transfer $F_i \in \mathcal{L}^H$ of ω_i' belongs to $\mathrm{Span}_{\mathbb{C}}(1, f_{56}, f_{78})^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$.
iii) As far as the transfer $\mathrm{Trans}_1^H : j_1[\Gamma_H, 1] \to \mathcal{L}$ is a $\mathbb{C}$-linear isomorphism, it suffices to establish the linear independence of the corresponding modular forms $\{\omega_{i_j};\, 1 \leq j \leq p\} \cup \{\omega_b;\, b \in B\}$. Evaluating the $\mathbb{C}$-linear combination $\sum\limits_{j=1}^{p} c_{i_j}\omega_{i_j} + \sum\limits_{b\in B} c_b\omega_b = 0$ at $\bar{\kappa}_{i_1}, \ldots, \bar{\kappa}_{i_p}$, one obtains $c_{i_j} = 0$, according to $\omega_{i_j}(\bar{\kappa}_{i_s}) = \delta_j^s$ and $\omega_b(\bar{\kappa}_{i_j}) = 0$, $b \in B$, $1 \leq j \leq p$. Then $\sum\limits_{b\in B} \omega_b = 0$ requires the vanishing of all c_b, due to the linear independence of B.
iv) If H_o is a subgroup of H then $\mathcal{L}^H$ is a subspace of $\mathcal{L}^{H_o}$, $j_1[\Gamma_H, 1]$ is a subspace of $j_1[\Gamma_{H_o}, 1]$ and $\Psi^H = \mathrm{pr}^{\mathcal{L}}\Psi^{H_o}$, $\Phi^H = \mathrm{pr}^{\Gamma_H}\Phi^{H_o}$ for the projections $\mathrm{pr}^{\mathcal{L}} : \mathbb{P}(\mathcal{L}^{H_o}) \to \mathbb{P}(\mathcal{L}^H)$, $\mathrm{pr}^{\Gamma_H} : \mathbb{P}(j_1[\Gamma_{H_o}, 1]) \to \mathbb{P}(j_1[\Gamma_H, 1])$. That is why, it suffices to justify that $\mathrm{rk}\Phi^H = \mathrm{rk}\Psi^H = 2$ is maximal. Assume the opposite and consider $R_i, R_j : X/H \dashrightarrow \mathbb{P}^1$. The commutative diagram

$$\begin{array}{ccc} X/H & \xrightarrow{(R_i,R_j)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle R_j}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_2} & \\ \mathbb{P}^1 & & \end{array}$$

has surjective R_j, as far as $R_j \not\equiv \mathrm{const}$. If the image $C = (R_i, R_j)(X/H)$ is a curve, then the projection $\mathrm{pr}_2 : C \to \mathbb{P}^1$ has only finite fibers. In particular, $\mathrm{pr}_2^{-1}(\infty) = R_i((R_j)_\infty) \times \infty \supseteq R_i(T_j) \times \infty$ consists of finitely many points. However, $R_i(T_j) = \mathbb{P}^1$ as an image of the non-constant elliptic function R_i: $T_j \dashrightarrow \mathbb{P}^1$. The contradiction implies that $\dim_{\mathbb{C}} C = 2$ and $\mathrm{rk}\Psi^H = 2$.
v) The transposition of the holomorphic coordinates $(u, v) \in \mathbb{C}^2$ affects non-trivially the constructed σ-quotients. However, one can replace the equations $u - \mathrm{i}^k v = 0$ of T_k, $1 \leq k \leq 4$ by $v - \mathrm{i}^{-k}u = 0$ and repeat the above considerations with interchanged u, v. The dimension of $j_1[\Gamma_H, 1]$ and the rank of Φ^H are invariant under the transposition of the global holomorphic coordinates on $\widetilde{A_{-1}} = \mathbb{C}^2$. $\square$

With a slight abuse of notation, we write $g(f)$ instead of $g^*(f)$, for $g \in G_{-1}$, $f \in \mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$.

Lemma 5. *The generators* τ_{33}, I, J, θ *of* G_{-1} *act on the binary parallel and triangular* σ*-quotients from* Corollary 4 *as follows*

$$3\tau_{33}(f_{56}) = -f_{56}, \quad \tau_{33}(f_{78}) = -f_{78}$$
$$\tau_{33}(f_{157}) = -\mathrm{i}e^{\frac{\pi}{2}} f_{168}, \quad \tau_{33}(f_{168}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{157}, \quad \tau_{33}(f_{357}) = -\mathrm{i}e^{-\frac{\pi}{2}} f_{368}$$
$$\tau_{33}(f_{368}) = \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \quad \tau_{33}(f_{258}) = f_{267}, \quad \tau_{33}(f_{267}) = f_{258}$$
$$\tau_{33}(f_{458}) = -f_{467}, \quad \tau_{33}(f_{467}) = -f_{458}$$
$$I(f_{56}) = -\mathrm{i}f_{56}, \quad I(f_{78}) = f_{78}$$
$$I(f_{157}) = -\mathrm{i}f_{467}, \quad I(f_{168}) = -e^{-\frac{\pi}{2}} f_{458}, \quad I(f_{357}) = \mathrm{i}f_{267}$$
$$I(f_{368}) = -e^{\frac{\pi}{2}} f_{258}, \quad I(f_{258}) = \mathrm{i}f_{168}, \quad I(f_{267}) = -e^{-\frac{\pi}{2}} f_{157}$$
$$I(f_{458}) = -\mathrm{i}f_{368}, \quad I(f_{467}) = -e^{\frac{\pi}{2}} f_{357}$$
$$J(f_{56}) = f_{56}, \quad J(f_{78}) = -\mathrm{i}f_{78}$$
$$J(f_{157}) = -\mathrm{i}e^{\frac{\pi}{2}} f_{258}, \quad J(f_{168}) = f_{267}, \quad J(f_{357}) = \mathrm{i}e^{-\frac{\pi}{2}} f_{458}$$
$$J(f_{368}) = f_{467}, \quad J(f_{258}) = f_{357}, \quad J(f_{267}) = -\mathrm{i}e^{-\frac{\pi}{2}} f_{368}$$
$$J(f_{458}) = f_{157}, \quad J(f_{467}) = \mathrm{i}e^{\frac{\pi}{2}} f_{168}$$
$$\theta(f_{56}) = f_{78}, \quad \theta(f_{78}) = f_{56}$$
$$\theta(f_{157}) = -e^{\frac{\pi}{2}} f_{357}, \quad \theta(f_{168}) = -e^{-\frac{\pi}{2}} f_{368}, \quad \theta(f_{357}) = -e^{-\frac{\pi}{2}} f_{157}$$
$$\theta(f_{368}) = -e^{\frac{\pi}{2}} f_{168}, \quad \theta(f_{258}) = f_{267}, \quad \theta(f_{267}) = f_{258}$$
$$\theta(f_{458}) = f_{467}, \quad \theta(f_{467}) = f_{458}.$$

Proof: Making use of Lemma 2 and Corollary 2, one computes that

$$\tau_{33}\sigma(u-1) = -e^{\pi u + \pi \mathrm{i} u}\sigma(u+\omega_1-\omega_2), \quad \tau_{33}\sigma(u+\omega_1-\omega_2) = e^{-2\pi u}\sigma(u-1)$$
$$\tau_{33}\sigma(u-\omega_1) = -e^{\pi \mathrm{i} u}\sigma(u-\omega_2), \quad \tau_{33}\sigma(u-\omega_2) = -e^{-\pi u}\sigma(u-\omega_1)$$
$$\tau_{33}(\Sigma_1) = -\mathrm{i}e^{-\frac{\pi}{2}}\Sigma_1, \quad \tau_{33}(\Sigma_2) = e^{-\pi}\Sigma_2, \quad \tau_{33}(\Sigma_3) = \mathrm{i}e^{-\frac{\pi}{2}}\Sigma_3, \quad \tau_{33}(\Sigma_4) = \Sigma_4$$
$$\tau_{33}(\Sigma_5) = e^{-\pi u - \pi \mathrm{i} u}\Sigma_6, \quad \tau_{33}(\Sigma_6) = e^{\pi u + \pi \mathrm{i} u}\Sigma_5$$
$$\tau_{33}(\Sigma_7) = e^{-\pi v - \pi \mathrm{i} v}\Sigma_8, \quad \tau_{33}(\Sigma_8) = e^{\pi v + \pi \mathrm{i} v}\Sigma_7$$
$$I\sigma(u-1) = \mathrm{i}e^{-\pi u + \pi \mathrm{i} u}\sigma(u-1), \quad I\sigma(u+\omega_1-\omega_2) = -e^{\pi u}\sigma(u+\omega_1-\omega_2)$$
$$I\sigma(u-\omega_1) = -\mathrm{i}e^{\pi \mathrm{i} u}\sigma(u-\omega_2), \quad I\sigma(u-\omega_2) = \mathrm{i}\sigma(u-\omega_1)$$
$$I(\Sigma_1) = \mathrm{i}e^{-\pi \mathrm{i} u + \pi \mathrm{i} v}\Sigma_4, \quad I(\Sigma_2) = \mathrm{i}e^{-\pi \mathrm{i} u - \pi v}\Sigma_1$$
$$I(\Sigma_3) = \mathrm{i}e^{-\pi \mathrm{i} u - \pi \mathrm{i} v}\Sigma_2, \quad I(\Sigma_4) = \mathrm{i}e^{-\pi \mathrm{i} u + \pi v}\Sigma_3$$
$$I(\Sigma_5) = -e^{-\pi \mathrm{i} u}\Sigma_6, \quad I(\Sigma_6) = -e^{\pi \mathrm{i} u}\Sigma_5, \quad I(\Sigma_7) = \Sigma_7, \quad I(\Sigma_8) = \Sigma_8$$
$$J\sigma(v+\mu) = I\sigma(u+\mu)|_{u=v}, \qquad \mu \in \mathbb{C}$$

$$\begin{aligned} 4J(\Sigma_1) &= \Sigma_2, \quad J(\Sigma_2) = \Sigma_3, \quad J(\Sigma_3) = \Sigma_4, \qquad J(\Sigma_4) = \Sigma_1 \\ J(\Sigma_5) &= \Sigma_5, \quad J(\Sigma_6) = \Sigma_6, \quad J(\Sigma_7) = -\mathrm{e}^{-\pi \mathrm{i} v}\Sigma_8, \quad J(\Sigma_8) = -\mathrm{e}^{\pi \mathrm{i} v}\Sigma_7 \end{aligned}$$

$$\theta\sigma(u+\mu) = \sigma(v+\mu), \qquad \mu \in \mathbb{C}$$

$$\theta(\Sigma_1) = -\mathrm{i}\mathrm{e}^{\pi u + \pi \mathrm{i} v}\Sigma_3, \qquad \theta(\Sigma_2) = \Sigma_2$$

$$\begin{aligned} \theta(\Sigma_3) &= \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} u - \pi v}\Sigma_1, \quad \theta(\Sigma_4) = -\mathrm{e}^{\pi u - \pi \mathrm{i} u - \pi v + \pi \mathrm{i} v}\Sigma_4 \\ \theta(\Sigma_5) &= \Sigma_7, \quad \theta(\Sigma_6) = \Sigma_8, \quad \theta(\Sigma_7) = \Sigma_5, \quad \theta(\Sigma_8) = \Sigma_6. \end{aligned}$$

□

The following lemma is an immediate consequence of Lemma 2 and Corollary 1.

Lemma 6.

$$\frac{f_{157}}{\Sigma_1}\Big|_{T_1} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \frac{f_{168}}{\Sigma_1}\Big|_{T_1} = \mathrm{e}^{-\pi}, \quad \frac{f_{258}}{\Sigma_2}\Big|_{T_2} = \mathrm{e}^{-\pi}, \quad \frac{f_{267}}{\Sigma_2}\Big|_{T_2} = \mathrm{e}^{-\pi}$$

$$\frac{f_{357}}{\Sigma_3}\Big|_{T_3} = \mathrm{e}^{-\pi}, \quad \frac{f_{368}}{\Sigma_3}\Big|_{T_3} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \frac{f_{458}}{\Sigma_4}\Big|_{T_4} = -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}, \quad \frac{f_{467}}{\Sigma_4}\Big|_{T_4} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}}$$

$$\frac{f_{157} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0, \quad \frac{f_{258} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{458}}{\Sigma_5}\Big|_{T_5} = 0.$$

Lemma 7.

$$[(f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368})]|_{T_2} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\left(1 + c\mathrm{e}^{-\frac{\pi}{2}}\right)$$

$$\frac{\sigma((1+\mathrm{i})v + \omega_3)}{\sigma((1+\mathrm{i})v)}\left[\mathrm{e}^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + \mathrm{e}^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right]$$

is non-constant for all $c \in \mathbb{C} \setminus \{-\mathrm{e}^{\frac{\pi}{2}}\}$.

Proof: Note that

$$\begin{aligned} f(v) &= [(f_{157} - \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{168}) + c(f_{357} - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368})]|_{T_2} \\ &= \left[\mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\Sigma_1(-v, v) - c\mathrm{e}^{-\pi + \pi \mathrm{i} v}\Sigma_3(-v, v)\right] \\ &\quad \times [\Sigma_5(-v)\Sigma_7(v) + \Sigma_6(-v)\Sigma_8(v)] \\ &= \mathrm{i}\mathrm{e}^{-\frac{\pi}{2} - \pi v}\left(1 + c\mathrm{e}^{-\frac{\pi}{2}}\right)\frac{\sigma((1+\mathrm{i})v - \omega_3)}{\sigma((1+\mathrm{i})v)} \\ &\quad \times \left[\mathrm{e}^{(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} + \mathrm{e}^{-(1+\mathrm{i})\pi v}\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2}\right] \end{aligned}$$

making use of Lemma 2 and Corollary 1. Obviously, $f(v)$ has no poles outside $\mathbb{Q}(\mathrm{i})$. It suffices to justify that $\lim_{v\to 0} f(v) = \infty$, in order to conclude that $f(v) \not\equiv$ const. To this end, use $\sigma(\omega_2) = \mathrm{i}\sigma(\omega_1)$ to observe that

$$f(v)\sigma((1+\mathrm{i})v)\Big|_{v=0} = 2\,\mathrm{i}e^{-\frac{\pi}{2}}\left(1 + ce^{-\frac{\pi}{2}}\right)\sigma(\omega_3) \neq 0$$

whenever $c \neq -e^{\frac{\pi}{2}}$, while $\sigma((1+\mathrm{i})v)|_{v=0} = 0$. □

5. Basic Results

Lemma 8. *For $H = \langle IJ^2, \tau_{33}J^2\rangle$, $\langle I^2J, \tau_{33}I^2\rangle$ with rational A_{-1}/H and any $-\mathrm{Id} \in H \leq G_{-1}$, the map $\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1])$ is constant.*

Proof: By Lemma 4 (iv), the assertion for $\langle I^2J, \tau_{33}I^2\rangle$ is a consequence of the one for $\langle IJ^2, \tau_{33}J^2\rangle$. In the case of $H = \langle IJ^2, \tau_{33}J^2\rangle$, the space $\mathcal{L}^H$ is spanned by Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0$$

$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{267} - e^{\frac{\pi}{2}}f_{258} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368} + \mathrm{i}f_{467} + \mathrm{i}f_{458}$. The Γ_H-cusps are $\overline{\kappa}_1 = \overline{\kappa}_2 = \overline{\kappa}_3 = \overline{\kappa}_4$, $\overline{\kappa}_5 = \overline{\kappa}_6$ and $\overline{\kappa}_7 = \overline{\kappa}_8$. By Lemma 6, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157})|_{T_1} \neq \infty$. Therefore $R_H(f_{157}) \in \mathcal{L}_2^H = \mathbb{C}$ and $\mathrm{rk}\Phi^H = 0$.

It suffices to observe that $-\mathrm{Id}$ changes the signs of the $\mathbb{C}$-basis

$$f_{56},\ f_{78},\ f_{157},\ f_{258},\ f_{368},\ f_{467} \tag{1}$$

of $\mathcal{L} = \mathcal{L}_{A_{-1}}\left(T_{-1}^{(6,8)}, \left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}}\right)$. Then for $H_o = \langle -\mathrm{Id}\rangle$ the space $\mathcal{L}^{H_o}$ is generated by $R_{H_o}(1) = 1$. Any subgroup $H_o \leq H \leq G_{-1}$ decomposes into cosets $H = \cup_{i=1}^k h_iH_o$ and $R_H = \sum_{i=1}^k h_iR_{H_o}$ vanishes on (1). Thus, $\mathcal{L}^H = \mathbb{C}$ and $\mathrm{rk}\Phi^H = 0$. □

Note that $A_{-1}/\langle -\mathrm{Id}\rangle$ has 16 double points, whose minimal resolution is the Kummer surface X_{-1} of A_{-1}. Thus, $H \ni -\mathrm{Id}$ exactly when the minimal resolution Y of the singularities of A_{-1}/H is covered by a smooth model of X_{-1}. More precisely, all A_{-1}/H with $-\mathrm{Id} \in H$ have vanishing irregularity $0 \leq q(A_{-1}/H) \leq q(X_{-1}) = 0$. These are the Enriques $A_{-1}/\langle -\mathrm{Id}, \tau_{33}I^2\rangle$, all K3 quotients A_{-1}/H with $\langle \tau_{33}^n\rangle \neq H \leq K = \ker\det\mathcal{L}$, except $A_{-1}/\langle \tau_{33}(-\mathrm{Id})\rangle$ and the rational A_{-1}/H with $\tau_{33}IJ \in H$ for $0 \leq n \leq 1$ or $\langle -\mathrm{Id}, h_1\rangle \leq H$ for

$$h_1 \in \{I^{2m}J^{2-2m},\quad \tau_{33}^mI,\quad \tau_{33}^mJ,\quad \tau_{33}^mI^lJ^{-l}\theta\ ;\ 0 \leq m \leq 1,\quad 0 \leq l \leq 3\}.$$

Lemma 9. *The non-trivial subgroups* $H \not\ni -\mathrm{Id}$ *of* G_{-1} *are*

i) *cyclic of order two*

$$H_2(m,l) = \langle \tau_{33} I^{2m} J^{2l} \rangle \quad \textit{with} \quad 0 \le m, l \le 1$$

$$H_2^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle \quad \textit{with} \quad 0 \le n \le 1,\ 0 \le k \le 3,\ H_2' = \langle I^2 \rangle,\ \ H_2'' = \langle J^2 \rangle$$

ii) *cyclic of order four*

$$H_4'(n,m) = \langle \tau_{33}^n I J^{2m} \rangle \quad \textit{with} \quad 0 \le n, m \le 1$$

$$H_4''(n,m) = \langle \tau_{33}^n I^{2m} J \rangle \quad \textit{with} \quad 0 \le n, m \le 1$$

iii) *isomorphic to the* ***Klein group*** $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$H_{2\times 2}'(m) = \langle \tau_{33} J^{2m}, I^2 \rangle \quad \textit{with} \quad 0 \le m \le 1$$

$$H_{2\times 2}''(m) = \langle \tau_{33} I^{2m}, J^2 \rangle \quad \textit{with} \quad 0 \le m \le 1$$

$$H_{2\times 2}^\theta(k) = \langle I^k J^{-k} \theta, \tau_{33} \rangle \quad \textit{with} \quad 0 \le k \le 1$$

$$H_{2\times 2}^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^2 J^2 \rangle \quad \textit{with} \quad 0 \le n, k \le 1$$

iv) *isomorphic to* $\mathbb{Z}_4 \times \mathbb{Z}_2$

$$H_{4\times 2}'(m,l) = \langle I J^{2m}, \tau_{33} J^{2l} \rangle \quad \textit{with} \quad 0 \le m, l \le 1$$

$$H_{4\times 2}''(m,l) = \langle I^{2m} J, \tau_{33} I^{2l} \rangle \quad \textit{with} \quad 0 \le m, l \le 1.$$

Proof: If H is a subgroup of G_{-1}, which does not contain $-\mathrm{Id}$, then $H \subseteq S = \{g \in G_{-1};\ -\mathrm{Id} \not\in \langle g \rangle\}$. Decompose $G_{-1} = G'_{-1} \cup G'_{-1}\theta$ into cosets modulo the abelian subgroup

$$G'_{-1} = \{\tau_{33}^n I^k J^l\,;\, 0 \le n \le 1, 0 \le k, l \le 3\} \le G_{-1}.$$

The cyclic group, generated by $(\tau_{33}^n I^k J^l \theta)^2 = (IJ)^{k+l}$ does not contain $-\mathrm{Id} = (IJ)^2$ if and only if $k + l \equiv 0 (\mathrm{mod}\ 4)$. If $S^{(r)} = \{g \in S\,;\, g \text{ is of order } r\}$ then

$$S \cap G'_{-1}\theta = \{\tau_{33}^n I^k J^{-k}\theta\,;\, 0 \le n \le 1,\ \ 0 \le k \le 3\} = S^{(2)} \cap G'_{-1}\theta =: S_1^{(2)}$$

and $S \cap G'_{-1}\theta \subseteq S^{(2)}$ consists of elements of order two. Concerning $S \cap G'_{-1}$, observe that $(\tau_{33}^n I^k J^{k+2m})^2 = (IJ)^{2k} \in S$ for $0 \le n, m \le 1, 0 \le k \le 3$ requires $k = 2p$ to be even. Consequently

$$\{\tau_{33}^n I^k J^l\,;\, k \equiv l (\mathrm{mod}\ 2)\} \cap S$$
$$= \{\tau_{33} I^{2m} J^{2l},\ \ I^2,\ \ J^2\,;\, 0 \le m, l \le 1\} = S^{(2)} \cap G'_{-1} =: S_0^{(2)}$$

$$\{\tau_{33}^n I^k J^l\,;\, k \equiv l + 1 (\mathrm{mod}\ 2)\} \cap S$$
$$= \{\tau_{33}^n I^{2m+1} J^{2l}, \tau_{33}^n I^{2m} J^{2l+1}\,;\, 0 \le n, m, l \le 1\} = S^{(4)}.$$

In such a way, one obtains $S = \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)} \cup S^{(4)}$ of cardinality $|S| = 31$. If a subgroup H of G_{-1} is contained in S, then $|H| \leq |S| = 31$ divides $|G_{-1}| = 64$, i.e., $|H| = 1, 2, 4, 8$ or 16. The only subgroup $H < G_{-1}$ of $|H| = 1$ is the trivial one $H = \{\mathrm{Id}\}$. The subgroups $-\mathrm{Id} \not\in H < G_{-1}$ of order two are the cyclic ones, generated by $h \in S_0^{(2)} \cup S_1^{(2)}$. We denote $H_2(m,l) = \langle \tau_{33} I^{2m} J^{2l} \rangle$ for $0 \leq m, l \leq 1$, $H_2^{\theta}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ for $0 \leq n \leq 1$, $0 \leq k \leq 3$ and $H_2' = \langle I^2 \rangle$, $H_2'' = \langle J^2 \rangle$.

For any $h \in S^{(4)}$ one has $\langle h \rangle = \langle h^3 \rangle$, so that the subgroups $-\mathrm{Id} \not\in H \simeq \mathbb{Z}_4$ of G_{-1} are depleted by $H_4'(n,m) = \langle \tau_{33}^n I J^{2m} \rangle$, $H_4''(n,m) = \langle \tau_{33}^n I^{2m} J \rangle$ with $0 \leq n, m \leq 1$.

The subgroups $-\mathrm{Id} \not\in H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ of G_{-1} are generated by commuting $g_1, g_2 \in S^{(2)} = S_0^{(2)} \cup S_1^{(2)}$. If $g_1, g_2 \in S_1^{(2)}$ then $g_1 g_2 \in G'_{-1}$, so that one can always assume that $g_2 \in S_0^{(2)}$. Any $g_1 \neq g_2$ from $S_0^{(2)} \subset G'_{-1}$ generate the Klein group of order four. Moreover, if

$$S_{0,1}^{(2)} = \{\tau_{33} I^{2m} J^{2l};\, 0 \leq m, l \leq 1\}, \qquad S_{0,0}^{(2)} = \{I^2, J^2\}$$

then for any $g_1, g_2 \in S_{0,1}^{(2)}$ with $g_1 g_2 \in S$ there follows $g_1 g_2 \in S_{0,0}^{(2)}$. Thus, any $S_0^{(2)} \supset H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ has at least one generator $g_2 \in S_{0,0}^{(2)}$. The requirement $I^2 J^2 = -\mathrm{Id} \not\in H$ specifies that $g_1 \in S_{0,1}^{(2)}$. In the case of $g_2 = I^2$ there is no loss of generality to choose $g_1 = \tau_{33} J^{2m}$, in order to form $H'_{2\times 2}(m)$. Similarly, for $g_2 = J^2$ it suffices to take $g_1 = \tau_{33} I^{2m}$, while constructing $H''_{2\times 2}(m)$. In order to determine the subgroups $-\mathrm{Id} \not\in H = \langle g_1 \rangle \times \langle g_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ with $g_1 \in S_1^{(2)}$, $g_2 \in S_0^{(2)}$, note that $g_1 = \tau_{33}^n I^k J^{-k} \theta$ does not commute with I^2, J^2 and commutes with $g_2 = \tau_{33} I^{2m} J^{2l}$ if and only if $2m \equiv 2l \pmod 4$, i.e., $0 \leq m = l \leq 1$. Bearing in mind that $\langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^{2m} J^{2m} \rangle = \langle \tau_{33}^{n+1} I^{k+2m} J^{-k+2m} \theta, \tau_{33} I^{2m} J^{2m} \rangle$, one restricts the values of k to $0 \leq k \leq 1$. For $m = 0$ denote $H^{\theta}_{2\times 2}(k) = \langle I^k J^{-k} \theta, \tau_{33} \rangle$. For $m = 1$ put $H^{\theta}_{2\times 2}(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta, \tau_{33} I^2 J^2 \rangle$.

Let $-\mathrm{Id} \not\in H \subset S$ be a subgroup of order 8. The non-abelian such H are isomorphic to quaternionic group $\mathbb{Q}_8 = \langle s, t;\, s^4 = \mathrm{Id},\ s^2 = t^2,\ sts = t \rangle$ or to dihedral group $\mathbb{D}_4 = \langle s, t;\, s^4 = \mathrm{Id},\ t^2 = \mathrm{Id},\ sts = t \rangle$. Note that $s \in S^{(4)}$ and $sts = t$ require $st \neq ts$. As far as $S^{(4)} \cup S_0^{(2)} \subset G'_{-1}$ for the abelian group $G'_{-1} = \langle \tau_{33}, I, J \rangle$, it suffices to consider $t = \tau_{33}^n I^k J^{-k} \theta \in S_1^{(2)}$ and $s = \tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ with $0 \leq n, m, l \leq 1$, $0 \leq p, k \leq 3$. However, $sts = \tau_{33}^n I^{k+2l+1} J^{k+2l+1} \theta \neq t$ reveals the non-existence of a non-abelian group $-\mathrm{Id} \not\in H \leq G_{-1}$ of order eight.

The abelian groups $H \subset S = \{\mathrm{Id}\} \cup S^{(2)} \cup S^{(4)}$ of order eight are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $\mathbb{Z}_4 \times \mathbb{Z}_2 \simeq H \subset S$ is generated by $s =$

$\tau_{33}^m I^p J^{2l+1-p} \in S^{(4)}$ and $t \in S_0^{(2)}$, as far as $t' = \tau_{33}^n I^k J^{-k}\theta \in S_1^{(2)}$ has

$$st' = \tau_{33}^{m+n} I^{p+k} J^{2l+1-(p+k)}\theta \neq \tau_{33}^{m+n} I^{2l+1-(p-k)} J^{p-k}\theta = t's.$$

For $s = \tau_{33}^n I^{2m+1} J^{2l} \in S^{(4)}$ there holds $\langle s, t\rangle = \langle s^3, t\rangle$ and it suffices to consider $s = \tau_{33}^n I J^{2l}$. Further, $t \notin \langle s^2\rangle = \langle I^2\rangle$ and $s^2 t \neq -\mathrm{Id}$ specify that $t = \tau_{33} I^{2p} J^{2q}$ for some $0 \leq p, q \leq 1$. Replacing eventually t by $ts^2 = tI^2$, one attains $t = \tau_{33} J^{2q}$. On the other hand, the generator $s = \tau_{33} I J^{2l} \in S^{(4)}$ of $H = \langle s, t\rangle$ can be restored by $st = IJ^{2(l+q)}$, so that $H = H'_{4\times 2}(l, q) = \langle IJ^{2l}, \tau_{33} J^{2q}\rangle$ for some $0 \leq l, q \leq 1$. Exchanging I with J, one obtains the remaining groups $H''_{4\times 2}(l, q) = \langle I^{2l} J, \tau_{33} I^{2q}\rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$, contained in S.

If $-\mathrm{Id} \notin H \subset S$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then arbitrary different elements $s, t, r \in H$ of order two commute and generate H. For any $x \in S$ and $M \subseteq S$, consider the centralizer $C_M(x) = \{y \in M;\, xy = yx\}$ of x in M. Looking for $s \in S^{(2)}$, $t \in C_{S^{(2)}}(s)$ and $r \in C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)$, one computes that

$$C_{S^{(2)}}(\tau_{33}^n I^2) = C_{S^{(2)}}(\tau_{33}^n J^2) = S_0^{(2)}$$

$$C_{S^{(2)}}(\tau_{33} I^{2m} J^{2m}) = S^{(2)} = S_0^{(2)} \cup S_1^{(2)}$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m} J^{-2m}\theta) = \{\tau_{33}^p I^{2q} J^{-2q}\theta,\ \ \tau_{33} I^{2p} J^{2p};\, 0 \leq p, q \leq 1\}$$

$$C_{S^{(2)}}(\tau_{33}^n I^{2m+1} J^{-2m-1}\theta) = \{\tau_{33}^p I^{2q+1} J^{-2q-1}\theta,\ \ \tau_{33} I^{2p} J^{2p};\, 0 \leq p, q \leq 1\}.$$

Any subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \subset \{\mathrm{Id}\} \cup S_0^{(2)} \cup S_1^{(2)}$ intersects $S_1^{(2)}$, due to $|S_0^{(2)}| = 6$. That allows to assume that $s \in S_1^{(2)}$ and observe that

$$C_{S^{(2)}}(s) = \{s,\ \ (-\mathrm{Id})s,\ \ \tau_{33}s,\ \ \tau_{33}(-\mathrm{Id})s,\ \ \tau_{33},\ \ \tau_{33}(-\mathrm{Id})\}.$$

If $t = \tau_{33} I^{2p} J^{2p} \in C_{S^{(2)}}(s)$ then $C_{S^{(2)}}(t) = S^{(2)}$, so that

$$H \setminus \{\mathrm{Id}, s, t\} \subseteq [C_{S^{(2)}}(s) \cap C_{S^{(2)}}(t)] \setminus \{s, t\} = C_{S^{(2)}} \setminus \{s, t\} \tag{2}$$

with $5 = |H \setminus \{\mathrm{Id}, s, t\}| \leq |C_{S^{(2)}}(s) \setminus \{s, t\}| = 4$ is an absurd. For $t \in C_{S^{(2)}}(s) \setminus \{\tau_{33} I^{2p} J^{2p};\, 0 \leq p \leq 1\}$ one has $C_{S^{(2)}}(t) = C_{S^{(2)}}(s)$, which again leads to (2). Therefore, there is no subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$ of G_{-1}.

Concerning the non-existence of subgroups $-\mathrm{Id} \notin H \subset S$ of order 16, the abelian $-\mathrm{Id} \notin H \subset S$ of order 16 may be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Any $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ is generated by $s, t \in S^{(4)}$ with $s^2 \neq t^2$. Replacing, eventually, s by s^3 and t by t^3, one has $s = \tau_{33}^n I J^{2m}$, $t = \tau_{33}^p I^{2q} J$ with $0 \leq n, m, p, q \leq 1$. Then $s^2 t^2 = I^2 J^2 = -\mathrm{Id} \in H$ is an absurd. The groups $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are generated by $s \in S^{(4)}$ and $t, rinC_{S^{(2)}}(s)$ with $r \in C_{S^{(2)}}(t)$. In the case of $s = \tau_{33}^n I J^{2m}$, the centralizer $C_{S^{(2)}}(s) = S_0^{(2)}$. Bearing in mind that $s^2 = I^2$, one observes that $\langle t, r\rangle \cap \{I^2, J^2\} = \emptyset$. Therefore $t, r \in \{\tau_{33} I^{2p} J^{2q};\, 0 \leq p, q \leq 1\}$, whereas $tr \in \{\mathrm{Id}, I^2, J^2, -\mathrm{Id}\}$. That reveals the non-existence of $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq H \not\ni -\mathrm{Id}$. The groups $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

contain 15 elements of order two, while $|S^{(2)}| = 14$. Therefore there is no abelian group $-\mathrm{Id} \notin H \leq G_{-1}$ of order 16.

There are three non-abelian groups of order 16, which do not contain a non-abelian subgroup of order 8 and consist of elements of order 1, 2 or 4. If

$$\langle s, t\,;\, s^4 = e,\ \ t^4 = e,\ \ st = ts^3 \rangle \simeq H \subset S$$

then $s, t \in S^{(4)} \subset G'_{-1} = \langle \tau_{33}, I, J \rangle$ commute and imply that s is of order two. The assumption

$$\langle a, b, c\,;\, a^4 = e,\ \ b^2 = e,\ \ c^2 = e,\ \ cbca^2b = e,\ \ ba = ab,\ \ ca = ac \rangle \simeq H \subset S$$

requires $b, c \in C_{S^{(2)}}(a) = S_0^{(2)} = \{\tau_{33} I^{2m} J^{2l},\ \ I^2,\ \ J^2\,;\, 0 \leq m, l \leq 1\}$. Then b and c commute and imply that $cbca^2b = e = a^2 = e$. Finally, for

$$G_{4,4} = \langle s, t\,;\, s^4 = e,\ \ t^4 = e,\ \ stst = e,\ \ ts^3 = st^3 \rangle$$

there follows $s, t \in S^{(4)} \subset G'_{-1}$, whereas $st = ts$. Consequently, $s^2 = t^2$ and $G_{4,4} = \{s^i t^j\,;\, 0 \leq i \leq 3,\ \ 0 \leq j \leq 1\}$ is of order ≤ 8, contrary to $|G_{4,4}| = 16$. Thus, there is no subgroup $-\mathrm{Id} \notin H \leq G_{-1}$ of order 16. $\square$

Throughout, we use the notations $H_\alpha^\beta(\gamma)$ from Lemma 9 and denote by $\Gamma_\alpha^\beta(\gamma)$ the corresponding lattices with $\Gamma_\alpha^\beta(\gamma)/\Gamma_{-1}^{(6,8)} = H_\alpha^\beta(\gamma)$.

Theorem 5. *For the groups* $H = H'_{4\times 2}(p,q) = \langle I J^{2p}, \tau_{33} J^{2q} \rangle$, $H''_{4\times 2}(p,q) = \langle I^{2p} J, \tau_{33} I^{2q} \rangle$, $H'_4(1-m,m) = \langle \tau_{33}^{1-m} I J^{2m} \rangle$, $H''_4(1-m,m) = \langle \tau_{33}^{1-m} I^{2m} J \rangle$, $H'_{2\times 2}(1) = \langle \tau_{33} J^2, I^2 \rangle$, $H''_{2\times 2}(1) = \langle \tau_{33} I^2, J^2 \rangle$, $H^\theta_{2\times 2}(n,m) = \langle \tau_{33}^n I^m J^{-m} \theta$, $\tau_{33} I^2 J^2 \rangle$ *with* $0 \leq p, q \leq 1$, $(p,q) \neq (1,1)$ *and* $0 \leq n, m \leq 1$ *the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^1$$

is dominant and not globally defined. The Baily-Borel compactifications $\widehat{\mathbb{B}/\Gamma_H}$ *are birational to ruled surfaces with elliptic bases whenever* $H = H'_{4\times 2}(0,0)$, $H''_{4\times 2}(0,0)$, $H'_4(1,0)$ *or* $H''_4(1,0)$. *The remaining ones are rational surfaces.*

Proof: According to Lemma 4(v), it suffices to prove the theorem for $H'_{4\times 2}(p,q)$ with $(p,q) \neq (1,1)$, $H'_4(1-m,m)$ $H'_{2\times 2}(1)$ and $H^\theta_{2\times 2}(n,m)$.

If $H = H'_4(1,0) = \langle \tau_{33} I \rangle$, then $\mathcal{L}^H$ is generated by $1 \in \mathbb{C}$ and Reynolds operators

$$R_H(f_{56}) = 0,\ \ R_H(f_{78}) = 0,\ \ R_H(f_{157}) = f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{258} + \mathrm{i}\mathrm{e}^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{458}$$

$$R_H(f_{168}) = f_{168} - \mathrm{i} f_{267} + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} f_{368} + \mathrm{e}^{-\frac{\pi}{2}} f_{467} = \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}} R_H(f_{368}).$$

There are four $\Gamma'_4(1,0)$-cusps : $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5$, $\bar{\kappa}_6$, $\bar{\kappa}_7 = \bar{\kappa}_8$. Applying Lemma 4 ii) to $T_1 \subset (R_H(f_{157}))_\infty, R_H(f_{168})_\infty \subseteq \sum_{i=1}^{8} T_i$, one concludes that $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^2$ and $\Phi^{H'_4(1,0)}$ is a dominant map to $\mathbb{P}(\mathcal{L}^H) \simeq \mathbb{P}^1$. Since $R_H(f_{157})|_{T_6} \neq \infty$, the entire $[\Gamma'_4(1,0),1]$ vanishes at $\bar{\kappa}_6$ and $\Phi^{H'_4(1,0)}$ is not defined at $\bar{\kappa}_6$.

The group $H = H'_{4\times 2}(0,0) = \langle I, \tau_{33}\rangle$ contains $F = H'_4(1,0)$ as a subgroup of index two with non-trivial coset representative I. Therefore $R_H(f_{56}) = R_F(f_{56}) + IR_F(f_{56}) = 0$, $R_H(f_{78}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,0)} \leq 1$. Due to

$$R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} - e^{\frac{\pi}{2}} f_{258} - e^{\frac{\pi}{2}} f_{267} + f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{458} - \mathrm{i}f_{467}$$

$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$. Lemma 6 provides $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Therefore $\dim_{\mathbb{C}} \mathcal{L}^H = 2$ and $\Phi^{H'_{4\times 2}(0,0)}$ is a dominant map to $\mathbb{P}^1$. The $\Gamma_{4\times 2}(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \overline{\kappa_3} = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Again from Lemma 6, $\left.\frac{f_{157} - e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{458}}{\Sigma_5}\right|_{T_5} = 0$, so that $R_H(f_{157})$ is regular over $T_5 + T_6$. As a result, $\Phi^{H'_{4\times 2}(0,0)}$ is not defined at $\bar{\kappa}_5 = \bar{\kappa}_6$.

For $H = H'_4(0,1) = \langle IJ^2\rangle$, the space $\mathcal{L}^H$ is spanned by 1 and Reynolds operators

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{267} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + \mathrm{i}f_{467}$$

$$R_H(f_{168}) = f_{168} + \mathrm{i}f_{258} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} + e^{-\frac{\pi}{2}} f_{458} = \mathrm{i}R_H(f_{258}).$$

The $\Gamma'_4(0,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$, $\bar{\kappa}_7$ and $\bar{\kappa}_8$. Note that $T_1 \subset (R_H(f_{157}))_\infty$, $(R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$, in order to conclude that $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157}))$ by Lemma 4 ii). Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_4(0,1)}$ is a dominant map to $\mathbb{P}^1$. Lemma 6 supplies $\left.\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$ and justifies that $\Phi^{H'_4(0,1)}$ is not defined at $\bar{\kappa}_5$.

For $H = H'_{4\times 2}(1,0) = \langle IJ^2, \tau_{33}\rangle$ note that $R_H(f_{56}) = 0$, $R_H(f_{78}) = 0$, as far as $H'_4(1,0)$ is a subgroup of $H'_{4\times 2}(1,0)$. Further,

$$R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} + e^{\frac{\pi}{2}} f_{267} + e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + f_{368} + \mathrm{i}f_{467} - \mathrm{i}f_{458}$$

has a pole over $\sum_{i=1}^{4} T_i$, according to $\left.\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ by Lemma 6 and the transitiveness of the $H'_4(1,0)$-action on $\{\kappa_i;\ 1 \leq i \leq 4\}$. Therefore $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{157})) \simeq \mathbb{C}^2$ and $\Phi^{H'_{4\times 2}(1,0)}$ is a dominant map to $\mathbb{P}^1$. One computes immediately that the $\Gamma'_{4\times 2}(1,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Again from Lemma 6, $\left.\frac{f_{157} + e^{\frac{\pi}{2}} f_{258} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i}f_{458}}{\Sigma_5}\right|_{T_5} = 0$, $R_H(f_{157})$ has no pole at $T_5 + T_6$ and $\Phi^{H'_{4\times 2}(1,0)}$ is not defined at $\bar{\kappa}_5 = \bar{\kappa}_6$.

If $H = H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ then

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 4f_{78}, \quad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - f_{368}$$

$$R_H(f_{258}) = f_{258} - f_{267} - \mathrm{i}e^{-\frac{\pi}{2}} f_{467} - \mathrm{i}e^{-\frac{\pi}{2}} f_{458} \quad \text{and} \quad 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_{2\times 2}(1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 reveals that $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}e^{\frac{\pi}{2}}f_{357}-f_{368}}{\Sigma_3}\Big|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = \frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, so that $R_H(f_{157})$, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78})$ and $\mathcal{L}^H \simeq \mathbb{C}^2$. As a result, $\Phi^{H'_{2\times 2}(1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$ and $\bar{\kappa}_2$.
For the group $H = H'_{4\times 2}(0,1) = \langle I, \tau_{33}J^2\rangle$, containing $H'_{2\times 2}(1) = \langle I^2, \tau_{33}J^2\rangle$ there follows $R_H(f_{56}) = 0$ and $\mathrm{rk}\Phi^{H'_{4\times 2}(0,1)} \leq 1$. Therefore $R_H(f_{78}) = 8f_{78}$,

$$R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + e^{\frac{\pi}{2}}f_{258} - e^{\frac{\pi}{2}}f_{267} + \mathrm{i}e^{\frac{\pi}{2}}f_{357} - f_{368} - \mathrm{i}f_{458} - \mathrm{i}f_{467}$$

and $1 \in \mathbb{C}$ span $\mathcal{L}^H$. The $\Gamma'_{4\times 2}(0,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6, $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, so that $R_H(f_{157}) \in \mathrm{Span}_{\mathbb{C}}(1, f_{78}) \simeq \mathbb{C}^2$. Thus, $\Phi^{H'_{4\times 2}(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$.
If $H = H^{\theta}_{2\times 2}(0,0) = \langle \theta, \tau_{33}I^2J^2\rangle$ then $\mathcal{L}^H$ is spanned by $1 \in \mathbb{C}$,

$$R_H(f_{56}) = 2(f_{56} + f_{78}), \quad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} - e^{\frac{\pi}{2}}f_{357} - \mathrm{i}f_{368}$$

and $R_H(f_{467}) = 2(f_{467} + f_{458})$, due to $R_H(f_{258}) = 0$. The $\Gamma^{\theta}_2(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 provides $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = 0$, $\frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{157}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. Therefore $\Phi^{H^{\theta}_2(0,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_4$.
For $H = H^{\theta}_{2\times 2}(0,1) = \langle IJ^{-1}\theta, \tau_{33}I^2J^2\rangle$ one has

$$R_H(f_{56}) = 2(f_{56} + \mathrm{i}f_{78}), \qquad R_H(f_{157}) = 0, \qquad R_H(f_{168}) = 0$$

$R_H(f_{368}) = 2(f_{368} - \mathrm{i}e^{\frac{\pi}{2}}f_{357})$, $R_H(f_{258}) = f_{258} - f_{267} - e^{-\frac{\pi}{2}}f_{458} - e^{-\frac{\pi}{2}}f_{467}$.
The $\Gamma^{\theta}_{2\times 2}(0,1)$-cusps are $\bar{\kappa}_1$, $\overline{\kappa_3}$, $\overline{\kappa_2} = \overline{\kappa_4}$, $\overline{\kappa_5} = \overline{\kappa_6} = \overline{\kappa_7} = \overline{\kappa_8}$. Lemma 6 implies that $\frac{f_{368}-\mathrm{i}e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, $\frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$, $\frac{f_{458}+f_{467}}{\Sigma_4}\Big|_{T_4} = 0$, whereas $R_H(f_{368}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}$. Consequently, $\Phi^{H^{\theta}_{2\times 2}(0,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_4$.
In the case of $H = H^{\theta}_{2\times 2}(1,0) = \langle \tau_{33}\theta, \tau_{33}I^2J^2\rangle$, the Reynolds operators are

$$R_H(f_{56}) = 2(f_{56} - f_{78}), \qquad R_H(f_{157}) = f_{157} + \mathrm{i}e^{\frac{\pi}{2}}f_{168} + \mathrm{i}f_{368} + e^{\frac{\pi}{2}}f_{357}$$

$$R_H(f_{258}) = 2(f_{258} - f_{267}), \qquad R_H(f_{458}) = 0, \ \ R_H(f_{467}) = 0.$$

The $\Gamma^{\theta}_{2\times 2}(1,0)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 yields $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}}f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{\mathrm{i}f_{368}+e^{\frac{\pi}{2}}f_{357}}{\Sigma_3}|_{T_3} = \frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$. Consequently, $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Bearing in mind that $R_H(f_{56})|_{T_5} =$

∞, one concludes that $\Phi^{H^{\theta}_{2\times 2}(1,0)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$.

Finally, for $H = H^{\theta}_{2\times 2}(1,1) = \langle \tau_{33} I J^{-1}\theta, \tau_{33} I^2 J^2 \rangle$ one has

$$R_H(f_{56}) = 2(f_{56} - \mathrm{i} f_{78}), \quad R_H(f_{157}) = 2(f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168}), \quad R_H(f_{357}) = 0$$

$$R_H(f_{368}) = 0 \quad \text{and} \quad R_H(f_{258}) = f_{258} - f_{267} + e^{-\frac{\pi}{2}} f_{467} + e^{-\frac{\pi}{2}} f_{458}.$$

The $\Gamma^{\theta}_{2\times 2}(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \overline{\kappa_8}$. Lemma 6 implies that $\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = \frac{f_{258} - f_{267}}{\Sigma_2}\Big|_{T_2} = 0$, so that $R_H(f_{157}), R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56})) \simeq \mathbb{C}^2$. As a result, $\Phi^{H^{\theta}_{2\times 2}(1,1)}$ is a dominant map to $\mathbb{P}^1$, which is not defined at $\bar{\kappa}_1$, $\bar{\kappa}_3$ and $\bar{\kappa}_2$. □

Theorem 6. *If $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$, $H''_{2\times 2}(0) = \langle \tau_{33}, J^2 \rangle$, $H^{\theta}_{2\times 2}(n) = \langle I^n J^{-n}\theta, \tau_{33} \rangle$ with $0 \le n \le 1$, $H'_4(n,n) = \langle \tau_{33}^n I J^{2n} \rangle$, $H''_4(n,n) = \langle \tau_{33}^n I^{2n} J \rangle$ with $0 \le n \le 1$ or $H_2(1,1) = \langle \tau_{33} I^2 J^2 \rangle$ then the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^2$$

is dominant and not globally defined. The surface $\widehat{\mathbb{B}/\Gamma_H}$ is K3 *for $H = H_2(1,1)$, rational for $H = H'_4(1,1)$, $H''_4(1,1)$ and ruled with an elliptic base for all the other aforementioned H.*

Proof: By Lemma 4 v), it suffices to consider $H'_{2\times 2}(0)$, $H^{\theta}_{2\times 2}(n)$, $H'_4(n,n)$ and $H_2(1,1)$.

In the case of $H = H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$, $\mathcal{L}^H$ is spanned by

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} + f_{368}$$

$$R_H(f_{258}) = f_{258} + f_{267} - \mathrm{i}e^{-\frac{\pi}{2}} f_{458} + \mathrm{i}e^{-\frac{\pi}{2}} f_{467} \quad \text{and} \quad 1 \in \mathbb{C}.$$

The $\Gamma'_{2\times 2}(0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Lemma 6 provides $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, whereas $R_H(f_{157})|_{T_1} = \infty$. Similarly, $\frac{f_{258} + f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$ suffices for $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent, according to Lemma 4 iii) and constitute a $\mathbb{C}$-basis for $\mathcal{L}^H$. In order to assert that $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, we use that $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$ by Lemma 7 with $c = \mathrm{i}e^{\frac{\pi}{2}}$. Lemma 6 provides $\frac{f_{157} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0$, in order to conclude that $R_H(f_{157})|_{T_5} \neq \infty$ and the entire $[\Gamma'_{2\times 2}(0), 1]$ vanishes at $\bar{\kappa}_5$. Therefore $\Phi^{H'_{2\times 2}(0)}$ is a dominant map to $\mathbb{P}([\Gamma'_{2\times 2}(0), 1]) = \mathbb{P}^2$, which is not defined at $\bar{\kappa}_5$.

For $H = H^{\theta}_{2\times 2}(0) = \langle \theta, \tau_{33} \rangle$, the Reynolds operators are

$$R_H(f_{56}) = 0, \quad R_H(f_{78}) = 0, \quad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168} - e^{\frac{\pi}{2}} f_{357} + \mathrm{i} f_{368}$$

$$R_H(f_{258}) = 2(f_{258} + f_{267}), \qquad R_H(f_{467}) = 0$$

generate $\mathcal{L}^H$. The $\Gamma^\theta_{2\times2}(0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. According to Lemma 6, $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, so that $R_H(f_{157})|_{T_1} = \infty$. Further, $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$ and the lemma provides $R_H(f_{258})|_{T_2} = \infty$. Therefore 1, $R_H(f_{157})$, $R_H(f_{258})$ are linearly independent and $\mathcal{L}^H \simeq \mathbb{C}^3$ by Lemma 4 iii). We claim that

$$R_H(f_{258})|_{T_1} = -2e^{-\pi \mathrm{i} v} \frac{\sigma((1+\mathrm{i})v + \omega_3)}{\sigma((1+\mathrm{i})v)} \left[\frac{\sigma(v-\omega_1)^2}{\sigma(v-\omega_2)^2} + e^{2\pi(1+\mathrm{i})v} \frac{\sigma(v-\omega_2)^2}{\sigma(v-\omega_1)^2} \right]$$

is non-constant. On one hand, $R_H(f_{258})|_{T_1}$ has no poles on $\mathbb{C} \setminus \mathbb{Q}(\mathrm{i})$. On the other hand, $\left[\frac{1}{2} R_H(f_{258})\Big|_{T_1}\right] \sigma((1+\mathrm{i})v)\Big|_{v=0} = -\sigma(\omega_3)\left[\frac{1}{\mathrm{i}^2} + \mathrm{i}^2\right] \neq 0$, so that $\lim_{v\to0} [R_H(f_{258})|_{T_1}] = \infty$. According to Lemma 4 iv), $R_H(f_{157})|_{T_1} = \infty$ and $R_H(f_{258})|_{T_1} \not\equiv \mathrm{const}$ suffice for $\Phi^{H^\theta_{2\times2}(0)}$ to be a dominant map to $\mathbb{P}^2$. The entire $\mathcal{L}^H$ takes finite values on T_4, so that $\Phi^{H^\theta_{2\times2}(0)}$ is not defined at $\bar{\kappa}_4$.

Concerning $H = H^\theta_{2\times2}(1) = \langle IJ^{-1}\theta, \tau_{33}\rangle$, one computes that

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0, \qquad R_H(f_{157}) = 2(f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168})$$

$$R_H(f_{368}) = 0, \quad R_H(f_{258}) = f_{258} + f_{267} - e^{-\frac{\pi}{2}} f_{458} + e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma^\theta_{2\times2}(1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$ and $\bar{\kappa}_5 = \bar{\kappa}_6 = \bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6 we have $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$ and $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$. Therefore $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{258})|_{T_2} = \infty$ and 1, $R_H(f_{157})$, $R_H(f_{258})$ constitute a $\mathbb{C}$-basis of $\mathcal{L}^H$, according to Lemma 4 iii). Applying Lemma 7 with $c = 0$, one concludes that $R_H(f_{157})|_{T_2} \not\equiv \mathrm{const}$. Then Lemma 4 iv) implies that $\Phi^{H^\theta_{2\times2}(1)}$ is a dominant map to $\mathbb{P}^2$. The lack of $f \in \mathcal{L}^H$ with $f|_{T_3} = \infty$ reveals that $\Phi^{H^\theta_{2\times2}(1)}$ is not defined at $\bar{\kappa}_3$.

If $H = H'_4(0,0) = \langle I\rangle$ then the Reynolds operators are

$$R_H(f_{56}) = 0, \;\; R_H(f_{78}) = 4f_{78}, \;\; R_H(f_{157}) = f_{157} - e^{\frac{\pi}{2}} f_{267} + \mathrm{i}e^{\frac{\pi}{2}} f_{357} - \mathrm{i}f_{467}$$

$$R_H(f_{168}) = f_{168} - \mathrm{i}f_{258} + \mathrm{i}e^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{458} \quad \text{and} \quad R_H(1) = 1 \in \mathbb{C}$$

span $\mathcal{L}^H$. The $\Gamma'_4(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$, $\bar{\kappa}_7$ and $\bar{\kappa}_8$. According to Lemma 4 ii), the inclusions $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157})$. Therefore $\mathcal{L}^H \simeq \mathbb{C}^3$. Observe that $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$, in order to apply Lemma 4 iv) and assert that $\Phi^{H'_4(0,0)}$ is a dominant map to $\mathbb{P}^2$. As far as $\frac{f_{157}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\Big|_{T_5} = 0$ by

Lemma 6, the abelian function $R_H(f_{157})$ has no pole on T_5. Therefore $\Phi^{H_4'(0,0)}$ is not defined at $\bar{\kappa}_5$.
For $H_4'(1,1) = \langle \tau_{33} I J^2 \rangle$ the Reynolds operators are

$$R_h(f_{56}) = 0, \quad R_H(f_{78}) = 4f_{78}, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{258} + ie^{\frac{\pi}{2}} f_{357} - if_{458}$$

$$R_H(f_{168}) = f_{168} + if_{267} + ie^{-\frac{\pi}{2}} f_{368} - e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_4'(1,1)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}_3 = \bar{\kappa}_4$, $\bar{\kappa}_5$, $\bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. Due to $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq \sum_{i=1}^{8} T_i$, Lemma 4 ii) applies to provide $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Thus, $\mathcal{L}^H \simeq \mathbb{C}^3$. According to Lemma 4 iv), $R_H(f_{78})|_{T_1} = 4\Sigma_{12}(v) \not\equiv \mathrm{const}$ suffices for $\Phi^{H_4'(1,1)}$ to be a dominant rational map to $\mathbb{P}^2$. Further, $\left.\frac{f_{157}+ie^{\frac{\pi}{2}} f_{357}}{\Sigma_5}\right|_{T_5} = 0$ by Lemma 6 implies that $R_H(f_{157})$ has no pole over T_5 and $\Phi^{H_4'(1,1)}$ is not defined at $\bar{\kappa}_5$.
If $H = H_2(1,1) = \langle \tau_{33} I^2 J^2 \rangle$ then $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \quad R_H(f_{56}) = 2f_{56}, \quad R_H(f_{78}) = 2f_{78}, \quad R_H(f_{157}) = f_{157} + ie^{\frac{\pi}{2}} f_{168}$$

$$R_H(f_{368}) = f_{368} - ie^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{258}) = f_{258} - f_{267}, \quad R_H(f_{467}) = f_{467} + f_{458}.$$

The $\Gamma_2(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\kappa}_3$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By Lemma 6 one has $\left.\frac{f_{157}+ie^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\right|_{T_1} = \left.\frac{f_{368}-ie^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\right|_{T_3} = \left.\frac{f_{258}-f_{267}}{\Sigma_2}\right|_{T_2} = \left.\frac{f_{467}+f_{458}}{\Sigma_4}\right|_{T_4} = 0$. Thus, $R_H(f_{157}), R_H(f_{368}), R_H(f_{258}), R_H(f_{467}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{78}))$ and $\mathcal{L}^H \simeq \mathbb{C}^3$. Bearing in mind that $R_H(f_{56})|_{T_5} = \infty$, $R_H(f_{78})|_{T_5} \not\equiv \mathrm{const}$, one applies Lemma 4 iv) and concludes that $\Phi^{H_2(1,1)}$ is a dominant map to $\mathbb{P}^2$. Since $\mathcal{L}^H$ has no pole over $\sum_{i=1}^{4} T_i$, the map $\Phi^{H_2(1,1)}$ is not defined at $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\kappa}_4$. □

Let us recall from Hacon and Pardini's [1] that the geometric genus $p_g(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^2)$ of a smooth minimal surface X of general type is at most 4. The next theorem provides a smooth toroidal compactification $Y = \left(\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}\right)'$ with abelian minimal model $A_{-1}/\langle \tau_{33} \rangle$ and $\dim_{\mathbb{C}} H^0(Y, \Omega_Y^2(T')) = 5$.

Theorem 7. i) *For $H = H_2' = \langle I^2 \rangle$, $H_2'' = \langle J^2 \rangle$, $H_2(n, 1-n) = \langle \tau_{33} I^{2n} J^{2-2n} \rangle$ or $H_2^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 3$ the logarithmic-canonical map*

$$\Phi^H : \widehat{\mathbb{B}/\Gamma_H} \dashrightarrow \mathbb{P}([\Gamma_H, 1]) = \mathbb{P}^3$$

has maximal $\mathrm{rk}\Phi^H = 2$. *For $H \neq H_2(n, 1-n)$ the rational map Φ^H is not globally defined and $\widehat{\mathbb{B}/\Gamma_H}$ are ruled surfaces with elliptic bases. In the case of $H = H_2(n, 1-n)$ the surface $\widehat{\mathbb{B}/\Gamma_H}$ is hyperelliptic.*

ii) *For* $H = H_2(0,0) = \langle \tau_{33} \rangle$ *the smooth surface* $\left(\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}\right)'$ *has abelian minimal model* $A_{-1}/\langle \tau_{33} \rangle$ *and the logarithmic-canonical map*

$$\Phi^{\langle \tau_{33} \rangle} : \widehat{\mathbb{B}/\Gamma_{\langle \tau_{33} \rangle}} \dashrightarrow \mathbb{P}([\Gamma_{\langle \tau_{33} \rangle}, 1]) = \mathbb{P}^4$$

is of maximal $\mathrm{rk}\Phi^{\langle \tau_{33} \rangle} = 2$.

Proof: i) By Lemma 4 v), it suffices to prove the statement for H_2', $H_2(1,0)$ and $H_2^\theta(n,k) = \langle \tau_{33}^n I^k J^{-k} \theta \rangle$ with $0 \leq n \leq 1$, $0 \leq k \leq 2$.
Note that H_2', $H_2(1,0)$ are subgroups of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$ and $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$. By Lemma 4 iv) that suffices for $\mathrm{rk}\Phi^{H_2'} = \mathrm{rk}\Phi^{H_2(1,0)} = 2$.
In the case of $H = H_2' = \langle I^2 \rangle$, the Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 2f_{78}$$

$$R_H(f_{157}) = f_{157} + \mathrm{ie}\frac{\pi}{2} f_{357}, \qquad R_H(f_{168}) = f_{168} + \mathrm{ie}^{-\frac{\pi}{2}} f_{368}$$

$$R_H(f_{258}) = f_{258} - \mathrm{ie}^{-\frac{\pi}{2}} f_{458}, \qquad R_H(f_{267}) = f_{267} + \mathrm{ie}^{-\frac{\pi}{2}} f_{467}.$$

The Γ_2'-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5$, $\bar{\kappa}_6$, $\bar{\kappa}_7$ and $\bar{\kappa}_8$. According to Lemma 4 ii), the inclusions $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}))$. Similarly, from $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ there follows $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{258}))$. As a result, one concludes that the space of the invariants $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{78}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over T_6, the rational map $\Phi^{H_2'}$ is not defined at $\bar{\kappa}_6$.
If $H = H_2(1,0) = \langle \tau_{33} I^2 \rangle$, then $\mathcal{L}^H$ is spanned by

$$1 \in \mathbb{C}, \qquad R_H(f_{56}) = 2f_{56}, \qquad R_H(f_{78}) = 0$$

$$R_H(f_{157}) = f_{157} + f_{368}, \qquad R_H(f_{258}) = f_{258} + \mathrm{ie}^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2(1,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$, $\bar{\kappa}_7 = \bar{\kappa}_8$. According to Lemma 4 iii), the inclusions $T_1 + T_3 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 + T_4 \subset (R_H(f_{258}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$ suffice for the linear independence of 1, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{258})$.
Further, observe that $H_2^\theta(n,0) = \langle \tau_{33}^n \theta \rangle$ are subgroups of $H^\theta_{2\times 2}(0) = \langle \tau_{33}, \theta \rangle$ with $\mathrm{rk}\Phi^{H^\theta_{2\times 2}(0)} = 2$. Therefore $\mathrm{rk}\Phi^{H_2^\theta(n,0)} = 2$ by Lemma 4 iv).
If $H = H_2^\theta(0,0) = \langle \theta \rangle$ then

$$R_H(f_{56}) = f_{56} + f_{78}, \quad R_H(f_{157}) = f_{157} - \mathrm{e}^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{368}) = f_{368} - \mathrm{e}^{\frac{\pi}{2}} f_{168}$$

$$R_H(f_{258}) = f_{258} + f_{267}, \qquad R_H(f_{467}) = f_{467} + f_{458}.$$

The $\Gamma_2^\theta(0,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$ and $\bar{\kappa}_6 = \bar{\kappa}_8$. According to Lemma 4 ii), $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ implies $R(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R(f_{157}))$. Lemma 6 supplies $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$ and $\frac{f_{467}+f_{458}}{\Sigma_4}\Big|_{T_4} = 0$. Therefore $R_H(f_{258})|_{T_2} = \infty$ and $R_H(f_{467}) \subset \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Thus, $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. The entire $[\Gamma_2^\theta(0,0), 1]$ vanishes at $\bar{\kappa}_4$ and $\Phi^{H_2^\theta(0,0)}$ is not globally defined.

For $H = H_2^\theta(1,0) = \langle \tau_{33}\theta \rangle$ the space $\mathcal{L}^H$ is generated by

$$1 \in \mathbb{C}, \qquad R_H(f_{56}) = f_{56} - f_{78}$$

$$R_H(f_{157}) = f_{157} + \mathrm{i} f_{368}, \quad R_H(f_{258}) = 2f_{258}, \quad R_H(f_{467}) = 0.$$

The $\Gamma_2^\theta(1,0)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$ and $\bar{\kappa}_6 = \bar{\kappa}_7$. Making use of $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ and $T_2 \subset (R_H(f_{258}))_\infty \subset T_2 + \sum_{\alpha=5}^{8} T_\alpha$, one applies Lemma 4 iii), in order to conclude that

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

The abelian functions from $\mathcal{L}^H$ have no poles along T_4, so that $\Phi^{H_2^\theta(1,0)}$ is not defined at $\bar{\kappa}_4$.

Observe that $H_2^\theta(n,1) = \langle \tau_{33}^n IJ^{-1}\theta \rangle$ are subgroups of $H_{2\times 2}^\theta(1) = \langle \tau_{33}, IJ^{-1}\theta \rangle$ with $\mathrm{rk}\Phi^{H_{2\times 2}^\theta(1)} = 2$, so that $\mathrm{rk}\Phi^{H_2^\theta(n,1)} = 2$ as well.

More precisely, Reynolds operators for $H = H_2^\theta(0,1) = \langle IJ^{-1}\theta \rangle$ are

$$R_H(f_{56}) = f_{56} + \mathrm{i} f_{78}, \quad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}, \quad R_H(f_{368}) = f_{368} - \mathrm{i}e^{\frac{\pi}{2}} f_{357}$$

$$R_H(f_{258}) = f_{258} - e^{-\frac{\pi}{2}} f_{458}, \qquad R_H(f_{267}) = f_{267} + e^{-\frac{\pi}{2}} f_{467}.$$

The Γ_2^θ-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$, $\bar{\kappa}_6 = \bar{\kappa}_7$. By Lemma 6 one has $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{368} - \mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = 0$, whereas $R_H(f_{157})|_{T_1} = \infty$, $R_H(f_{368}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$. Applying Lemma 4 ii) to the inclusions $T_2 \subset (R_H(f_{258}))_\infty, (R_H(f_{267}))_\infty \subseteq T_2 + T_4 + \sum_{\alpha=5}^{8} T_\alpha$, one concludes that $R_H(f_{267}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{258}))$. Altogether

$$\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4.$$

Since $\mathcal{L}^H$ has no pole over T_3, the rational map $\Phi^{H_2^\theta(0,1)}$ is not defined at $\bar{\kappa}_3$.

If $H = H_2^\theta(1,1) = \langle \tau_{33} IJ^{-1}\theta \rangle$ then

$$R_H(f_{56}) = f_{56} - \mathrm{i} f_{78}, \qquad R_H(f_{157}) = 2f_{157}$$

$$R_H(f_{368}) = 0, \qquad R_H(f_{258}) = f_{258} + e^{-\frac{\pi}{2}} f_{467}.$$

The $\Gamma_2^\theta(1,1)$-cusps are $\bar{\kappa}_1$, $\bar{\kappa}_3$, $\bar{\kappa}_2 = \bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$ and $\bar{\kappa}_6 = \bar{\kappa}_8$. Making use of $R_H(f_{157})|_{T_1} = \infty$, $T_H(f_{258})|_{T_2} = \infty$, one applies Lemma 4 iii), in order to conclude that $\mathcal{L}^H = \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}), R_H(f_{258})) \simeq \mathbb{C}^4$. Since $\mathcal{L}^H$ has no pole over T_3, the rational map $\Phi^{H_2^\theta(1,1)}$ is not defined at $\bar{\kappa}_3$.

Reynolds operators for $H = H_2^\theta(0,2) = \langle I^2 J^2 \theta \rangle$ are

$$R_H(f_{56}) = f_{56} - f_{78}, \quad R_H(f_{157}) = f_{157} + e^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{168}) = f_{168} + e^{-\frac{\pi}{2}} f_{368}$$

$$R_H(f_{258}) = f_{258} - f_{267}, \qquad R_H(f_{467}) = f_{467} - f_{458}.$$

The $\Gamma_2^\theta(0,2)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_7$, $\overline{\kappa_6} = \overline{\kappa_8}$. Lemma 4 ii) applies to $T_1 \subset (R_H(f_{157}))_\infty, (R_H(f_{168}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$ to provide $R_H(f_{168}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}), R_H(f_{157}))$. By Lemma 6 one has $\frac{f_{258}-f_{267}}{\Sigma_2}\Big|_{T_2} = 0$ and $\frac{f_{467}-f_{458}}{\Sigma_4}\Big|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. As a result, $R_H(f_{258}) \in \mathrm{Span}_{\mathbb{C}}(1, R_H(f_{56}))$ and $R_H(f_{467})|_{T_4} = \infty$. Lemma 4 iii) reveals that $1 \in \mathbb{C}$, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$ form a $\mathbb{C}$-basis of $\mathcal{L}^H$. Since $\mathcal{L}^H$ has no pole over T_2, the rational map $\Phi^{H_2^\theta(0,2)}$ is not defined over $\bar{\kappa}_2$.

In the case of $H = H_2^\theta(1,2) = \langle \tau_{33} I^2 J^2 \theta \rangle$ one has

$$R_H(f_{56}) = f_{56} + f_{78}, \qquad R_H(f_{157}) = f_{157} - \mathrm{i} f_{368}$$

$$R_H(f_{258}) = 0, \qquad R_H(f_{467}) = 2 f_{467}.$$

The $\Gamma_2^\theta(1,2)$-cusps are $\bar{\kappa}_1 = \bar{\kappa}_3$, $\bar{\kappa}_2$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_8$ and $\bar{\kappa}_6 = \bar{\kappa}_7$. Lemma 4 iii) applies to $T_1 \subset (R_H(f_{157}))_\infty \subseteq T_1 + T_3 + \sum_{\alpha=5}^{8} T_\alpha$, $T_4 \subset (R_H(f_{467}))_\infty \subseteq T_4 + T_6 + T_7$, in order to justify the linear independence of 1, $R_H(f_{56})$, $R_H(f_{157})$, $R_H(f_{467})$. Since $\mathcal{L}^H \simeq \mathbb{C}^4$ has no pole over T_2, the rational map $\Phi^{H_2^\theta(1,2)}$ is not defined at $\bar{\kappa}_2$.

ii) For $H = H_2(0,0) = \langle \tau_{33} \rangle$ one has the following Reynolds operators

$$R_H(f_{56}) = 0, \qquad R_H(f_{78}) = 0, \qquad R_H(f_{157}) = f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}$$

$$R_H(f_{258}) = f_{258} + f_{267}, \quad R_H(f_{368}) = f_{368} + \mathrm{i}e^{\frac{\pi}{2}} f_{357}, \quad R_H(f_{467}) = f_{467} - f_{458}.$$

There are six $\Gamma_{\langle \tau_{33} \rangle}$-cusps: $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\kappa}_3$, $\bar{\kappa}_4$, $\bar{\kappa}_5 = \bar{\kappa}_6$ and $\bar{\kappa}_7 = \bar{\kappa}_8$. By the means of Lemma 6 one observes that $\frac{f_{157} - \mathrm{i}e^{\frac{\pi}{2}} f_{168}}{\Sigma_1}\Big|_{T_1} = -2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{258}+f_{267}}{\Sigma_2}\Big|_{T_2} = 2e^{-\pi} \neq 0$, $\frac{f_{368}+\mathrm{i}e^{\frac{\pi}{2}} f_{357}}{\Sigma_3}\Big|_{T_3} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$, $\frac{f_{467}-f_{458}}{\Sigma_4}\Big|_{T_4} = 2\mathrm{i}e^{-\frac{\pi}{2}} \neq 0$. Therefore $T_i \subset (R_H(f_{i,\alpha_i,\beta_i}))_\infty \subseteq T_i + \sum_{\delta=5}^{8} T_\delta$ for $1 \leq i \leq 4$, $(\alpha_1, \beta_1) = (5,7)$, $(\alpha_2, \beta_2) = (5,8)$, $(\alpha_3, \beta_3) = (6,8)$, $(\alpha_4, \beta_4) = (6,7)$. According to Lemma 4 iii), that suffices for 1, $R_H(f_{157})$, $R_H(f_{258})$, $R_H(f_{368})$, $R_H(f_{467})$ to be a $\mathbb{C}$-basis of $\mathcal{L}^H$. Bearing in mind that $H_2(0,0) = \langle \tau_{33} \rangle$ is a subgroup of $H'_{2\times 2}(0) = \langle \tau_{33}, I^2 \rangle$ with $\mathrm{rk}\Phi^{H'_{2\times 2}(0)} = 2$, one concludes that $\mathrm{rk}\Phi^{\langle \tau_{33} \rangle} = 2$. □

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